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α-paracompact subsets and well-situated subsets


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α-PARACOMPACT SUBSETS AND WELL-SITUATED SUBSETS

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(Received November 20, 1985)

INTRODUCTION

In Section 1 we study α-paracompact subsets, defined by C. E. Aull. We obtain some covering properties of α-paracompact subsets which are similar to the properties of paracompact spaces. In particular, we characterize α-paracompact subsets in regular spaces. Moreover, we study the behaviour of α-paracompact subsets under perfect mappings.

In Section 2 we consider R. Telgársky's well-situated subsets. The properties of α-paracompact subsets of Section 1 yield properties of well-situated subsets. Well-situated subsets are related to Tamano's problem (i.e.: to give an intrinsic description of $T_2$ spaces $X$ such that $X \times Y$ is paracompact for each paracompact $T_2$ space $Y$) which remains open.

In Section 3 we solve a problem of Telgársky. We establish that in the realm of $T_2$ spaces, the class $\mathcal{H}^*$ is perfect.

1. α-PARACOMPACT SUBSETS

C. E. Aull in [1] defined the notion of an α-paracompact subset. A subset $E$ of a topological space $X$ is said to be $\alpha$-paracompact in $X$ if every covering of $E$ by open subsets of $X$ has a refinement by open subsets of $X$, locally finite in $X$, which covers $E$. We continue in this paper the study of α-paracompact subsets. We omit the proofs in this section.

1.1. Proposition. Let $X$ be a topological space. Then:

1) If $X$ is $T_2$, $E$ is an α-paracompact subset in $X$ and $F$ is a closed subset of $E$, then $F$ is α-paracompact in $X$.

2) If $\{E_j\}_{j \in J}$ is a set of subsets of $X$, locally finite in $X$ and such that $E_j$ is α-paracompact in $X$ for every $j \in J$ and there exists a locally finite family of open

*) Part of this paper is contained in the author's Doctoral Thesis written under the supervision of Professor E. Outerelo. This paper has been published in a shorted version in Quest.
subsets \( \{U_j\}_{j \in J} \) of \( X \) such that \( E_j \subseteq U_j \) for every \( j \in J \), then \( \bigcup E_j \) is \( \alpha \)-paracompact in \( X \). In particular, every finite union of \( \alpha \)-paracompact subsets is \( \alpha \)-paracompact.

1.2. Remark. In Proposition 1.1, point 2), the hypothesis "\( \{U_j\}_{j \in J} \) is a locally finite family" cannot be replaced by the hypothesis "\( \{U_j\}_{j \in J} \) is a locally finite set". Indeed, in the Niemytski plane \( X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \) (in which for \( y_0 > 0 \) the neighbourhoods of \( (x_0, y_0) \) are the usual neighbourhoods in the plane relativized with respect to \( X \), while for \( y_0 = 0 \) the neighbourhoods of \( (x_0, 0) \) consist of open circles with center \( (x_0, y) \) and radius \( y \) with the point \( (x_0, 0) \) for each \( y > 0 \)), \( \{E_q\}_{q \in Q} \) where \( E_q = \{(q, 0)\} \) is a locally finite set of \( \alpha \)-paracompact subsets of \( X \) such that \( \bigcup E_q \) is not \( \alpha \)-paracompact ([1] p. 50), and \( \{U_q\}_{q \in Q} \) where \( U_q = X \) for each \( q \in Q \) is a locally finite set such that \( E_q \subseteq U_q \) for each \( q \in Q \).

1.3. Theorem. Let \( X \) be a regular space and \( E \) a subset of \( X \). The following conditions are equivalent:

a) \( E \) is an \( \alpha \)-paracompact subset in \( X \).

b) 1) Every covering \( \mathcal{U} \) of \( E \) by open subsets of \( X \) has a refinement \( \mathcal{V} \) by open subsets of \( X \), \( \sigma \)-locally finite in \( X \), which covers \( E \), and 2) Every open subset \( U \) of \( X \) such that \( E \subseteq U \) has an open subset \( V \) such that \( E \subseteq V \subseteq \overline{V} \subseteq U \).

c) Every covering \( \mathcal{U} \) of \( E \) by open subsets of \( X \) has a refinement \( \mathcal{A} = \{A_s\}_{s \in S} \) by arbitrary sets of \( X \), locally finite in \( X \), such that \( E \subseteq (\bigcup A_s)^0 \).

d) Every covering \( \mathcal{U} \) of \( E \) by open subsets of \( X \) has a refinement \( \mathcal{F} = \{F_j\}_{j \in J} \) by closed subsets of \( X \), locally finite in \( X \), such that \( E \subseteq (\bigcup F_j)^0 \).

