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TOLERANCES IN CONGRUENCE PERMUTABLE ALGEBRAS

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Let $\mathbf{A}$ be an algebra and *Con* $\mathbf{A}$ its congruence lattice. Congruences $\Theta, \Phi \in Con\ \mathbf{A}$ are *permutable* if

$$\Theta \,.\, \Phi = \Phi \,.\, \Theta \,.$$

An algebra $\mathbf{A}$ is *congruence permutable* if this equality is satisfied for each two congruences $\Theta, \Phi \in Con\ \mathbf{A}$. A variety $\mathscr{V}$ is *congruence permutable* if each $\mathbf{A} \in \mathscr{V}$ has this property. An algebra $\mathbf{A}$ is *congruence distributive* if *Con* $\mathbf{A}$ is a distributive lattice. $\mathbf{A}$ is *arithmetic* if it is congruence permutable and congruence distributive. A variety $\mathscr{V}$ is *arithmetic* if each $\mathbf{A} \in \mathscr{V}$ has this property.

By a *tolerance* on an algebra $\mathbf{A}$ is meant a reflexive and symmetric binary relation on $\mathbf{A}$ which has the *substitution property* with respect to all operations of $\mathbf{A}$, i.e. it is a subalgebra of the direct product $\mathbf{A} \times \mathbf{A}$. The set of all tolerances on $\mathbf{A}$ forms the algebraic lattice *Tol* $\mathbf{A}$ (with respect to set inclusion). Denote by $T(a, b)$ the least tolerance in *Tol* $\mathbf{A}$ containing the pair $\langle a, b\rangle$ of elements a, b of $\mathbf{A}$. An algebra $\mathbf{A}$ is *tolerance trivial* if each $T \in Tol\ \mathbf{A}$ is a congruence on $\mathbf{A}$. $\mathbf{A}$ is *principal tolerance trivial* (see [2]) if $T(a, b) = \Theta(a, b)$ for each $a, b \in \mathbf{A}$. A variety $\mathscr{V}$ is (*principal*) *tolerance trivial* if each $\mathbf{A} \in \mathscr{V}$ has this property. An algebra $\mathbf{A}$ is *tolerance permutable* if

$$T \,.\, S = S \,.\, T$$

for each $T, S \in Tol\ \mathbf{A}$.

For the proof of the following proposition, see e.g. [2] or [10]:

Proposition 1. *A variety $\mathscr{V}$ is tolerance trivial if and only if $\mathscr{V}$ is congruence permutable.*

Hence, investigations on tolerances are reasonable only in congruence non-permutable varieties. On the other hand, the following question is natural:

Question 1. *Is the Proposition* 1 *true also for single algebras?*

Let us explain the situation. A ternary function $p(x, y, z)$ is called a *Mal'cev-function* if

$$p(x, y, y) = x \,, \quad p(x, x, y) = y \,.$$

A ternary function $m(x, y, z)$ is called a *Pixley-function* if

$$m(x, y, y) = m(y, y, x) = m(x, y, x) = x .$$

Thus every Pixley-function is a Mal'cev-function.

The following statement was proven by A. I. Mal'cev (A. F. Pixley):

A variety $\mathscr{V}$ is congruence permutable (arithmetic) if and only if there exists a ternary polynomial which is a Mal'cev (Pixley)-function.

H.-P. Gumm asks in [9] if these theorems holds also for single algebras.

If there exists a ternary polynomial $p(x, y, z)$ in an algebra $\mathbf{A}$ which is a Mal'cev-function in $\mathbf{A}$ and if $T \in Tol\ \mathbf{A}$, then $\langle a, b\rangle \in T$ and $\langle b, c\rangle \in T$ imply

$$\langle a, c\rangle = \langle p(a, b, b), p(b, b, c)\rangle \in T ,$$

thus T is transitive and hence $T \in Con\ \mathbf{A}$, i.e. $\mathbf{A}$ is tolerance trivial. The Question 1 can be modified:

Does there exist a ternary polynomial which is a Mal'cev-function in every congruence permutable algebra?

