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NATURAL TRANSFORMATIONS OF SECOND TANGENT AND COTANGENT FUNCTORS

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Modugno and Stefani introduced an intrinsic isomorphism between TT^*M and T^*TM for every manifold M, [7]. From the categorical point of view, this is a natural equivalence between the functors TT^* and T^*T, which are defined on the category $\mathcal{M}_n$ of all n-dimensional manifolds and their local diffeomorphisms. In the present paper, we analytically determine all natural transformations of TT^* into T^*T and we interpret them geometrically. We deduce that the natural transformation by Modugno and Stefani can be distinguished in a similar way as the canonical involution of the second tangent bundle can be characterized among all natural transformations of functor TT into itself. Since a basic construction of the symplectic geometry gives a natural equivalence between TT^* and T^*T^*, any two of the functors TT^*, T^*T and T^*T^* are naturally equivalent. On the other hand, functor TT is naturally equivalent to none of them. In particular, this proves the fact that, in contradictinction of the cotangent bundles, there is no natural symplectic structure on the tangent bundles. — All manifolds and maps are assumed to be infinitely differentiable.

1. Differential equations for the natural transformations of TT^* into T^*T. We shall use the concept of a lifting functor by Nijenhuis, [8], in a slightly modified form. Let $\mathcal{M}_n$ denote the category of n-dimensional manifolds and their local diffeomorfisms. A lifting functor F is a functor from $\mathcal{M}_n$ into the category of fibred manifolds transforming every n-dimensional manifold M into a fibred manifold FM over M and every local diffeomorphism $f: M \to N$ into a fibred manifold morphism $Ff: FM \to FN$ over f. The construction of the cotangent spaces is a lifting functor, provided we define $T^*f: T^*M \to T^*N$ in such a way that $T^*_x f: T^*_x M \to T^*_{f(x)} N$ is the inverse map of the dual map $(T_x f)^*: T^*_{f(x)} N \to T^*_x M$. Then TT^* and T^*T are two second order lifting functors. According to a general theory, [9], [2], if F and G are two r-th order lifting functors, then $F_0 R^n$ and $G_0 R^n$ are L^r_n-spaces, where L^r_n means the group of all invertible r-jets on R^n with source and target 0, and the natural transformations $F \to G$ are bijectively related with the L^r_n-equivariant maps $F_0 R^n \to G_0 R^n$.

In our case we first determine the actions of L^2_n on $TT^*_0 R^n = S$ and $T^*T_0 R^n = W$.

The canonical coordinates x^i on R^n induces the additional coordinates p_i on T^*R^n and $\xi^i = dx^i$, $\pi_i = dp_i$ on TT^*R^n. If we evaluate the effect of a change of coordinates on R^n and pass to 2-jets, we find that the equations of the action of L_n^2 on S are

(1) $$\bar{p}_i = \tilde{a}_i^j p_j\,, \quad \bar{\xi}^i = \tilde{a}_j^i \xi^j\,, \quad \bar{\pi}_i = \tilde{a}_i^j \pi_j - a_{jk}^l \tilde{a}_l^m \tilde{a}_i^j p_m \xi^k\,,$$

where a_j^i, a_{jk}^i are the canonical coordinates of an element of L_n^2 and $\tilde{a}_j^i$ means the inverse matrix of a_j^i. Further, if η^i are the induced coordinates on TR^n, then the expression $\varrho_i\, dx^i + \sigma_i\, d\eta^i$ determines the additional coordinates ϱ_i, σ_i on T^*TR^n. Then the action of L_n^2 on W is

(2) $$\bar{\eta}^i = a_j^i \eta^j\,, \quad \bar{\varrho}_i = \tilde{a}_i^j \varrho_j - a_{jk}^l \tilde{a}_l^m \tilde{a}_i^j \sigma_m \eta^k\,, \quad \bar{\sigma}_i = \tilde{a}_i^j \sigma_j\,.$$

Our aim is to find all L_n^2-equivalent maps $S \to W$. Any map $\lambda: S \to W$ has the form

(3) $$\eta^i = f^i(p, \xi, \pi)\,, \quad \varrho_i = g_i(p, \xi, \pi)\,, \quad \sigma_i = h_i(p, \xi, \pi)\,.$$

If λ is L_n^2-equivariant, then for every vector $A = (A_j^i, A_{jk}^i)$ of the Lie algebra l_n^2 of L_n^2 the corresponding fundamental vector fields A_S on S and A_W on W must be λ-related. This gives the following system of partial differential equations for f^i, g_i and h_i with parameters A_j^i, A_{jk}^i

