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ON EXTENSION OF VECTOR POLYMEASURES

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(Received November 28, 1985)

Vector and operator valued polymeasures were introduced in [2], where some of their basic properties needed for the subsequent integration theory were deduced. In this paper we treat the problem of extension of a vector polymeasure from a Cartesian product of rings to the corresponding Cartesian product of generated σ-rings. Since, by definition, a non trivial vector polymeasure has no further extension to the generated product σ-ring, our setting of the extension problem for vector polymeasures is the right one. On the other hand, it is also important to know whether such further extension to the product σ-ring exists. For this latter problem we refer to References [1], [20], [21], [32] and [40] of the paper [2].

For uniform vector polymeasures, see Definition 1 in [2]; in Theorem 2 we give a necessary and sufficient condition for their extension. The proof of this result is based on Kluvánek’s theorem on extension of vector measures, see [8] or Section 1.5 in [1]. Under the assumption of existence of control polymeasures, see Section 3 in [2], in Theorem 5 we give a necessary and sufficient condition for extension of a not necessarily uniform vector polymeasure. Hence solutions of Problem 1 from [2] is of great importance for the extension problem for vector polymeasures.

We adopt the notations from [2] without repeating their meanings.

Our first theorem reduces the extension problem for operator valued polymeasures which are separately countably additive in the strong operator topology to the extension problem for vector polymeasures.

**Theorem 1.** Let $\Gamma_0: \mathcal{R}_1 \times \ldots \times \mathcal{R}_d \to L^d(X_1, \ldots, X_d; Y)$ be an operator valued $d$-polymeasure separately countably additive in the strong operator topology and for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$ let there be a separately countably additive vector $d$-polymeasure $\gamma_{(\omega)}: \sigma(\mathcal{R}_1) \times \ldots \times \sigma(\mathcal{R}_d) \to Y$ such that $\gamma_{(x_0)}(R_i) = \Gamma_0(R_i)(x_i)$ for each $(R_i) \in \mathcal{X}\mathcal{R}_i$. Put $\Gamma(A_i) = \gamma_{(x_0)}(A_i)$ for $(A_i) \in \mathcal{X}\sigma(\mathcal{R}_i)$ and $(x_i) \in \mathcal{X}X_i$. Then $\Gamma: \mathcal{R}_1 \times \ldots \times \mathcal{R}_d \to L^d(X_1, \ldots, X_d; Y)$, and $\Gamma$ is the operator valued $d$-polymeasure separately countably additive in the strong operator topology which extends $\Gamma_0$.

**Proof.** Let $(A_i) \in \mathcal{X}\sigma(\mathcal{R}_i)$. Then the $d$-linearity of $\Gamma(A_i)$ follows immediately from the $d$-linearity of $\Gamma_0(R_i)$ for each $(R_i) \in \mathcal{X}\mathcal{R}_i$ by the application of Lemma 4.
from [2]. The separate continuity of \( \Gamma(A_t) \) is a consequence of the following facts: For a fixed \((x_i) \in X X_t\) the set \( \{\gamma(x_0)(A_t), (A_t) \in \chi(\mathcal{P})\} \) is bounded in \( Y \) by the Nikodým uniform boundedness theorem for polymeasures, see (N) in [2]. Hence the set \( \{\Gamma(R), (R) \in \chi(\mathcal{P})\} \subset L^d(X_1, \ldots, X_d; Y) \) is pointwise bounded. Thus by the uniform boundedness principle for operators, see [5], this set is norm bounded in \( L^d(X_1, \ldots, X_d; Y) \). From here even the uniform boundedness of \( \{\Gamma(A^t), (A^t) \in \chi(\mathcal{P})\} \) follows by Lemma 4 in [2]. The theorem is proved.

Let us recall that a set function \( \nu: \mathcal{F} \rightarrow Y \) is called \emph{exhaustive} (also \emph{strongly bounded}, or \emph{strongly additive}) if \( \nu(A) \geq 0 \) for any sequence of pairwise disjoint sets \( A_n \in \mathcal{R}, n = 1, 2, \ldots \). The next extension theorem for uniform vector polymeasures is based on Kluvanek's extension theorem for vector measures, see [8], or Section I.5 in [1].