Remark. Theorem 1.3 implies Corollary 3 and Theorem 4 of [1].

1.4. Proposition. Let \( X \) be a regular space and \( E \) an \( \alpha \)-paracompact subset in \( X \). Then:

1) Every covering \( \mathcal{U} \) of \( E \) by open subsets of \( X \) has a refinement by open subsets of \( X \), barycentric in \( X \), which covers \( E \).

2) Every covering \( \mathcal{U} \) of \( E \) by open subsets of \( X \) has a star refinement by open subsets of \( X \), which covers \( E \).

1.5. Proposition. Let \( X \) be a regular space and \( E \) an \( \alpha \)-paracompact subset in \( X \). Then for every family \( \{F_s\}_{s \in S} \) of subsets of \( E \), locally finite (discrete) in \( X \), there is a family \( \{U_s\}_{s \in S} \) of open subsets of \( X \), locally finite (discrete) in \( X \) and such that \( F_s \subseteq U_s \) for every \( s \in S \).

We pass now to the study of the behaviour of the \( \alpha \)-paracompact subsets under perfect mappings.

1.6. Proposition. Let \( X \) and \( X' \) be topological spaces and \( f: X \to X' \) a perfect mapping. If \( E' \) is an \( \alpha \)-paracompact subset in \( X' \) then \( f^{-1}(E') \) is an \( \alpha \)-paracompact subset in \( X \).
1.7. **Remark.** Proposition 1.6 implies that if $X$ and $Y$ are topological spaces, $E$ is an $\alpha$-paracompact subset in $X$ and $Y$ is compact, then $E \times Y$ is an $\alpha$-paracompact subset in $X \times Y$.

However, if $X$ and $Y$ are topological spaces, $E$ is an $\alpha$-paracompact subset in $X$ and $F$ is an $\alpha$-paracompact subset in $Y$, $E \times F$ is not necessarily an $\alpha$-paracompact subset in $X \times Y$. Indeed, $Q$ is an $\alpha$-paracompact subset in the Michael line $(R, T)$, $R \setminus Q$ is an $\alpha$-paracompact subset in $R \setminus Q$, but $Q \times (R \setminus Q)$ is not an $\alpha$-paracompact subset in $(R, T) \times (R \setminus Q)$. (Since the sets $Q \times (R \setminus Q)$ and $C = \{(x, x) | x \in R \setminus Q\}$ are disjoint closed sets which are not strongly separated, it follows from Theorem 5 in [1] that $Q \times (R \setminus Q)$ is not an $\alpha$-paracompact subset.)

1.8. **Proposition.** Let $X$ and $X'$ be topological spaces, where $X$ is regular, let $f$ be a perfect mapping from $X$ onto $X'$ and $E'$ a subset of $X'$. If $f^{-1}(E')$ is an $\alpha$-paracompact set in $X$ then $E'$ is an $\alpha$-paracompact subset in $X'$.

1.9. **Remark.** In Proposition 1.8 the hypothesis “$f$ is a mapping onto” cannot be omitted. Indeed, let $(R, T)$ be the Michael line. Then the mapping $i: Q \times (R \setminus Q) \rightarrow (R, T) \times (R \setminus Q)$ is a perfect mapping but is not onto, $Q \times (R \setminus Q)$ is an $\alpha$-paracompact subset in $Q \times (R \setminus Q)$ with the usual topology, and $Q \times (R \setminus Q)$ is not an $\alpha$-paracompact subset in $(R, T) \times (R \setminus Q)$ (see 1.7).

1.10. **Proposition.** Let $X$ and $X'$ be topological spaces and $f$ a perfect and open mapping from $X$ onto $X'$; if $E$ is an $\alpha$-paracompact subset in $X$ then $f(E)$ is an $\alpha$-paracompact subset in $X'$.