The answer was given by H.-P. Gumm [9]:

For an algebra $\mathbf{A}$ (with the support A_s) we denote

$$I_n(\mathbf{A}) = \{h: A_s^n \to A_s;\ h \text{ is idempotent and compatible with all congruences on } \mathbf{A} \text{ and } h \text{ is not a projection}\} .$$

Proposition 2 (Theorem 2.2 in [9]). *Let $\mathbf{A}$ be an algebra with $I_n(\mathbf{A}) \neq \emptyset$ for some $n \geqq 2$. Then either $I_2(\mathbf{A}) \neq \emptyset$ or $I_3(\mathbf{A})$ contains a Mal'cev-function.*

Hence, if we should avoid to algebras with a Mal'cev-function among its ternary polynomials, we have to find it among algebras with idempotent binary functions compatible with congruences. The representatives of such algebras are e.g. lattices. The following example answers Question 1:

Example 1. *The modular lattice M_{3-2} in Fig. 1 is congruence permutable but not tolerance trivial.*

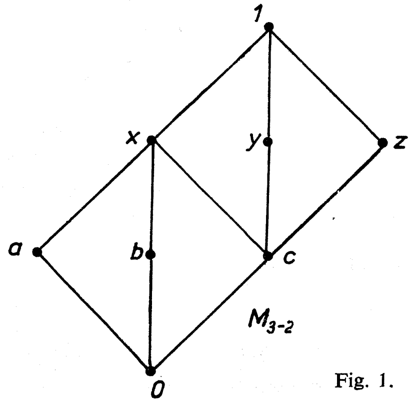

Fig. 1.

Since M_{3-2} is a simple lattice (i.e. $Con\, M_{3-2} = \{\omega, \iota\}$, where ω is the identity and ι the full relation), M_{3-2} is congruence permutable. Moreover, $Tol\, M_{3-2} = \{\omega, \iota, T\}$, where T is determined by its two blocks:

$$B_1 = \{0, a, b, c, x\}\,, \quad B_2 = \{c, x, y, z, 1\}\,.$$

The previous example is too special since it is a simple algebra. Better examples are algebras whose *Tol* **A** (and, hence, also *Con* **A**) are chains. If *Con* **A** is a chain, then **A** is evidently congruence permutable. It was proven in [6] that for any integer $n \geqq 1$ there exists a lattice L such that $Tol\, L$ is an n element chain

$$\omega \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_{n-2} \subseteq \iota\,,$$

where $T_2, T_4, \ldots$ are congruences. The example of a such lattice for $n = 3$ is M_{3-2} and for $n = 5$ it is the following

Example 2. *The lattice L in Fig.* 2 *is congruence permutable, non-simple and tolerance non-trivial.*

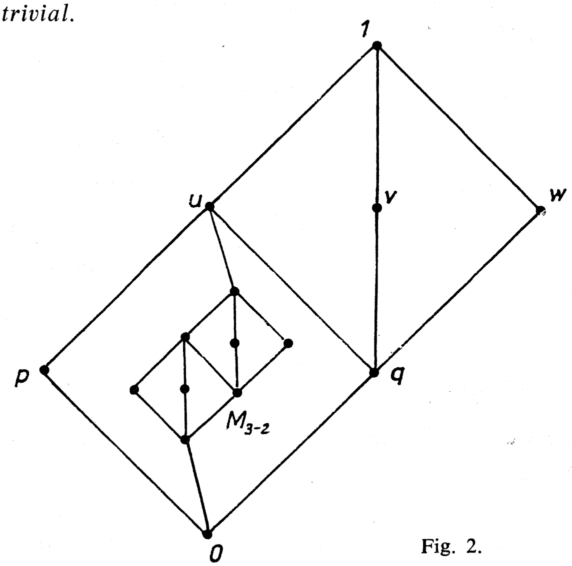

Fig. 2.