**Theorem 2.** Let \( \gamma_0: \mathcal{R}_1 \times \ldots \times \mathcal{R}_d \rightarrow Y \) be a uniform vector \( d \)-polymeasure. Then there is a unique uniform vector \( d \)-polymeasure \( \gamma: \sigma(\mathcal{R}_1) \times \ldots \times \sigma(\mathcal{R}_d) \rightarrow Y \) which extends \( \gamma_0 \) if and only if \( \gamma_0 \) is separately uniformly exhaustive.

**Proof.** By the extension theorem of Kluvanek, see [8], or Section I.5 in [1], for each \((A_2, \ldots, A_d) \in \mathcal{R}_2 \times \ldots \times \mathcal{R}_d\) there is a unique countably additive extension \( \gamma_1(\cdot, A_2, \ldots, A_d): \sigma(\mathcal{R}_1) \rightarrow Y \) of \( \gamma_0 \). According to Theorem 11 in [3], or to [9], these extensions \( \gamma_1(\cdot, A_2, \ldots, A_d): \sigma(\mathcal{R}_1) \rightarrow Y \) are uniformly countably additive. Hence by the well known result of Theorem IV.9.2 in [5], see also Theorem 3.10 in [6] and Theorem 7 in [3], there is a countably additive measure \( \lambda_1: \sigma(\mathcal{R}_1) \rightarrow [0, +\infty) \) such that the vector measures \( \gamma_1(\cdot, A_2, \ldots, A_d), (A_2, \ldots, A_d) \in \mathcal{R}_2 \times \ldots \times \mathcal{R}_d \), are uniformly absolutely continuous with respect to \( \lambda_1 \) on \( \sigma(\mathcal{R}_1) \). Now, by Theorem D in § 13 in [7], for each \( A_1 \in \sigma(\mathcal{R}_1) \) there is a sequence \( A_{1,n} \in \mathcal{R}_1, n = 1, 2, \ldots \) such that \( \lambda_1(A_1 \Delta A_{1,n}) \rightarrow 0 \). But then \( \gamma_1(A_{1,n}, A_2, \ldots, A_d) \rightarrow \gamma_1(A_1, A_2, \ldots, A_d) \) uniformly with respect to \( (A_2, \ldots, A_d) \in \mathcal{R}_2 \times \ldots \times \mathcal{R}_d \) by the above mentioned uniform absolute continuity. From here and from the assumption that \( \gamma_0 \) is a uniform vector \( d \)-polymeasure on \( \mathcal{R}_1 \times \ldots \times \mathcal{R}_d \) we easily see that \( \gamma_1(\cdot): \sigma(\mathcal{R}_1) \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_d \rightarrow Y \) is a uniform vector \( d \)-polymeasure.

Repeating the argument, we successively obtain extensions \( \gamma_{1,2}: \sigma(\mathcal{R}_1) \times \sigma(\mathcal{R}_2) \times \mathcal{R}_3 \times \ldots \times \mathcal{R}_d \rightarrow Y \) which are uniformly absolutely continuous with respect to \( \sigma(\mathcal{R}_1) \times \ldots \times \sigma(\mathcal{R}_d) \rightarrow Y \). From Corollary 2 of Theorem 2 in [2] we know that each vector \( d \)-polymeasure \( \gamma: 2^N \times \ldots \times 2^N \rightarrow Y (N = \{1, 2, \ldots\}) \) is uniform. Hence we have also

**Corollary 1.** A scalar bimeasure \( \beta_0: \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow K \) can be uniquely extended to a necessarily uniform scalar bimeasure \( \beta: \sigma(\mathcal{R}_1) \times \sigma(\mathcal{R}_2) \rightarrow K \) if and only if \( \beta_0 \) is separately uniformly exhaustive on \( \mathcal{R}_1 \times \mathcal{R}_2 \).
Corollary 2. Let $\Phi$ be the ring of all finite subsets of $N = \{1, 2, \ldots\}$. Then a vector $d$-polymeasure $\gamma_0: \Phi \times \ldots \times \Phi \rightarrow Y$ can be uniquely extended to a necessarily uniform vector $d$-polymeasure $\gamma: 2^N \times \ldots \times 2^N \rightarrow Y$ if and only if $\gamma_0$ is separately uniformly exhaustive on $\Phi \times \ldots \times \Phi$.

