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Small directed graphs as neighbourhood graphs


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LOCAL PROPERTIES OF GRAPHS WERE STUDIED BY MANY AUTHORS. THE FIRST HINT TO THEIR STUDY WAS GIVEN BY A PROBLEM OF A. A. Zykov AND B. A. Trahtenbrot AT THE SYMPOSIUM ON GRAPH THEORY IN SMOLENICE [1] IN 1963. SURVEY PAPERS ON THESE PROPERTIES WERE WRITTEN BY J. Sedláček [2], [3].

MOST RESULTS ON LOCAL PROPERTIES OF GRAPHS CONCERN UNDIRECTED GRAPHS. LOCAL PROPERTIES OF DIRECTED GRAPHS WERE STUDIED IN [4] AND [5]; THIS PAPER CONTINUES THIS STUDY.


LET $H$ BE THE CLASS OF ALL DIGRAPHS $H$ WITH THE PROPERTY THAT THERE EXISTS A DIGRAPH $G$ SUCH THAT $N_G(v) \cong H$ FOR EACH VERTEX $v$ OF $G$. THE PROBLEM TO DETERMINE $H$ IS THE DIGRAPH VARIANT OF THE MENTIONED PROBLEM FROM SMOLENICE. WE SHALL STUDY DIGRAPHS WHICH BELONG TO $H$ AND HAVE AT MOST THREE VERTICES.

BEFORE FORMULATING A THEOREM WE INTRODUCE AN AUXILIARY CONCEPT. LET $m, n$ BE POSITIVE INTEGERS. BY $V(m, n)$ WE DENOTE THE SET OF ALL $n$-DIMENSIONAL VECTORS $(v_1, \ldots, v_n)$ WHERE $v_i \in \{0, 1, \ldots, m - 1\}$ FOR $i = 1, \ldots, n$. IF WE PERFORM ADDITIONS OR Subtractions WITH COORDINATES OF THESE VECTORS, WE CONSIDER THEM MODULO $m$.

**THEOREM.** LET $H$ BE A DIRECTED GRAPH WITH AT MOST THREE VERTICES. THEN $H \in H$ IF AND ONLY IF THE NUMBER OF DOUBLE EDGES OF $H$ IS NOT 2.

**Remark.** By a double we mean a pair of edges which join the same pair of vertices and are directed oppositely to each other.

**Proof.** In Fig. 1 we see all possible directed graphs with at most three vertices
and with exactly two double edges. Suppose that $H' \in \mathcal{H}$; let $G'$ be the corresponding graph from the definition of $H$. Then $G'$ contains at least one induced subgraph isomorphic to $H'$; let its centre be $u_1$, its other vertices $u_2, u_3$. The graph $N_G(u_1)$ contains $u_2$ and $u_3$. As $N_G(u_1) \cong H'$, there exists a vertex $u_4$ in $N_G(u_1)$ joined by double edges with $u_2$ and $u_3$. The graph $N_G(u_2)$ contains the vertices $u_1, u_4$; they are joined by an edge. As $H$ contains only double edges, there must be a double edge joining $u_1$ and $u_4$. Now $N_G(u_2)$ must contain the third vertex $v$ joined by a double edge with exactly one of the vertices $u_1, u_4$. It cannot be $u_3$, because it is joined by double edges with both $u_1, u_4$. Thus $v$ is different from all $u_1, u_2, u_3, u_4$ and is joined by a double edge with $u_1$ and $u_4$. But then $u_1$ or $u_4$ has the degree at least 4 and its neighbourhood graph has at least 4 vertices, which is a contradiction.

Now suppose that $H'' \in \mathcal{H}$ and let $G''$ be the corresponding graph. The graph $G''$ contains an induced subgraph isomorphic to $H''$; let the common end vertex of two double edges in it be $u_1$, the initial vertex of the simple edge $u_2$ and its terminal vertex $u_3$. Again the graph $N_{G''}(u_1)$ contains a vertex $u_4$ joined by double edges with $u_2$ and $u_3$. The graph $N_{G''}(u_2)$ contains $u_1, u_3, u_4$ and double edges between $u_1$ and between $u_3$ and $u_4$. Thus $u_1$ and $u_4$ must be joined by a simple edge. The graph $N_{G''}(u_3)$ contains the vertices $u_1, u_4$ joined by a simple edge; therefore there must exist a vertex $v$ in $N_{G''}(u_3)$ joined by double edges with $u_1$ and $u_4$. It cannot be $u_2$, because then $u_2$ and $u_3$ would be joined by a double edge. Thus $v$ is different from all $u_1, u_2, u_3, u_4$ and is joined by double edges with $u_1$ and $u_4$. But then $u_1$ has the degree at least 4, which is a contradiction.