Clearly $Tol\, L$ is a chain $\omega \subseteq T_1 \subseteq \Theta \subseteq T_3 \subseteq \iota$, where T_1 is equal to T from Example 1 on the small M_{3-2} (which substitutes the element b of the big M_{3-2}) and equal to the identity for other elements; Θ is a congruence collapsing the small M_{3-2} only; T_3 is determined by the two blocks:

$$B_1 = \{0, p, q, u, small\, M_{3-2}\}\,, \quad B_2 = \{q, u, v, w, 1\}\,.$$

With respect to the previous examples, we have the following question:

Question 2. *Does there exist a congruence permutable algebra which is not tolerance trivial and Con* **A** *is not a chain?*

Let $\mathscr{C}$ be a class of algebras of the same type and closed under finite products. $\mathscr{C}$ has *directly decomposable congruences* (*tolerances*) if for each $\mathbf{A}, \mathbf{B} \in \mathscr{C}$ and every $\Theta \in Con\ \mathbf{A} \times \mathbf{B}$ ($T \in Tol\ \mathbf{A} \times \mathbf{B}$) there exist $\Theta_1 \in Con\ \mathbf{A}$, $\Theta_2 \in Con\ \mathbf{B}$ ($T_1 \in Tol\ \mathbf{A}$, $T_2 \in Tol\ \mathbf{B}$) such that $\Theta = \Theta_1 \times \Theta_2$ ($T = T_1 \times T_2$). Such classes were investigated in [7] and [4].

Theorem 1. *Let $\mathscr{C}$ be a class of algebras with directly decomposable congruences. If there exists a simple algebra $\mathbf{A} \in \mathscr{C}$ which is not tolerance trivial, then $\mathbf{B} = \mathbf{A} \times \mathbf{A}$ satisfies:*

(i) *$\mathbf{B}$ is congruence permutable;*
(ii) *$\mathbf{B}$ is not tolerance trivial;*
(iii) *$Con\ \mathbf{B}$ is not a chain.*

Proof. Suppose $\mathbf{A} \in \mathscr{C}$ is simple, i.e. $Con\ \mathbf{A} = \{\omega_1, \iota_1\}$ and there exists $T \in Tol\ \mathbf{A}$ which is not a congruence. Put $\mathbf{B} = \mathbf{A} \times \mathbf{A}$. Since $\mathscr{C}$ has directly decomposable congruences, $Con\ \mathbf{B} = \{\omega, \iota, \Theta, \Phi\}$, where $\omega = \omega_1 \times \omega_2$, $\iota = \iota_1 \times \iota_2$, $\Theta = \iota_1 \times \omega_2$, $\Phi = \omega_1 \times \iota_2$. Thus (iii) is satisfied. Since all projections of all congruences are ω_i, ι_i also (i) is evident. Moreover, the tolerances $T_1 = T \times \omega_2$, $T_2 = T \times \iota_2$, $T_3 = \omega_1 \times T$, $T_4 = \iota_1 \times T$ and $T_5 = T \times T$ are not congruences, i.e. (ii) holds.

Theorem 2. *Let $\mathscr{C}$ be a class of algebras with directly decomposable tolerances. If there exists a simple algebra $\mathbf{A} \in \mathscr{C}$ such that Tol $\mathbf{A}$ contains exactly one tolerance which is not a congruence, then $\mathbf{B} = \mathbf{A} \times \mathbf{A}$ satisfies* (i), (ii), (iii) *of Theorem* 1 *and*

(iv) *$\mathbf{B}$ is tolerance permutable.*

Proof. Let $\mathbf{A} \in \mathscr{C}$ be simple and $Tol\ \mathbf{A} = \{\omega, \iota, T\}$ (where T is not a congruence). Put $\mathbf{B} = \mathbf{A} \times \mathbf{A}$. Since $\mathscr{C}$ has directly decomposable tolerances, it has also directly decomposable congruences. By Theorem 1, $\mathbf{B}$ satisfies (i), (ii), (iii). Moreover, direct decomposability of tolerances implies that

$$Tol\ \mathbf{B} = \{\omega, \iota, T_1, T_2, T_3, T_4, T_5\},$$

where T_i are constructed in the same way as in the previous proof. Since every but one of each $T \in Tol\ \mathbf{B}$ has every projection equal to ω or ι, $\mathbf{B}$ has permutable tolerances.