Unfortunately the author knows no further types of uniform polymeasures on Cartesian products of $\sigma$-rings, except the trivial case of polymeasures with bounded variations (clearly uniform) which have extensions with bounded variations.

Theorem 3. Let $Y$ be a reflexive Banach space, and let $\gamma_0: R_1 \times \ldots \times R_d \rightarrow Y$ be a vector $d$-polymeasure such that the scalar $d$-polymeasure $y^*\gamma_0$ can be extended to a $d$-polymeasure on $\sigma(R_1) \times \ldots \times \sigma(R_d)$ for each $y^* \in Y^*$. Then there is a unique vector $d$-polymeasure $\gamma: \sigma(R_1) \times \ldots \times \sigma(R_d) \rightarrow Y$ which extends $\gamma_0$.

Proof. Let $(A_i) \in \mathcal{X}(R_i)$ be fixed, and for $y^* \in Y^*$ put $\gamma(A_i)(y^*) = \gamma_{y^*}(A_i)$, where $\gamma_{y^*}$ is the unique extension of $y^*\gamma_0$. By Lemma 4 in [2], $\gamma(A_i): Y^* \rightarrow K$ is clearly linear. Since for each $y^* \in Y^*$ the range of the scalar $d$-polymeasure $\gamma_{y^*}$ is bounded on $\mathcal{X}(R_i)$, see the Nikodým theorem (N) in [2], the uniform boundedness principle for operators implies that the set $\{\gamma_{y^*}(R_i)\}$ embedded in $Y^{**}$ is bounded, see Theorem II.3.20 in [5]. Hence there is a constant $C > 0$ such that $|\gamma_{y^*}(R_i)(y^*)| \leq C \cdot |y^*|$ for each $(R_i) \in \mathcal{X}(R_i)$ and each $y^* \in Y^*$. But then by Lemma 4 in [2] we have the inequality $|\gamma(A_i)(y^*)| = |\gamma_{y^*}(A_i)| \leq C \cdot |y^*|$ for each $(A_i) \in X\sigma(R_i)$ and each $y^* \in Y^*$. Thus $\gamma(A_i) \in Y^{**} = Y$. Now the separate countable additivity of $\gamma$ follows from the Orlicz-Pettis theorem, see the beginning of Section 1 in [2]. The theorem is proved.

Since polymeasures with bounded variations (the variation of a polymeasure was introduced in Definition 3 in [2]) are evidently uniform, using Theorem 2 we obtain

Corollary 1. Let $Y$ be a reflexive Banach space, and let $\gamma_0: R_1 \times \ldots \times R_d \rightarrow Y$ be a vector $d$-polymeasure such that the variation $\nu(y^*\gamma_0, \ldots)$ is a bounded $d$-polymeasure on $R_1 \times \ldots \times R_d$ for each $y^* \in Y^*$. Then there is a unique vector $d$-polymeasure $\gamma: \sigma(R_1) \times \ldots \times \sigma(R_d) \rightarrow Y$ which extends $\gamma_0$.

Using Corollary 1 of Theorem 2 we also obtain

Corollary 2. Let $Y$ be a reflexive Banach space. Then a vector bimeasure $\beta_0: R_1 \times \times R_2 \rightarrow Y$ can be uniquely extended to a vector bimeasure $\beta: \sigma(R_1) \times \sigma(R_2) \rightarrow Y$ if and only if for each $y^* \in Y^*$ the scalar bimeasure $y^*\beta_0: R_1 \times R_2 \rightarrow K$ is separately uniformly exhaustive.

It is an interesting open problem whether Theorem 3 remains valid if $Y$ is a Banach space not containing an isomorphic copy of the space $c_0$.

We shall say that a set function $v_1: R \rightarrow Y_1$ is $(\delta \rightarrow \varepsilon)$-absolutely continuous with respect to a set function $v: R \rightarrow Y$ if $|v_1(A)| < \varepsilon$ whenever $A \in R$ and $v(A) < \delta$, where $\overline{v}(A) = \sup \{|v(B)|, B \in A \cap R\}$. For countably additive vector measures on a $\sigma$-ring the $(\delta \rightarrow \varepsilon)$-absolute continuity coincides with the (we may say $0 \rightarrow 0$)
absolute continuity of Definition 2 in [2], see [3]. The next theorem is a generalization of Theorem 12 from [3].