1.11. **Remark.** Let $X$ and $X'$ be topological spaces and $f$ a perfect mapping from $X$ onto $X'$; if $E$ is an $\alpha$-paracompact subset in $X$, $f(E)$ is not necessarily $\alpha$-paracompact in $X'$. Indeed, let $(R, T)$ be the Michael line, $j_1: Q \times (R \setminus Q) \rightarrow Q \times (R \setminus Q) + + (R, T) \times (R \setminus Q)$ and $j_2: (R, T) \times (R \setminus Q) \rightarrow Q \times (R \setminus Q) + (R, T) \times \times (R \setminus Q)$. Then the mapping onto $f: Q \times (R \setminus Q) + (R, T) \times (R \setminus Q) \rightarrow (R, T) \times \times (R \setminus Q)$ such that

$$f(j_1(x, y)) = (x, y) \quad \text{if} \quad (x, y) \in Q \times (R \setminus Q)$$

$$f(j_2(x, y)) = (x, y) \quad \text{if} \quad (x, y) \in R \times (R \setminus Q)$$

is a perfect mapping, $f(j_1(Q \times (R \setminus Q))) = Q \times (R \setminus Q)$, $j_1(Q \times (R \setminus Q))$ is an $\alpha$-paracompact subset in $Q \times (R \setminus Q) + (R, T) \times (R \setminus Q)$ and $Q \times (R \setminus Q)$ is not an $\alpha$-paracompact subset in $(R, T) \times (R \setminus Q)$ (1.7).

2. **WELL-SITUATED SUBSETS**

The concept of a well-situated subset was introduced by R. Telgársky in [4]. Using the notion of an $\alpha$-paracompact subset, H. W. Martin phrased the definition of a well-situated subset of a space $X$ as follows: a subset $E$ of a space $X$ is said
to be well-situated in $X$ if for every paracompact $T_2$ space $Y$, $E \times Y$ is an $\alpha$-paracompact subset in $X \times Y$\([3]\)).

If $E$ is a well-situated subset of a space $X$ then $E$ is an $\alpha$-paracompact subset in $X$, but $Q$ is $\alpha$-paracompact in $(R, T)$, the Michael line, and $Q$ is not a well-situated subset in $(R, T)$ (cf. 1.7).

From Section 1 we easily obtain the following theorems.

2.1. Proposition. Let $X$ be a topological space. Then:

1) If $X$ is $T_2$, $E$ is a well-situated subset in $X$ and $F$ is a closed subset of $E$, then $F$ is a well-situated subset in $X$.

2) If $\{E_j\}_{j \in J}$ is a set of subsets of $X$, locally finite in $X$ and such that $E_j$ is a well-situated subset in $X$ for every $j \in J$, and there exists a locally finite family of open subsets $\{U_j\}_{j \in J}$ of $X$ such that $F_j \subset U_j$ for every $j \in J$, then $\bigcup E_j$ is a well-situated subset in $X$. In particular, every finite union of well-situated subsets is well-situated.

2.2. Theorem. Let $X$ be a regular space and $E$ a subset of $X$. The following conditions are equivalent:

a) $E$ is a well-situated subset in $X$.

b) For every paracompact $T_2$ space $Y$, 1) Every covering $\mathcal{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathcal{V}$ by open subsets of $X \times Y$, $\sigma$-locally finite in $X \times Y$, which covers $E \times Y$, and 2) Every open subset $U$ of $X \times Y$ such that $E \times Y \subset U$ has an open subset $V$ such that $E \times Y \subset V \subset V \subset U$.

c) For every paracompact $T_2$ space $Y$, every covering $\mathcal{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathcal{A} = \{A_s\}_{s \in S}$ by arbitrary sets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset \bigcup A_s^0$.

d) For every paracompact $T_2$ space $Y$, every covering of $E \times Y$ by open subsets of $X \times Y$ has a refinement $\mathcal{F} = \{F_j\}_{j \in J}$ by closed subsets of $X \times Y$, locally finite in $X \times Y$, such that $E \times Y \subset \bigcup F_j^0$.