The following corollary gives the affirmative answer to Question 2:

Corollary 1. *There exists a congruence permutable modular lattice which is tolerance permutable and not tolerance trivial and whose congruence lattice is not a chain.*

Proof. By [4], the class of all lattices has directly decomposable tolerances. Moreover, the lattice M_{3-2} in Fig. 1 satisfies the hypotheses of Theorem 2.

Another example of a lattice satisfying the hypotheses of Theorem 2 is the lattice M_{3-4} in Fig. 3.

Denote by I the two element lattice. The examples of lattices satisfying Corollary 1 are e.g. also lattices $M_{3-2} \times I$, $M_{3-2} \times M_{3-2}$, $M_{3-4} \times I$, $M_{3-4} \times M_{3-4}$, $M_{2-3} \times M_{3-4}$.

Question 3. *Does there exist a lattice L satisfying* (i), (ii), (iii) *of Theorem* 1 *but not* (iv) *of Theorem* 2?

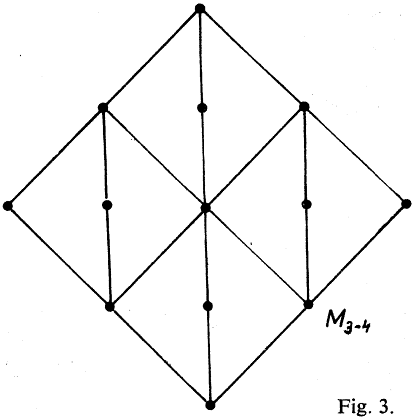

Fig. 3.

The answer is affirmative:

Example 3. *The modular lattice A in Fig. 4 is simple and hence congruence permutable. A is not tolerance trivial; there exist e.g. tolerance* T_1, T_2 (*see Fig.* 5) *which are not congruences.* Prove that T_1, T_2 *are not permutable.* Clearly $\langle a, c\rangle \in$

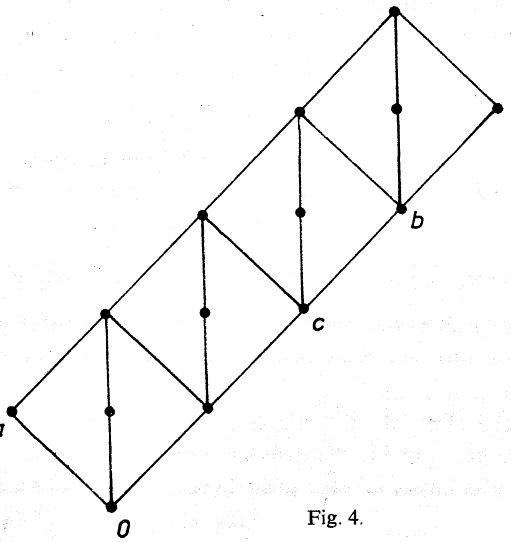

Fig. 4.

$\in T_1$, $\langle c, b\rangle \in T_2$, thus $\langle a, b\rangle \in T_1 . T_2$. On the contrary, $\langle a, b\rangle \notin T_2 . T_1$. Now, put $L = A \times I$. Clearly *L satisfies* (i), (ii), (iii) *but not* (iv) *of Theorem* 2.