**Theorem 4.** Let \( γ: σ(Ω_1) \times \ldots \times σ(Ω_d) → Y \) be a vector \( d \)-polymeasure and let \( λ_i: σ(Ω_i) → [0, +∞) \), \( i = 1, \ldots, d \), be countably additive measures. Then \( γ \) is separately (\( δ → ε \))-absolutely continuous with respect to the polymeasure \( λ_1 \times \ldots \times λ_d \) on \( σ(Ω_1) \times \ldots \times σ(Ω_d) \) if and only if its restriction to \( Ω \times \ldots \times Ω \) is separately (\( δ → ε \))-absolutely continuous with respect to the polymeasure \( λ_1 \times \ldots \times λ_d : Ω_1 \times \ldots \times Ω_d → [0, +∞) \).

**Proof.** The necessity implication is trivial. Conversely, let \( γ: R_1 \times \ldots \times Ω \rightarrow Y \) be separately (\( δ → ε \))-absolutely continuous with respect to \( λ_1 \times \ldots \times λ_d : Ω \rightarrow [0, +∞) \). According to Theorem 12 in [3], \( γ(\cdot, R_2, \ldots, R_d): σ(Ω_1) \rightarrow Y \) is (\( δ → ε \))-absolutely continuous with respect to \( λ_1: σ(Ω_1) → [0, +∞) \) on \( σ(Ω_1) \) for each \( (R_2, \ldots, R_d) \in Ω_2 \times \ldots \times Ω_d \). Hence if \( N_1 ∈ σ(Ω_1) \) and \( λ_1(N_1) = 0 \), then \( γ(N_1, R_2, \ldots, R_d) = 0 \) for each \( (R_2, \ldots, R_d) \in Ω_2 \times \ldots \times Ω_d \). Now let \( (A_2, \ldots, A_d) ∈ \ σ(Ω_2) \times \ldots \times σ(Ω_d) \). Since \( γ(\cdot, \ldots, A_d): σ(Ω_2) \times \ldots \times σ(Ω_d) → Y \) is a vector \((d-1)\)-polymeasure, by Lemma 4 in [2] there are \( A_i, n ∈ Ω_i \), \( i = 2, \ldots, d \), \( n = 1, 2, \ldots \) such that

\[
γ(N_1, A_2, \ldots, A_d) = \lim_{n→∞} γ(N_1, A_2, n \ldots, A_d, n) = 0 .
\]

Hence \( γ(\cdot, A_2, \ldots, A_d): σ(Ω_1) → (0-0) \); hence equivalently (\( δ → ε \))-absolutely continuous with respect to \( λ_1: σ(Ω_1) → [0, +∞) \) on \( σ(Ω_1) \) for each \( (A_2, \ldots, A_d) ∈ \ σ(Ω_2) \times \ldots \times σ(Ω_d) \). By symmetry in coordinates, analogous assertions hold for \( i = 2, \ldots, d \). The theorem is proved.

**Theorem 5.** Let \( γ_0: Ω_1 \times \ldots \times Ω \rightarrow Y \) be a vector \( d \)-polymeasure and suppose that there are countably additive measures \( λ_i: σ(Ω_i) → [0, +∞) \), \( i = 1, \ldots, d \), such that \( γ_0 \) is separately (\( δ → ε \))-absolutely continuous with respect to \( λ_1 \times \ldots \times λ_d \) on \( Ω_1 \times \ldots \times Ω \). Then there is a unique separately countably additive extension \( γ: σ(Ω_1) \times \ldots \times σ(Ω_d) → Y \) of \( γ_0 \) if and only if the following condition holds:

(1) if \( A_i, n ∈ Ω_i \), \( i = 1, \ldots, d \), \( n = 1, 2, \ldots \), and \( γ(\cdot, A_2, n \ldots, A_d, n) \) converges a.e. \( λ_1 - (λ_i: σ(Ω_i) → [0, +∞)) \) on \( T_i \) for each \( i = 1, \ldots, d \) then

\[
\lim_{n_1, \ldots, n_d → ∞} γ(A_1, n_1 \ldots, A_d, n_d) ∈ Y \text{ exists}. \]

In that case \( λ_1 \times \ldots \times λ_d \) is a control \( d \)-polymeasure for \( γ \) in the sense of Definition 4 in [2].