(4) $$A_j^i f^j = -\frac{\partial f^i}{\partial p_j} A_j^k p_k + \frac{\partial f^i}{\partial \xi^j} A_k^j \xi^k - \frac{\partial f^i}{\partial \pi_j} (A_j^k \pi_k + A_{jk}^l p_l \xi^k)\,,$$

(5) $$-A_i^j g_j - A_{ij}^k h_k f^j = -\frac{\partial g_i}{\partial p_j} A_j^k p_k + \frac{\partial g_i}{\partial \xi^j} A_k^j \xi^k - \frac{\partial g_i}{\partial \pi_j} (A_j^k \pi_k + A_{jk}^l p_l \xi^k)\,,$$

(6) $$-A_i^j h_j = -\frac{\partial h_i}{\partial p_j} A_j^k p_k + \frac{\partial h_i}{\partial \xi^j} A_k^j \xi^k - \frac{\partial h_i}{\partial \pi_j} (A_j^k \pi_k + A_{jk}^l p_l \xi^k)\,.$$

Let $\langle p, \xi \rangle = p_i \xi^i$ denote the value of p at ξ.

Proposition 1. *The general solution of the system* (4)–(6) *is*

(7) $$\begin{aligned} \eta^i &= F(\langle p, \xi \rangle)\, \xi^i\,, \\ \varrho_i &= F(\langle p, \xi \rangle)\, H(\langle p, \xi \rangle)\, \pi_i + G(\langle p, \xi \rangle)\, p_i\,, \\ \sigma_i &= H(\langle p, \xi \rangle)\, p_i\,, \end{aligned}$$

where $F(t)$, $G(t)$ and $H(t)$ are three arbitrary smooth functions of one variable.

Proof. Setting $A_j^i = 0$ in (4), we obtain $\partial f^i / \partial \pi_j = 0$. Then (4) are the differential equations of an L_n^1-equivalent map $f^i(p, \xi): R^{n*} \times R^n \to R^n$. By Proposition 2 of [6], it holds $f^i = F(\langle p, \xi \rangle)\, \xi^i$, where F is an arbitrary smooth function of one variable. Setting $A_j^i = 0$ in (6), we get $\partial h_i / \partial \pi_j = 0$. .Then (6) are the differential equations of an L_n^1-equivalent map $h_i(p, \xi): R^{n*} \times R^n \to R^{n*}$. From the proof of Proposition 2 of [6] it follows $h_i = H(\langle p, \xi \rangle)\, p_i$, where H is an arbittary smooth function of one variable. Then (5) with $A_j^i = 0$ reads

(8) $$-A_{ji}^k F(\langle p, \xi \rangle)\, H(\langle p, \xi \rangle)\, \xi^j\, p_k = -\frac{\partial g_i}{\partial \pi_j}\, p_l \xi^k A_{jk}^l\,.$$

Setting in the i-th equation $A^i_{ii} = 1$ and all other A's equal to zero, we obtain $\partial g_i/\partial \pi_i = F(\langle p, \xi\rangle)\, H(\langle p, \xi\rangle)$. For $A^i_{ij} = 1$, $i \neq j$, and all other A's equal to zero, we get $\partial g_i/\partial \pi_j = 0$. This implies

$$(9) \qquad g_i = F(\langle p, \xi\rangle)\, H(\langle p, \xi\rangle)\, \pi_i + \bar{g}_i(p, \xi)\,,$$

where $\bar{g}_i$ are some functions of p and ξ. Since the function $F(\langle p, \xi\rangle)\, H(\langle p, \xi\rangle) = c(p, \xi)$ is invariant, it holds $(\partial c/\partial p_i)\, A^j_i p_j - (\partial c/\partial \xi^i)\, A^i_j \xi^j = 0$. Substituting (9) into (5) and using the latter relation, we obtain

$$(10) \qquad A^j_i \bar{g}_j = \frac{\partial \bar{g}_i}{\partial p_j}\, A^k_j p_k - \frac{\partial \bar{g}_i}{\partial \xi^j}\, A^j_k \xi^k\,.$$

These are the differential equations for an L^1_n-equivariant map $R^{n*} \times R^n \to R^{n*}$, so that we have $\bar{g}_i = G(\langle p, \xi\rangle)\, p_i$ similarly as above. Conversely, it is easy to check that map (7) is L^2_n-equivariant, so that it satisfies (4)–(6). QED.