In [9], H.-P. Gumm tries to find congruence identities for which there exists a local

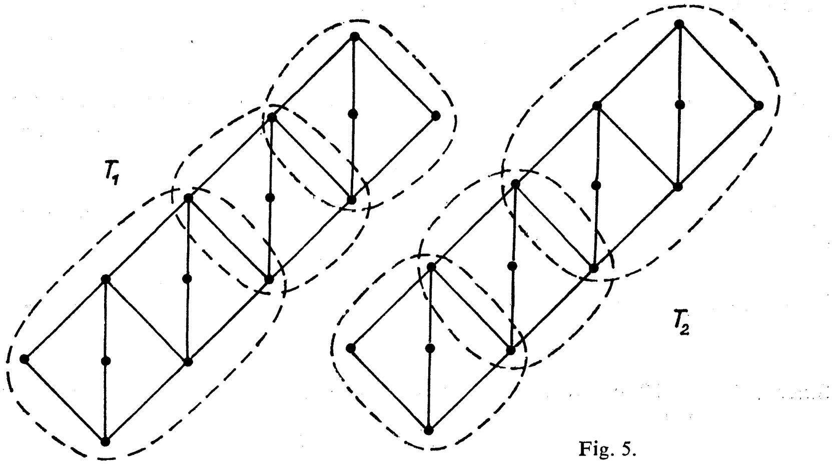

Fig. 5.

version of the Mal'cev theory. His result is that the only one is the arithmeticity. However, this local theorem does not contain the same assertion as the corresponding one for varieties. He gives only the following theorem of Pixley:

Proposition 3 (Theorem 1.3 in [9]). *Let* **A** *be an algebra with Con* **A** *finite. Then* **A** *is arithmetic if and only if there is a Pixley-function on* **A** *which is compatible with all congruences on* **A**.

The assumption of finiteness of *Con* **A** can be weakened but the great difference with the corresponding theorem for varieties is that the mentioned Pixley-function $m(x, y, z)$ need not be a polynomial on **A** but only a function compatible with all congruences on **A**. As it was shown, if this Pixley-function $m(x, y, z)$ is a polynomial, **A** is tolerance trivial. If $m(x, y, z)$ is not a polynomial, then **A** need not be tolerance trivial. Since lattices are congruence distributive (see e.g. [8]), all foregoing examples are arithmetical algebras in which any Pixley-function compatible with all congruences is not a polynomial. Moreover, these Pixley-functions are not compatible with tolerances on **A** (since this case implies the tolerance triviality).

It is easy to prove that the product $T_1 . T_2$ of $T_1, T_2 \in$ *Tol* **A** is a tolerance on **A** if and only if $T_1 . T_2 = T_2 . T_1$ (see e.g. Theorem 3 in [1]). Hence, *Tol* **A** is a semigroup with respect to the relation product if and only if **A** is tolerance permutable. This fact can serve as a motivation of our further investigations. As the corollary we obtain the following implication:

$$T \in Tol\ \mathbf{A} \Rightarrow T^p \in Tol\ \mathbf{A} \quad \text{for every integer} \quad p \geqq 1\,.$$

If an algebra $\boldsymbol{A}$ has permutable tolerances, it has clearly also permutable congruences. In the general case, the converse implication is not true. We can prove only:

Theorem 3. *Let an algebra* $\boldsymbol{A}$ *has permutable congruences. Then for each* $T_1, T_2 \in \in Tol\ \boldsymbol{A}$ *there exists an integer* $p \geqq 1$ *such that*

$$T_1 . T_2 \subseteq T_1^p . T_2^p .$$

Proof. Suppose $T_1, T_2 \in Tol\ \boldsymbol{A}$ and $\langle x, y \rangle \in T_1 . T_2$. Denote by $t(T)$ the transitive hull of T, i.e. $t(T)$ is the least congruence on $\boldsymbol{A}$ containing T (see e.g. [5]). Then

$$\langle x, y \rangle \in t(T_1 . T_2) .$$

By Theorem 3 in [5], $t(T_1 . T_2) = t(T_1) . t(T_2)$. The congruence permutability implies

