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ON VERONESE SURFACES

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**0.** The following result is known, see [1]: Let  $(M, g)$  be a closed, connected 2-dimensional manifold with curvature  $K$ . Let (i)  $\frac{1}{3} \leq K \leq 1$  or (ii)  $\frac{1}{6} \leq K \leq \frac{1}{3}$  resp. If  $\sigma: M \rightarrow S^N(1)$  is a minimal isometric immersion then  $K = 1$  or  $K = \frac{1}{3}$  in the case (i) and  $K = \frac{1}{3}$  or  $K = \frac{1}{6}$  in the case (ii). For  $K = \frac{1}{3}$ ,  $\sigma(M) \subset S^4(1) \subset S^N(1)$  is a Veronese surface.

Here, I study minimal immersions  $\sigma: M \rightarrow S^4(1)$ . To each such immersion, I associate a normal vector bundle of  $\sigma(M)$  and its curvature  $k$ . If  $K$  and  $k$  satisfy certain inequalities,  $\sigma(M)$  is a Veronese surface as well.

**1.** Let  $M$  be a 2-dimensional manifold,  $\sigma: M \rightarrow S^4(1)$  an immersion into the 4-dimensional unit sphere of the real Euclidean space  $\mathbb{R}^5$ . To each point  $m_0 \in M$ , let us associate an orthonormal frame  $\{m; v_1, \dots, v_5\}$  of  $\mathbb{R}^5$  such that  $m = \sigma(m_0)$ ;  $v_1, v_2 \in T_m(\sigma(M))$ ;  $m + v_5 =$  the center of  $S^4(1)$ . Then we have the fundamental equations of our moving frames

$$(1.1) \quad \begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2, \\ dv_1 &= \omega^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 + \omega^1 v_5, \quad dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 + \omega^2 v_5, \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4, \quad dv_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_4, \\ dv_5 &= -\omega^1 v_1 - \omega^2 v_2 \end{aligned}$$

with the integrability conditions ( $\omega_i^j = -\omega_j^i$ )

$$(1.2) \quad d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j.$$

From

$$(1.3) \quad \omega^3 = \omega^4 = \omega^5 = 0,$$

we get

$$(1.4) \quad \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0$$

and the existence of the functions  $a_1, \dots, b_3$  such that

$$(1.5) \quad \begin{aligned} \omega_1^3 &= a_1 \omega^1 + a_2 \omega^2, & \omega_1^4 &= b_1 \omega^1 + b_2 \omega^2, \\ \omega_2^3 &= a_2 \omega^1 + a_3 \omega^2, & \omega_2^4 &= b_2 \omega^1 + b_3 \omega^2. \end{aligned}$$

From (1.1) and (1.5), we get

$$(1.6) \quad \Delta m = (a_1 + a_3)v_3 + (b_1 + b_3)v_4 + 2v_5,$$

$\Delta$  being the Laplace operator. The mapping  $\sigma$  is called a *minimal immersion* if the vector  $\Delta m$  is a multiple of  $v_5$ , i.e., if

$$(1.7) \quad a_1 + a_3 = b_1 + b_3 = 0.$$

In what follows, let us restrict ourselves to minimal immersions.

Around the point  $m$ , consider a field of tangent unit vectors

$$(1.8) \quad t = xv_1 + yv_2; \quad x^2 + y^2 = 1.$$

By  $\nabla$ , we denote the symbol of the covariant differentiation associated to the induced metric

$$(1.9) \quad ds^2 = (\omega^1)^2 + (\omega^2)^2.$$

Then it is easy to see that

$$(1.10) \quad \nabla_t t = (\cdot)v_1 + (\cdot)v_2 + (\cdot)v_5 + \{a_1(x^2 - y^2) + 2a_2xy\}v_3 + \\ + \{b_1(x^2 - y^2) + 2b_2xy\}v_4.$$

Each unit vector  $t$  (1.8) is thus mapped into the point

$$(1.11) \quad m + \xi v_3 + \eta v_4; \quad \xi = a_1(x^2 - y^2) + 2a_2xy, \quad \eta = b_1(x^2 - y^2) + 2b_2xy$$

of the plane  $v_m = \{m; v_3, v_4\}$  of the *normal bundle*  $\nu$  of  $\sigma(M)$ . The points (1.11) form, for  $m$  fixed and all  $t$ 's, the ellipse

$$(1.12) \quad (b_1^2 + b_2^2)\xi^2 - 2(a_1b_1 + a_2b_2)\xi\eta + (a_1^2 + a_2^2)\eta^2 = (a_1b_2 - a_2b_1)^2,$$

the so-called *indicatrix of normal curvature*.