If F is constant, there exists an underlying natural transformation of T into itself expressed by the first line of (7). Similarly, if H is constant, there exists an underlying natural transformation of T^* into itself expressed by the third line of (7).

2. Geometrical interpretation of the analytic results. We first explain a simple geometric construction of one isomorphism of TT^*M into T^*TM. Let $q\colon T^*M \to M$ be the bundle projection and $i\colon TTM \to TTM$ be the canonical involution. Every $A \in TT^*M$ is a vector tangent to a curve $(x^i(t), a_i(t))$ at $t = 0$. If B is any vector of $T_{Tq(A)}TM$, then iB is tangent to a curve $(x^i(t), b^i(t))$ with the same $x^i(t)$. Hence we can evaluate $\langle a(t), b(t)\rangle$ for every t and the derivative

$$(11) \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \langle a(t), b(t)\rangle = \frac{\mathrm{d}a_i(0)}{\mathrm{d}t}\, b^i(0) + a_i(0)\, \frac{\mathrm{d}b^i(0)}{\mathrm{d}t} = \pi_i\, \mathrm{d}x^i + p_i \mathrm{d}\eta^i$$

depends only on A and B. This determines a linear map $T_{Tq(A)}TM \to R$, i.e. an element of T^*TM. Thus we obtain a natural equivalence $s\colon TT^* \to T^*T$,

$$(12) \qquad \xi^i = \eta^i\,, \quad \varrho_i = \pi_i\,, \quad \sigma_i = p_i$$

corresponding to the constant values $F = 1$, $G = 0$, $H = 1$. This equivalence was constructed in another way by Modugno and Stefani, [7].

Further we show that for any constant values $F = f$, $G = g$, $H = h$, (7) can be determined by a simple modification of the previous construction. The vector $fA \in TT^*M$ is tangent to the curve $(x^i(ft), a_i(ft))$. If B is any vector of $T_{fTq(A)}TM$, then iB is tangent to a curve $(x^i(ft), b^i(t))$. Then we define an element $s_{(f,g,h)}A \in T^*TM$ by

$$(13) \qquad \langle s_{(f,g,h)}A, B\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \langle a(ft), hb(t)\rangle + g\langle a(0), b(0)\rangle\,.$$

The coordinate expression of (13) is $(fh\pi_i + gp_i)\, \mathrm{d}x^i + hp_i\, \mathrm{d}\eta^i$ and our construction implies $\eta^i = f\xi^i$. This gives (7) with constant coefficients.

Even in the general case (7) can be interpreted in such a way. Let $\pi\colon TT^*M \to T^*M$ be the bundle projection. Every $A \in TT^*M$ determines $Tq(A) \in TM$ and $\pi(A) \in T^*M$

over the same base point in M. Then we take the values of F, G and H at $\langle \pi(A), Tq(A) \rangle$ and apply the latter construction.

It is remarkable that the natural equivalence s by Modugno and Stefani can be distinguished by another geometrical way analogously to a property of the canonical involution of the second tangent functor. According to [3], [4], the natural transformations of TT into itself form a four-parameter family. Every vector field ξ on M induces a vector field $\mathcal{T}\xi$ on M by means of flows, i.e. $\exp t\mathcal{T}\xi = T(\exp t\xi)$, [5]. If we compare the results of [4] and [5], we deduce immediately that i is the natural transformation $TT \to TT$ satisfying $i_M \circ T\xi = \mathcal{T}\xi$ for every vector field on M, where $T\xi\colon TM \to TTM$ is the tangent map of $\xi\colon M \to TM$. Let $\omega\colon M \to T^*M$ be any 1-form on M. Then $\langle \omega, \xi \rangle\colon M \to R$ and the second component of the tangent map $T\langle \omega, \xi \rangle\colon TM \to TR = R \times R$ will be denoted by $\delta\langle \omega, \xi \rangle$. We have $T\omega\colon TM \to$ $\to TT^*M$, so that $sT\omega\colon TM \to T^*TM$ and $\langle sT\omega, \mathcal{T}\xi \rangle\colon TM \to R$.

Proposition 2. *s is the only natural transformation $TT^* \to T^*T$ over the identity transformation of T satisfying*

$$\langle sT\omega, \mathcal{T}\xi \rangle = \delta\langle \omega, \xi \rangle \tag{14}$$

for every vector field ξ and every 1-form ω.