**Proof.** Let \( (R_1, \ldots, R_d, n) ∈ Ω_1 \times \ldots \times Ω_{d-1} \). Since by assumption \( γ_0(R_1, \ldots, R_{d-1}, .): Ω_d → Y \) is (\( δ → ε \))-absolutely continuous with respect to the bounded measure \( λ_d: Ω_d → [0, +∞) \), by Klüvánek's theorem on extension, see [8] or Section I.5 in [1], there is a unique countably additive extension \( γ_d(R_1, \ldots, R_{d-1}, .): σ(Ω_d) → Y \) of \( γ_0(R_1, \ldots, R_{d-1}, .): Ω_d → Y \). According to Theo-
Suppose (1) and let $(A_1, \ldots, A_d) \in \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_d)$. Due to Theorem D in §13 and Theorem D in §22 in [7] there are $A_{i,n} \in \mathcal{H}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ such that $x_{A_{i,n}}(\cdot) \to x_{A_i}(\cdot) \lambda_i$ a.e. on $T_i$ for each $i = 1, \ldots, d$. Hence by (1) we may unambiguously define $\gamma(A_1, \ldots, A_d) = \lim_{n_1,\ldots,n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d,n_d})$. (If $A_{i,n} \in \mathcal{H}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ and $x_{A_{i,n}}(\cdot) \to x_{A_i}(\cdot) \lambda_i$ a.e. on $T_i$, then $x_{A_{i,n}}(\cdot) \to x_{A_i}(\cdot) \lambda_i$ a.e. on $T_i$, where $A_{i,n} = A_{i,n}$ for $n$ odd and $= A_{i,n}$ for $n$ even.)

Clearly $\gamma$ extends $\gamma_0$, so it remains to show that $\gamma$ is separately countably additive. Since our assumptions are symmetric with respect to the coordinates, it is enough to prove that $\gamma(A_1, \ldots, A_{d-1}, \cdot) : \sigma(\mathcal{H}_d) \to Y$ is countably additive for each $(A_1, \ldots, A_{d-1}) \in \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_{d-1})$.

Let $(A_1, \ldots, A_{d-1}) \in \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_{d-1})$ be fixed. Then for each $A_d \in \sigma(\mathcal{H}_d)$,

$$\gamma(A_1, \ldots, A_d) = \lim_{n_1,\ldots,n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d,n_d}) =$$

$$= \lim_{n_1,\ldots,n_{d-1} \to \infty} \lim_{n_d \to \infty} \gamma_0(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}}, A_{d,n_d}) =$$

$$= \lim_{n_1,\ldots,n_{d-1} \to \infty} \gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(A_d),$$

since $x_{A_{d,n}}(\cdot) \to x_{A_d}(\cdot) \lambda_d$ a.e. on $T_d$, and since $\gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(\cdot) : \sigma(\mathcal{H}_d) \to Y$ is $(\delta \to \varepsilon)$-absolutely continuous with respect to $\lambda_d$. Consequently, since $\gamma_d(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}})(\cdot) : \sigma(\mathcal{H}_d) \to Y$, $n_1, \ldots, n_{d-1} = 1, 2, \ldots$ are countably additive vector measures, the countable additivity of $\gamma(A_1, \ldots, A_{d-1}, \cdot) : \sigma(\mathcal{H}_d) \to Y$ follows from the (VHSN) theorem, see Theorem I.4.8 in [1]. Hence the sufficiency is proved.