The *Gauss curvature*  $K$  and the *curvature of the normal bundle*  $k$  are defined by

$$(1.13) \quad d\omega_1^2 = -K\omega^1 \wedge \omega^2, \quad d\omega_3^4 = -k\omega^1 \wedge \omega^2$$

resp; we get

$$(1.14) \quad K = 1 - a_1^2 - a_2^2 - b_1^2 - b_2^2, \quad k = 2(a_1b_2 - a_2b_1).$$

The *Veronese surface* is defined as follows: In the Euclidean 3-space  $\mathbb{R}^3$ , consider orthonormal coordinates  $(x, y, z)$  and the mapping  $S^2(\sqrt{3}) \rightarrow S^4(1)$  given by

$$(1.15) \quad u_1 = \frac{1}{3}\sqrt{3} \cdot yz, \quad u_2 = \frac{1}{3}\sqrt{3} \cdot xz, \quad u_3 = \frac{1}{3}\sqrt{3} \cdot xy, \\ u_4 = \frac{1}{6}\sqrt{3} \cdot (x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

$(u_1, \dots, u_5)$  being orthonormal coordinates in  $\mathbb{R}^5$ . To each point of the Veronese surface, we may associate orthonormal frames such that we get (1.1) with

$$(1.16) \quad \omega_1^3 = -\omega_2^4 = \frac{1}{3}\sqrt{3} \cdot \omega^2, \quad \omega_1^4 = \omega_2^3 = \frac{1}{3}\sqrt{3} \cdot \omega^1, \quad \omega_3^4 = -2\omega_1^2;$$

see [2]. For our Veronese surface, we get

$$(1.17) \quad K = \frac{1}{3}, \quad k = -\frac{2}{3}.$$

2. We are going to prove the following (auxiliary)

**Theorem 1.** *Let  $\sigma: M \rightarrow S^4(1)$ ,  $\dim M = 2$ , be a minimal immersion; let  $M$  be compact. If*

$$(2.1) \quad 2K > k$$

*on  $M$ , the indicatrices of normal curvature are circles.*

**Proof.** Let us start with the equations (1.5) + (1.7). The differential consequences being

$$(2.2) \quad \begin{aligned} (da_1 - 2a_2\omega_1^2 - b_1\omega_3^4) \wedge \omega^1 + (da_2 + 2a_1\omega_1^2 - b_2\omega_3^4) \wedge \omega^2 &= 0, \\ (da_2 + 2a_1\omega_1^2 - b_2\omega_3^4) \wedge \omega^1 - (da_1 - 2a_2\omega_1^2 - b_1\omega_3^4) \wedge \omega^2 &= 0, \\ (db_1 - 2b_2\omega_1^2 + a_1\omega_3^4) \wedge \omega^1 + (db_2 + 2b_1\omega_1^2 + a_2\omega_3^4) \wedge \omega^2 &= 0, \\ (db_2 + 2b_1\omega_1^2 + a_2\omega_3^4) \wedge \omega^1 - (db_1 - 2b_2\omega_1^2 + a_1\omega_3^4) \wedge \omega^2 &= 0, \end{aligned}$$

we get the existence of functions  $\alpha_1, \dots, \beta_2$  such that

$$(2.3) \quad \begin{aligned} da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2, \\ db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2, \\ da_2 + 2a_1\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 - \alpha_1\omega^2, \\ db_2 + 2b_1\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 - \beta_1\omega^2. \end{aligned}$$