Proof. Let $x^i = x^i$, $p_i = a_i(x)$ be the coordinate expression of ω, so that the additional coordinate expression of $T\omega$ is $\xi^i = \xi^i$, $\pi_i = (\partial a_i/\partial x^j)\, \xi^j$. This is transformed by (7) into

$$\eta^i = \xi^i\,, \quad \varrho_i = H \frac{\partial a_i}{\partial x^j}\, \xi^j + G a_i\,, \quad \sigma_i = H a_i\,,$$

where $F = 1$ follows from the assumption that our natural transformation is over the identity of T. If $b^i(x)$ are the coordinate components of a vector field ξ, then the coordinate expression of $\mathcal{T}\xi$ is $\mathrm{d}x^i = b^i(x)$, $\mathrm{d}\xi^i = (\partial b^i/\partial x^j)\, \xi^j$, see [4]. Hence (14) requires

$$\left(H \frac{\partial a_i}{\partial x^j}\, \xi^j + G a_i\right) b^i + H a_i \frac{\partial b_i}{\partial x^j}\, \xi^j = \left(\frac{\partial a_i}{\partial x^j}\, b^i + a_i \frac{\partial b^i}{\partial x^j}\right) \xi^j\,. \tag{15}$$

Since a_i and b^i are arbitrary, this implies $G = 0$, $H = 1$. QED.

3. The second tangent and cotangent functors. The iterated cotangent functor T^*T^* is also a second order lifting functor. If x^i, w_i are the usual coordinates on T^*M and the additional coordinates ζ_i, μ^i on T^*T^*M are given by the expression $\zeta_i\, \mathrm{d}x^i +$ $+ \mu^i\, \mathrm{d}w_i$, then the action of L^2_n on $T^*T^*_0R^n$ is

$$\bar{w}_i = \tilde{a}^j_i w_j\,, \quad \bar{\zeta}_i = \tilde{a}^j_i \zeta_j + a^i_{kl} \tilde{a}^l_i a^m_j \mu^k w_m\,, \quad \bar{\mu}^i = a^i_j \mu^j\,. \tag{16}$$

The problem of finding of all natural transformations between any two of the functors TT^*, T^*T and T^*T^* can be reduced to (7), if we take into account a classical geometrical construction of a natural equivalence between TT^* and T^*T^*. Consider the Liouville 1-form on T^*M, [1]. The exterior differential $\mathrm{d}\omega = \Omega$ endows T^*M

with a natural symplectic structure. A basic fact of the symplectic geometry is that Ω determines a bijection t_M between the tangent and cotangent bundles of T^*M by

$$t_M(X) = X \lrcorner \Omega \tag{17}$$

where $X \lrcorner \Omega$ means the inner product of a vector X with Ω. Since the coordinate expression of ω is $w_i \, dx^i$, (17) leads to the following equations of t

$$w_i = p_i, \quad \zeta_i = \pi_i, \quad \mu^i = -\xi^i. \tag{18}$$

Using (1) and (16), we can check even formally that t is a natural equivalence between TT^* and T^*T^*.

Since s: $TT^* \to T^*T$ is also a natural equivalence, $t \circ s^{-1}$: $T^*T \to T^*T^*$ is another natural equivalence, the coordinate expression of which is

$$w_i = \sigma_i, \quad \zeta_i = \varrho_i, \quad \mu^i = -\eta^i. \tag{19}$$

This proves

Proposition 3. *The natural transformations between any two of the functors TT^*, T^*T and T^*T^* depend on three arbitrary smooth functions of one variable. Their coordinate expressions can be deduced from* (7) *by means of* (12), (18) *and* (19).

Functor TT is not of this type, since its natural transformations into itself depend on four real parameters. This is related with the fact that TT is defined on the whole category $\mathcal{M}$ of all manifolds and all smooth maps and is product preserving. According to a recent result, [3], the restriction of such a functor to connected manifolds is determined by a Weil algebra. (In particular, TT is determined by $D \otimes D$, where D is the algebra of dual numbers.) The natural transformations of any two Weil functors are in bijection with the homomorphisms of the corresponding Weil algebras, so that they depend on a finite member of real parameters. We further remark that the natural transformations between two Weil functors are algebraic, while there are three arbitrary smooth functions in (7).

Since the natural transformations of TT into itself are essentially different from the natural transformations of T^*T into itself, there is no natural equivalence between TT and T^*T. This implies that there is no natural symplectic structure on TM.

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