2.3. Proposition. Let $X$ be a regular space and $E$ a well-situated subset in $X$. Then:

1) For every paracompact $T_2$ space $Y$, every covering $\mathcal{U}$ of $E \times Y$ by open subsets of $X \times Y$ has a refinement by open subsets of $X \times Y$, barycentric in $X \times Y$, which covers $E \times Y$.

2) For every paracompact $T_2$ space $Y$, every covering of $E \times Y$ by open subsets of $X \times Y$ has a star refinement by open subsets of $X \times Y$ which covers $E \times Y$.

2.4. Proposition. Let $X$ be a regular space and $E$ a well-situated subset in $X$. Then for every paracompact $T_2$ space $Y$, for every family $\{F_s\}_{s \in S}$ of subsets of $E \times Y$, locally finite (discrete) in $X \times Y$, there is a family $\{U_s\}_{s \in S}$ of open subsets of $X \times Y$, locally finite (discrete) in $X \times Y$ and such that $F_s \subset U_s$ for any $s \in S$.

2.5. Proposition. Let $X$ and $X'$ be topological spaces and $f : X \to X'$ a perfect
mapping. If $E'$ is a well-situated subset in $X'$ then $f^{-1}(E')$ is a well-situated subset in $X$.

**Proof.** For every paracompact $T_2$ space $Y$, $f \times 1Y: X \times Y \to X' \times Y$ is a perfect mapping. Now 1.6 implies that $f^{-1}(E')$ is well-situated.

**2.6. Proposition.** Let $X$ and $X'$ be $T_2$ topological spaces where $X'$ is paracompact, let $E$ be a well-situated subset in $X$ and $F$ a closed subset of $X'$. Then $E \times F$ is an $\alpha$-paracompact subset in $X \times X'$.

**Proof** follows from 1.1.

**Remark.** In [4] R. Telgársky denoted by $\Pi$ the class of all $T_2$ spaces $X$ such that $X \times Y$ is paracompact for each paracompact $T_2$ space $Y$. Let $X$ and $Y$ be $T_2$ topological spaces, $E$ a well-situated subset in $X$ and $Y \in \Pi$. Then $E \times Y$ is well-situated in $X \times Y$. (Indeed, for each paracompact $T_2$ space $Z$, $Y \times Z$ is paracompact and $T_2$, hence $(E \times Y) \times Z$ is $\alpha$-paracompact in $(X \times Y) \times Z$.)

**2.7. Proposition.** Let $X$ and $X'$ be topological spaces, where $X$ is regular, let $f$ be a perfect mapping from $X$ onto $X'$ and $E'$ a subset of $X'$. If $f^{-1}(E')$ is a well-situated subset in $X$ then $E'$ is a well-situated subset in $X'$.

**Proof.** For every paracompact $T_2$ space $Y$, $f \times 1Y$ is a perfect mapping from $X \times Y$ onto $X' \times Y$. It follows from 1.8 that $E'$ is well-situated.

**2.8. Remark.** In Proposition 2.7 the hypothesis "$f$ is a mapping onto" cannot be omitted. In deed, let $(R, T)$ be the Michael line. The mapping $i: Q \to (R, T)$ is perfect but is not onto, $Q \in \Pi$ ([4] p. 66) but $Q$ is not well-situated in $(R, T)$ (cf. 1.7).

**2.9. Proposition.** Let $X$ and $X'$ be topological spaces and $f$ a perfect and open mapping from $X$ onto $X'$: If $E$ is a well-situated subset in $X$ then $f(E)$ is a well-situated subset in $X'$.

**Proof.** For every paracompact $T_2$ space $Y$, $f \times 1Y$ is a perfect and open mapping from $X \times Y$ onto $X' \times Y$. Now 1.10 implies that $f(E)$ is well-situated.

**2.10. Remark.** Let $X$ and $X'$ be topological spaces and $f$ a perfect mapping from $X$ onto $X'$. If $E$ is a well-situated subset in $X$, $f(E)$ is not necessarily a well-situated subset in $X'$. (See 1.11).

3. THE CLASS $\Pi^*$

In [4] R. Telgársky denoted by $\Pi^*$ the class of all paracompact $T_2$ spaces which are well-situated in every paracompact $T_2$ space in which they are embedded as closed subsets.