From this,

$$(2.4) \quad \begin{aligned} d(a_1 + b_2) - (a_2 - b_1)(2\omega_1^2 - \omega_3^4) &= A_1\omega^1 + A_2\omega^2, \\ d(a_2 - b_1) + (a_1 + b_2)(2\omega_1^2 - \omega_3^4) &= A_2\omega^1 - A_1\omega^2; \\ A_1 &:= \alpha_1 + \beta_2, \quad A_2 := \alpha_2 - \beta_1. \end{aligned}$$

The exterior differentiation of (2.4) yields

$$(2.5) \quad \begin{aligned} \{dA_1 - A_2(3\omega_1^2 - \omega_3^4)\} \wedge \omega^1 + \{dA_2 + A_1(3\omega_1^2 - \omega_3^4)\} \wedge \omega^2 &= \\ = (2K - k)(a_2 - b_1)\omega^1 \wedge \omega^2, \\ \{dA_2 + A_1(3\omega_1^2 - \omega_3^4)\} \wedge \omega^1 - \{dA_1 - A_2(3\omega_1^2 - \omega_3^4)\} \wedge \omega^2 &= \\ = (k - 2K)(a_1 + b_2)\omega^1 \wedge \omega^2. \end{aligned}$$

The function  $f$  being defined by

$$(2.6) \quad 2f = (a_1 + b_2)^2 + (a_2 - b_1)^2,$$

we have

$$df = \{(a_1 + b_2)A_1 + (a_2 - b_1)A_2\}\omega^1 + \{(a_1 + b_2)A_2 - (a_2 - b_1)A_1\}\omega^2$$

and

$$(2.7) \quad d * df = 2\{A_1^2 + A_2^2 + (2K - k)f\}\omega^1 \wedge \omega^2.$$

The supposition (2.1) and the Stokes theorem (or the maximum principle) imply  $f \equiv 0$ , i.e.,

$$(2.8) \quad b_1 = a_2, \quad b_2 = -a_1.$$

Now, look at (1.12). QED.

3. Let us prove our main Theorems.

**Theorem 2.** *Let  $\sigma: M \rightarrow S^4(1)$ ,  $\dim M = 2$ , be a minimal immersion; let  $M$  be compact. If*

$$(3.1) \quad 2K > k \geq -2K$$

*on  $M$ , there are just two cases possible: (i)  $K = 1$ ,  $k = 0$ , and  $\sigma(M)$  is a great sphere; (ii)  $K = \frac{1}{3}$ ,  $k = -\frac{2}{3}$ , and  $\sigma(M)$  is the Veronese surface.*

Proof. Theorem 1 implies (2.8), and the equations (2.3) reduce to

$$(3.2) \quad \begin{aligned} da_1 - a_2(2\omega_1^2 + \omega_3^4) &= \alpha_1\omega^1 + \alpha_2\omega^2, \\ da_2 + a_1(2\omega_1^2 + \omega_3^4) &= \alpha_2\omega^1 - \alpha_1\omega^2. \end{aligned}$$

The differential consequences are

$$(3.3) \quad \begin{aligned} \{d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4)\} \wedge \omega^1 + \{d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4)\} \wedge \omega^2 &= \\ &= (2K + k) a_2 \omega^1 \wedge \omega^2, \\ \{d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4)\} \wedge \omega^1 - \{d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4)\} \wedge \omega^2 &= \\ &= -(2K + k) a_1 \omega^1 \wedge \omega^2, \end{aligned}$$

and we get the existence of functions  $\alpha_{ij}$  such that

$$(3.4) \quad \begin{aligned} d\alpha_1 - \alpha_2(3\omega_1^2 + \omega_3^4) &= \alpha_{11}\omega^1 + \alpha_{12}\omega^2, \\ d\alpha_2 + \alpha_1(3\omega_1^2 + \omega_3^4) &= \alpha_{21}\omega^1 + \alpha_{22}\omega^2; \\ \alpha_{21} - \alpha_{12} &= (2K + k) a_2, \quad \alpha_{11} + \alpha_{22} = (2K + k) a_1. \end{aligned}$$