Now consider the graphs without double edges. We shall determine the graphs $G$ for particular graphs $H$. If $H$ is an empty graph, then $G$ is any graph without edges. If $H$ consists of one vertex, then $G$ is any (directed) cycle. If $H$ has two vertices and no edge, then $G$ is the graph whose vertex set is $V(m, 2)$ for $m \geq 3$ and in which from each $(v_1, v_2)$ edges go to $(v_1 + 1, v_2)$ and $(v_1, v_2 + 1)$. If $H$ has two vertices

![Fig. 2.](image-url)
and one edge, then $G$ is the graph whose vertex set is $V(m, 1)$ for $m \geq 5$ and in which from each $(v_1)$ edges go to $(v_1 + 1)$ and $(v_1 + 2)$. All graphs with three vertices without double edges are in Fig. 2. The graph $G$ for $H$ will be denoted by $G_i$ for $i = 1, \ldots, 14$. The graph $G_1$ has the vertex $V(m, 3)$ for $m \geq 5$ and from each vertex $(v_1, v_2, v_3)$ edges go to $(v_1 + 1, v_2, v_3)$, $(v_1, v_2 + 1, v_3)$, $(v_1, v_2, v_3 + 1)$. The graphs $G_2, G_3, G_4, G_5$ have the vertex set $V(m, 2)$ for $m \geq 5$, $m$ even. In $G_2$ from each $(v_1, v_2)$ edges go to $(v_1 + 1, v_2)$, $(v_1, v_2 + 1)$, $(v_1, v_2 + 1)$. If both $v_1, v_2$ are even, then from $(v_1, v_2)$ edges go to $(v_1 - 1, v_2)$, $(v_1, v_2 - 1)$. If $v_1$ is even and $v_2$ is odd, then from $(v_1, v_2)$ edges go to $(v_1 - 1, v_2)$, $(v_1, v_2 + 1)$ and $(v_1 - 1, v_2 + 1)$. If $v_1$ is odd and $v_2$ is even, then from $(v_1, v_2)$ edges go to $(v_1 + 1, v_2 + 1)$. Further if $v_2$ is even, then edges go to $(v_1 + 1, v_2)$ and $(v_1 + 1, v_2 + 1)$, and if $v_2$ is odd, then to $(v_1 - 1, v_2)$ and $(v_1 - 1, v_2 + 1)$. The graphs $H_6$ and $H_7$ are tournaments and the assertion on them follows from the results in [4]. But we may say that $G_6$ has the vertex set $V(m, 1)$ for $m \geq 7$ and from $(v_1)$ edges go to $(v_1 + 1)$, $(v_1 + 2)$, $(v_1 + 3)$. The graph $G_7$ is a tournament on 7 vertices $u_1, \ldots, u_7$ in which from $u_i$ edges go to $u_{i+2}$, $u_{i+4}$, $u_{i+6}$ for $i = 1, \ldots, 7$, the sums being taken modulo 7.

Now we turn to graphs with one or three double edges. If $H$ is a graph with two vertices and one double edge, the corresponding graph $G$ is the complete digraph with three vertices. The digraphs with three vertices and one or three double edges are in Fig. 3. The graph $G_8$ is obtained from the undirected graph of a trilateral prism by replacing each undirected edge by a pair of oppositely directed edges. The graph $G_9$ has the vertex set $V(m, 1)$ for $m \geq 5$ and in it from the vertex $(v_1)$ edges go to $(v_1 - 1)$, $(v_1 + 1)$, $(v_1 + 2)$. The graph $G_{10}$ has the vertex set $V(m, 1)$
for $m \geq 6$, $m$ even. From $(v_1)$ the edges go also to $(v_1 - 1)$ and $(v_1 + 1)$; further for $v_1$ even an edge goes to $(v_1 + 2)$ and for $v_1$ odd an edge goes to $(v_1 - 2)$. The graph $G_{11}$ is in Fig. 4, the graph $G_{12}$ is in Fig. 5, the graph $G_{13}$ is in Fig. 6. The graph $G_{14}$ is the complete directed graph with four vertices.
References


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