Conversely, let $\gamma : \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_d) \to Y$ be the unique separately countably additive extension of $\gamma_0$. First we show that $\lambda_1 \times \cdots \times \lambda_d$ is a control $d$-polymeasure for $\gamma$ on $\sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_d)$. By symmetry in the coordinates it is enough to deduce that $A_d \in \sigma(\mathcal{H}_d)$ and $\lambda_d(A_d) = 0$ implies $\gamma(A_1, \ldots, A_d) = 0$ for each $(A_1, \ldots, A_{d-1}) \in \sigma(\mathcal{H}_1) \times \cdots \times \sigma(\mathcal{H}_{d-1})$. Hence let $A_d$ be such and let $(A_1, \ldots, A_{d-1})$ be fixed. According to Lemma 4 in [2] there are $A_{i,n} \in \mathcal{H}_i$, $i = 1, \ldots, d - 1$, $n = 1, 2, \ldots$ such that $\gamma(A_1, \ldots, A_d) = \lim_{n \to \infty} \gamma(A_{1,n}, \ldots, A_{d-1,n}, A_d)$. But $\gamma(A_{1,n}, \ldots, A_{d-1,n}, A_d) = 0$ for each $n = 1, 2, \ldots$ by the $(0 \to 0)$-absolute continuity of $\gamma_d(A_{1,n}, \ldots, A_{d-1,n})$ a.e. on $T$ with respect to $\lambda_d$, which was proved at the beginning of the proof. Now using this control $d$-polymeasure for $\gamma$ we immediately obtain condition (1) as a direct consequence of Theorem 1 in [2]. The theorem is proved.

Let us note that for $d = 1$ condition (1) in the just proved theorem is a consequence of the assumed $(\delta \to \varepsilon)$-absolute continuity of $\gamma_0$ with respect to $\lambda_1$. Whether the analogue holds for $d > 1$ remains an open problem.
Using Theorem 11 in [2] we immediately obtain the following

**Corollary.** Let each \( \mathcal{R}_i, i = 1, 2, ..., d, \) be a countable ring. Then a vector \( d \)-polymeasure \( \gamma_0: \mathcal{S}_1 \times ... \times \mathcal{S}_d \to Y \) can be uniquely extended to a vector \( d \)-polymeasure \( \gamma: \sigma(\mathcal{R}_1) \times ... \times \sigma(\mathcal{R}_d) \to Y \) if and only if there are countably additive measures \( \lambda_i: \sigma(\mathcal{R}_1) \to [0, +\infty), \) \( i = 1, ..., d, \) such that \( \gamma_0 \) is separately \((\delta \to \varepsilon)\)-absolutely continuous with respect to \( \lambda_1 \times ... \times \lambda_d \) on \( \mathcal{R}_1 \times ... \times \mathcal{R}_d, \) and (1) of Theorem 5 holds.

Let us note that if it turns out that each vector \( d \)-polymeasure \( \gamma: \mathcal{S}_1 \times ... \times \mathcal{S}_d \to Y \) has a control \( d \)-polymeasure, see Section 3 in [2], then the above corollary will be true for any rings \( \mathcal{R}_i, i = 1, ..., d. \)

In some situations the following result may be useful.

**Theorem 6.** Let \( 1 \leq d_1 < d \) be a positive integer, let \( \gamma_0: \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1} \times \times \mathcal{R}_{d_1+1} \times ... \times \mathcal{R}_d \to Y \) be a vector \( d \)-polymeasure, and let there be countably additive measures \( \lambda_i: \sigma(\mathcal{R}_1) \to [0, +\infty), \) \( i = d_1 + 1, ..., d, \) such that \( \gamma_0(A_1, ..., A_{d_1}) = \gamma(A_1, ..., A_d) \) for each \( (A_1, ..., A_{d_1}) \in \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1}. \) Then \( \gamma_0 \) can be uniquely extended to a vector \( d \)-holomeasure \( \gamma: \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1} \times \times \mathcal{R}_{d_1+1} \times ... \times \mathcal{R}_d \to Y \) if and only if \( \gamma_0(A_1, ..., A_{d_1}, ...) : \mathcal{R}_{d_1+1} \times ... \times \mathcal{R}_d \to Y \) can be extended to a \((d - d_1)\)-polymeasure \( \gamma_{d-d_1}(A_1, ..., A_{d_1}): \sigma(\mathcal{R}_{d_1+1}) \times ... \times \sigma(\mathcal{R}_d) \to Y \) for each \( (A_1, ..., A_{d_1}) \in \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1} \). (In that case \( \gamma(A_1, ..., A_d) = \gamma_{d-d_1}(A_1, ..., A_{d_1})(A_{d_1+1}, ..., A_d) \) for each \( (A_1, ..., A_d) \in \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1} \times \times \mathcal{R}_{d_1+1} \times ... \times \mathcal{R}_d \) by the uniqueness of extensions.)