R. Telgársky showed that $\Pi^*$ is a very wide class contained in the class $\Pi$, and raised the following questions:
1. Is the class $\Pi^*$ perfect? ([4], Problem 2.1.)
2. Does the class of all paracompact $C$-scattered spaces coincide with the class $\Pi^*$? ([4], Problem 2.3.)

In the present paper, we shall give an affirmative answer to question 1.

3.1. Proposition. Let $E$ and $E'$ be topological spaces where $E$ is $T_2$, and let $f$ be a perfect mapping from $E$ onto $E'$. If $E' \in \Pi^*$ then $E \in \Pi^*$.

Proof. Let $X$ be a paracompact $T_2$ space such that $E$ is embedded in $X$ as a closed subset. Let $j_1: X \to X + E'$, $j_2: E' \to X + E'$ be the embeddings of the subspaces $X$ and $E'$ in the sum $X + E'$, let $X \cup_f E'$ be the adjunction space determined by $X$, $E'$ and $f$ and let $q: X + E' \to X \cup_f E'$ be the natural quotient mapping. As the mapping $f$ is closed, $q$ is a continuous and closed mapping; since $X + E'$ is paracompact and $T_2$, $X \cup_f E'$ is paracompact (this follows from the Michael Theorem) and $T_2$. Further, $q \circ j_2: E' \to X \cup_f E'$ is a homeomorphic embedding and $(q \circ j_2)(E')$ is closed in $X \cup_f E'$. Since $E' \in \Pi^*$ and $X \cup_f E'$ is paracompact and $T_2$, $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$.

Let $\tilde{f} = q \circ j_1: X \to X \cup_f E'$. Clearly $\tilde{f}$ is a perfect mapping.

$$
\begin{array}{ccc}
X & \rightarrow & X \cup_f E' \\
\uparrow & & \uparrow q \circ j_2 \\
E & \rightarrow & E'
\end{array}
$$

Since $(q \circ j_2)(E')$ is a well-situated subset in $X \cup_f E'$, Proposition 2.5 implies that $\tilde{f}^{-1}(q \circ j_2)(E')$ is a well-situated subset in $X$. However,

$$
\tilde{f}^{-1}(q \circ j_2)(E') = j_1^{-1}(q^{-1}(q(j_2(E')))) = E.
$$

Thus $E$ is a well-situated subset in $X$. Hence $E \in \Pi^*$.

3.2. Proposition. Let $E$ and $E'$ be topological spaces and $f$ a perfect mapping from $E$ onto $E'$. If $E \in \Pi^*$ then $E' \in \Pi^*$.

Proof. Since $f$ is continuous, $G_f = \{(x, f(x)) \in E \times E' | x \in E\}$ is homeomorphic to $E$, hence $G_f \in \Pi^*$.

Since $E$ is $T_{3\alpha}$ and $f$ is a perfect mapping from $E$ onto $E'$, $G_f$ is a closed subset of $\beta E \times X'$ (see [5], proof of Theorem 3.10).

Let $X'$ be a paracompact $T_2$ space such that $E'$ is embedded in $X'$ as a closed subset. Then $G_f$ is a closed subset of $\beta E \times X'$ which is paracompact and $T_2$. Thus $G_f$ is a well-situated subset in $\beta E \times X'$.

The projection $p_2: \beta E \times X' \to X'$ is perfect and open, and $p_2(G_f) = E'$. It follows from 2.9 that $E'$ is a well-situated subset in $X'$.

3.3. Theorem. In the realm of $T_2$ spaces, the class $\Pi^*$ is perfect (i.e., if $E$ and $E'$ are topological spaces where $E$ is $T_2$, and $f$ is a perfect mapping from $E$ onto $E'$ then $E \in \Pi^*$ if and only if $E' \in \Pi^*$).

Proof follows from 3.1 and 3.2.
Added in proofs. The author learned, after writing this paper, that J. D. Wine [in: Locally paracompact spaces, Glasnik Mat., 10 (30) (1975), 351–357] has obtained Proposition 1.1.2), and that I. Kovacević [in: Subsets and paracompactness, Zbornik Radova PMF Univ. u Novom Sadu, ser Mat. 14 (1984), 79–87] has obtained also the implication a) $\Rightarrow$ b) of the Theorem 1.3. The author thanks to Professors J. D. Wine and I. Kovacević for making available their papers to him.

References


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