For the function  $g$  defined by

$$(3.5) \quad 2g = a_1^2 + a_2^2,$$

we get

$$(3.6) \quad \begin{aligned} dg &= (a_1\alpha_1 + a_2\alpha_2)\omega^1 + (a_1\alpha_2 - a_2\alpha_1)\omega^2 \\ d * dg &= \{2(\alpha_1^2 + \alpha_2^2) + (2K + k)(a_1^2 + a_2^2)\} \omega^1 \wedge \omega^2. \end{aligned}$$

The supposition  $2K + k \geq 0$  and the Stokes theorem (or the maximum principle as well) imply  $\alpha_1 = \alpha_2 = 0$  and  $a_1^2 + a_2^2 = \text{const}$ .

First of all, let  $a_1 = a_2 = 0$ . Then the equations (1.1) reduce to

$$(3.7) \quad \begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2, \quad dv_1 = \omega_1^2 v_2 + \omega^1 v_5, \quad dv_2 = -\omega_1^2 v_1 + \omega^2 v_5, \\ dv_3 &= \omega_3^4 v_4, \quad dv_4 = -\omega_3^4 v_3, \quad dv_5 = -\omega^1 v_1 - \omega^2 v_2, \end{aligned}$$

and  $\sigma(M)$  is the sphere  $S^2(1)$  in the (fixed) space  $\mathbb{R}^3$  through the center of  $S^4(1)$  spanned by the vectors  $v_1, v_2, v_5$ .

Now, let  $a_1^2 + a_2^2 \neq 0$ . Then

$$(3.8) \quad 2K + k = 0$$

from the integral formula based on (3.6). From (2.8) and (1.14),

$$(3.9) \quad K = 1 - 2(a_1^2 + a_2^2), \quad k = -2(a_1^2 + a_2^2).$$

Inserting this into (3.8), we get  $a_1^2 + a_2^2 = \frac{1}{3}$  and  $K = \frac{1}{3}$ ,  $k = -\frac{2}{3}$ . To the points of our surface  $\sigma(M)$ , let us associate the frames  $\{m; v_1, v_2, v_3^*, v_4^*, v_5\}$  with

$$(3.10) \quad v_3^* = \sqrt{3} \cdot (a_2 v_3 - a_1 v_4), \quad v_4^* = \sqrt{3} \cdot (a_1 v_3 + a_2 v_4).$$

By a direct calculation, we get the fundamental equations (1.1) with (1.16), we have just to replace  $v_3, v_4$  by  $v_3^*, v_4^*$  resp. Thus  $\sigma(M)$  is the Veronese surface. QED.

**Theorem 3.** Let  $\sigma: M \rightarrow S^4(1)$ ,  $\dim M = 2$ , be a minimal immersion; let  $M$  be compact. Suppose, on  $M$ ,  $K > 0$  and

$$(3.11) \quad -\frac{7}{3} \min_M K < \min_M k \leq \max_M k \leq -2 \max_M K.$$

Then  $\sigma(M)$  is the Veronese surface.

*Proof.* From  $K > 0$  and (3.11), we get (2.1), and we may use the equations (3.2) and (3.4). The prolongation of (3.4<sub>1,2</sub>) yields

$$(3.12) \quad \begin{aligned} & \{d\alpha_{11} - (\alpha_{12} + 3\alpha_{21})\omega_1^2 - \alpha_{21}\omega_3^4\} \wedge \omega^1 + \\ & + \{d\alpha_{12} + (\alpha_{11} - 3\alpha_{22})\omega_1^2 - \alpha_{22}\omega_3^4\} \wedge \omega^2 = (3K + k)\alpha_2\omega^1 \wedge \omega^2, \\ & \{d\alpha_{21} + (3\alpha_{11} - \alpha_{22})\omega_1^2 + \alpha_{11}\omega_3^4\} \wedge \omega^1 + \\ & + \{d\alpha_{22} + (\alpha_{21} + 3\alpha_{12})\omega_1^2 + \alpha_{12}\omega_3^4\} \wedge \omega^2 = -(3K + k)\alpha_1\omega^1 \wedge \omega^2 \end{aligned}$$