**Proof.** The necessity is obvious. Conversely, suppose that the extensions \( \gamma_{d-d_1}(A_1, ..., A_{d_1}): \sigma(\mathcal{R}_{d_1+1}) \times ... \times \sigma(\mathcal{R}_d) \to Y \) exist for each \( (A_1, ..., A_{d_1}) \in \mathcal{S}_1 \times ... \times \mathcal{S}_{d_1}. \) Define \( \gamma \) by the equality in the brackets above. We have to show that \( \gamma(A_1, ..., A_{d_1+1}, ..., A_d): \mathcal{S}_1 \times ... \times \mathcal{S}_d \to Y \) is separately countably additive for each \( (A_{d_1+1}, ..., A_d) \in \sigma(\mathcal{R}_{d_1+1}) \times ... \times \sigma(\mathcal{R}_d). \) Let \( A_i \in \sigma(\mathcal{R}_1), \) \( i = d_1 + 1, ..., d, \) according to Theorem D in §13 in [7] there are \( A_{i,n} \in \mathcal{R}_i, \) \( i = d_1 + 1, ..., d, n = 1, 2, ..., \) such that \( \lambda_i(A_i \Delta A_{i,n}) \to 0. \) Hence \( \gamma(A_1, ..., A_d, A_{d_1+1,n}, ..., A_{d,n}) \to \gamma(A_1, ..., A_d) \) for each \( (A_1, ..., A_d) \in \mathcal{S}_1 \times ... \times \mathcal{S}_d, \) by Theorem 12 in [2]. Since by assumption \( \gamma(A_1, ..., A_{d_1+1}, ..., A_{d,n}): \mathcal{S}_1 \times ... \times \mathcal{S}_d \to Y \) are \( d_1 \)-polymeasures for each \( n = 1, 2, ..., \) the (VHSN)-theorem for polymeasures, see the beginning in [2], implies the required separate countable additivity of \( \gamma(A_1, ..., A_{d_1+1}, ..., A_d): \mathcal{S}_1 \times ... \times \mathcal{S}_d \to Y. \) The theorem is proved.

Using Theorem 2 in [4], Theorem 10 and Corollary of Theorem 14 in [2] we easily obtain our last

**Theorem 7.** Let \( T_i, i = 1, ..., d, \) be locally compact Hausdorff topological spaces, and let \( C_{T_i}(C_{T_i},) \) denote the class of compact \( (\sigma_0, d) \) subsets of \( T_i. \) Then

1) for each separately regular vector Borel \( d \)-polymeasure
\( \gamma: \sigma(\mathcal{C}_1) \times \ldots \times \sigma(\mathcal{C}_d) \to Y \) and for each \( (A_i) \in X \sigma(\mathcal{C}_i) \) there is a \( d \)-tuple \( (B_i) \in X \sigma(\mathcal{C}_0, \ldots, \mathcal{C}_i) \) such that \( \gamma(A_i) = \gamma(B_i) \);

2) each vector Baire \( d \)-holymeasure \( \gamma_0: X \sigma(\mathcal{C}_0) \to Y \), which has a control \( d \)-polymeasure, can be uniquely extended to a separately regular vector Borel \( d \)-polymeasure \( \gamma: X \sigma(\mathcal{C}_1) \to Y \), and

3) each uniform vector Baire \( d \)-polymeasure \( \gamma_0: X \sigma(\mathcal{C}_0, \ldots, \mathcal{C}_i) \to Y \) can be uniquely extended to a separately uniformly regular vector Borel \( d \)-polymeasure \( \gamma: X \sigma(\mathcal{C}_1) \to Y \).

The analogues hold if \( \sigma(\mathcal{C}_i) \) is replaced by \( \sigma(\mathcal{U}_i) \), \( i = 1, \ldots, d \), where \( \mathcal{U}_i \) denotes the class of all open subsets of \( T_i \).

References


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