$$\langle x, y \rangle \in t(T_2) . t(T_1) ,$$

i.e. there exists an element $z \in \boldsymbol{A}$ such that

$$\langle x, z \rangle \in t(T_2) , \quad \langle z, y \rangle \in t(T_1) .$$

Since $t(T) = \bigcup_{n=1}^{\infty} T^n$ for every $T \in Tol\ \boldsymbol{A}$, there exist integers $r \geqq 1$ and $s \geqq 1$ with

$$\langle x, z \rangle \in T_2^r , \quad \langle z, y \rangle \in T_1^s .$$

Put $p = max(r, s)$. Then $\langle x, z \rangle \in T_2^p$, $\langle z, y \rangle \in T_1^p$ proving

$$\langle x, y \rangle \in T_2^p . T_1^p .$$

Theorem 4. *Let* $\boldsymbol{A}$ *be a principal tolerance trivial algebra.* $\boldsymbol{A}$ *has permutable tolerances if and only if it has permutable congruences.*

Proof. Let $\boldsymbol{A}$ be principal tolerance trivial and congruence permutable algebra. Suppose $T_1, T_2 \in Tol\ \boldsymbol{A}$ and $\langle a, b \rangle \in T_1 . T_2$. Then there exists an element $c \in \boldsymbol{A}$ with

$$\langle a, c \rangle \in T_1 , \quad \langle c, b \rangle \in T_2 , \quad \text{i.e.}$$

$$T(a, c) \subseteq T_1 , \quad T(c, b) \subseteq T_2 .$$

Hence $\Theta(a, c) \subseteq T_1$, $\Theta(c, b) \subseteq T_2$ and

$$\langle a, b \rangle \in \Theta(a, c) . \Theta(c, b) = \Theta(c, b) . \Theta(a, c) \subseteq T_2 . T_1 .$$

We have $T_1 . T_2 \subseteq T_2 . T_1$; the converse inclusion can be proved analogously.

The converse implication is trivial.

Corollary 2. *For a distributive lattice L, the following conditions are equivalent:*

(1) *L is tolerance permutable;*

(2) *L is congruence permutable (i.e. arithmetic);*

(3) *L is tolerance trivial;*

(4) *L is relatively complementary.*

Proof. By [2]. the variety of all distributive lattices is principal tolerance trivial. Hence (1) $\Leftrightarrow$ (2) follows by Theorem 4. By Corollary 2 in [3], we have (3) $\Leftrightarrow$ (4). The equivalence (2) $\Leftrightarrow$ (4) is well-known, see e.g. Exercises 35 and 36 of § 1 in [8].

References

[1] *Chajda I.:* Systems of equations and tolerance relations, Czech. Math. J., *25* (1975), 302—308.

[2] *Chajda I.:* Tolerance trivial algebras and varieties, Acta Sci. Math. (Szeged), *46* (1983), 35—40.

[3] *Chajda I., Niederle J., Zelinka B.:* On existence conditions for compatible tolerances, Czech. Math. J. *26* (1976), 304—311.

[4] *Chajda I., Nieminen J.:* Direct decomposability of tolerances on lattices, semilattices and quasilattices, Czech. Math. J., *32* (1982), 110—115.

[5] *Chajda I., Zelinka B.:* Permutable tolerances, Časop. Pěst. matem. *102* (1977), 217—220.

[6] *Chajda I., Zelinka B.:* Every finite chain is the tolerance lattice of some lattice, Glasnik. Matem. (Zagreb), *20* (1985), 3—6.

[7] *Fraser G. A., Horn A.:* Congruence relations in direct products, Proc. Amer. Math. Soc. *26* (1970), 390—394.

[8] *Grätzer G.:* General lattice theory, Academie-Verlag, Berlin 1978.

[9] *Gumm H.-P.:* Is there a Mal'cev theory for single algebras?, Algebra Univ. *8* (1978), 320—329.

[10] *Werner H.:* A Mal'cev condition for admissible relations, Algebra Univ. *3* (1973), 263.

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