and the existence of functions  $\alpha_{ijk}$  such that

$$(3.13) \quad \begin{aligned} d\alpha_{11} - (\alpha_{12} + 3\alpha_{21})\omega_1^2 - \alpha_{21}\omega_3^4 &= \alpha_{111}\omega^1 + \alpha_{112}\omega^2, \\ d\alpha_{12} + (\alpha_{11} - 3\alpha_{22})\omega_1^2 - \alpha_{22}\omega_3^4 &= \alpha_{121}\omega^1 + \alpha_{122}\omega^2, \\ d\alpha_{21} + (3\alpha_{11} - \alpha_{22})\omega_1^2 + \alpha_{11}\omega_3^4 &= \alpha_{211}\omega^1 + \alpha_{212}\omega^2, \\ d\alpha_{22} + (\alpha_{21} + 3\alpha_{12})\omega_1^2 + \alpha_{12}\omega_3^4 &= \alpha_{221}\omega^1 + \alpha_{222}\omega^2; \end{aligned}$$

$$(3.14) \quad \alpha_{121} - \alpha_{112} = (3K + k)\alpha_2, \quad \alpha_{212} - \alpha_{221} = (3K + k)\alpha_1.$$

The differential consequences of (3.4<sub>3,4</sub>) are then

$$(3.15) \quad \begin{aligned} \alpha_{211} - \alpha_{121} &= (2K_1 + k_1)a_2 + (2K + k)\alpha_2, \\ \alpha_{212} - \alpha_{122} &= (2K_2 + k_2)a_2 - (2K + k)\alpha_1, \\ \alpha_{111} + \alpha_{221} &= (2K_1 + k_1)a_1 + (2K + k)\alpha_1, \\ \alpha_{112} + \alpha_{222} &= (2K_2 + k_2)a_1 + (2K + k)\alpha_2, \end{aligned}$$

the first covariant derivatives of  $K$  and  $k$  being defined by

$$(3.16) \quad dK = K_1\omega^1 + K_2\omega^2, \quad dk = k_1\omega^1 + k_2\omega^2$$

resp.

By a direct calculation, we get, for each  $r \in \mathbb{R}$ ,

$$(3.17) \quad \begin{aligned} & \frac{1}{2}d * \{d(\alpha_1^2 + \alpha_2^2) - (2K + k + r)d(a_1^2 + a_2^2)\} = \\ & = \{\alpha_{11}^2 + \alpha_{21}^2 + \alpha_{12}^2 + \alpha_{22}^2 + (3K + k - 2r)(\alpha_1^2 + \alpha_2^2) - \\ & - (2K + k)(2K + k + r)(a_1^2 + a_2^2)\} \omega^1 \wedge \omega^2, \end{aligned}$$

and the corresponding integral formula (in the case of  $M$  being non-orientable, we pass to its universal covering  $M^* \rightarrow M$ ). Let us take

$$(3.18) \quad r = -\frac{1}{4}(\min_M K + \min_M k).$$

Then using (3.11),

$$(3.19) \quad \begin{aligned} 2K + k &\leq 2 \max_M K + \max_M k \leq 0, \\ 3K + k - 2r &= 3(K - \min_M K) + (k - \min_M k) + \\ &\quad + \frac{3}{2}(\min_M k + \frac{7}{3} \min_M K) > 0, \\ 2K + k + r &= 2(K - \min_M K) + (k - \min_M k) + \\ &\quad + \frac{3}{4}(\min_M k + \frac{7}{3} \min_M K) > 0. \end{aligned}$$

Because of this, our integral formula yields  $\alpha_{ij} = 0$  and

$$(3.20) \quad \alpha_1 = \alpha_2 = 0,$$

$$(3.21) \quad (2K + k)(a_1^2 + a_2^2) = 0.$$

Thus  $a_1^2 + a_2^2 = \text{const}$ . In the case  $a_1 = a_2 = 0$ , we get  $K = 1$ ,  $k = 0$ , a contradiction to (3.11). Thus  $a_1^2 + a_2^2 \neq 0$ , and (3.21) implies (3.8). Now follow the proof of the preceding Theorem. QED.

#### References

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