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PRODUCTS OF COARSE CONVERGENCE GROUPS

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INTRODUCTION

Minimal topological groups have been studied for more than fifteen years (cf. [8], [1]). Basic facts about coarse *FLUSH*-convergence groups (i.e. the sequential counterparts of minimal topological groups) were established in [4]. Further properties of coarse groups are given in [2] and [7]. In particular, answering a question from [4], it is proved in [7] that (under Continuum Hypothesis) an abelian coarse group need not be sequentially precompact. In the present paper we study products of coarse groups. In Section 1 we prove that coarseness is preserved by finite products (this answers another question from [4]) – recall that a product of two minimal topological groups need not be minimal (cf. [1]). In Section 2 we construct a sequence of coarse groups the product of which fails to be coarse. The first example showing that coarseness need not be preserved by countable products was given by D. Dikranjan (see [2]). In Section 3 we prove that the product of $\mathfrak{c} = 2^N$ copies of the two-point cyclic group equipped with the pointwise convergence of sequences fails to be coarse. This implies that the sequential coreflection of a compact (hence minimal) topological group need not be coarse. The results have been announced at the Conference on Generalized Functions and Convergence held in Szczyrk (Poland) in 1985.

In notation and terminology we follow [4]. To make the paper more self-contained, we repeat here some basic facts about coarse convergence groups.

By Z we denote the group of integers, by N the positive integers, by $Z(2) = Z/2Z$ the two-point cyclic group, and by MON the set of all monotone (one-to-one) mappings of N into N. If $S = \langle x_n \rangle$ is a sequence of points of a set G, i.e., if S is a mapping of N into G, then for $s \in MON$ the subsequence of S the n-th term of which is $x_{s(n)}$ is denoted by $S \circ s$. For each $x \in G$, the constant sequence generated by x will be denoted by $\langle x \rangle$. If G is a group and S and T are two sequences in G, then their product ST is the sequence the n-th term of which is $S(n)\, T(n)$ and the sequence S^{-1} is defined analogously. The neutral element of G will be denoted by e. If G is an abelian group then the additional notation will be used correspondingly and the neutral element will be denoted by 0.

By a convergence group we understand a group G equipped with a compatible sequential convergence $\mathfrak{G} \subset G^N \times G$ which satisfies the so-called *FLUSH*-axioms or, equivalently, $\mathscr{L}$-axioms (cf. [5]). Recall that $H(=\mathscr{L}_0)$ stands for the uniqueness of sequential limits, $S(=\mathscr{L}_1)$ means that for all $g \in G$ the constant sequence $\langle g \rangle$ converges to g, $F(=\mathscr{L}_2)$ means that if a sequence converges to a point, then each subsequence of the sequence converges to this point, $U(=\mathscr{L}_3)$ denotes the Urysohn axiom, and $L(=\mathscr{S}\mathscr{G})$ stands for the compatibility of $\mathfrak{G}$ with the group structure of G. We say that $\mathfrak{G}$ (and also G) is coarse if there is no $FLUSH$-convergence for G strictly larger than $\mathfrak{G}$ (i.e. having more convergent sequences than $\mathfrak{G}$).

Let G be a convergence group. We say that a sequence S of points of G has the property (C) if either of the two conditions holds true:

(C_1) for some $s \in MON$, the subsequence $S \circ s$ of S converges to the neutral element of G;

or

(C_2) for some $p \in G$, $p \neq e$, the constant sequence $\langle p \rangle$ is a product of finitely many sequences of the type $\langle g\, U(n)^{\varepsilon}\, g^{-1} \rangle$, where $g \in G$, $\varepsilon \in \{-1, 1\}$, and $U \in G^N$ either converges to e or it is a subsequence of S.

In [4] the following necessary and sufficient condition is given for a convergence group to be coarse.

Criterion 1. *A convergence group is coarse iff each sequence of points of the group has the property* (C).

Further (cf. [4]), if the group is abelian, then (C_2) in (C) is equivalent to the following condition:

(C_2'') for some $p \in G$, $p \neq 0$, some finite linear combination of subsequences of S, with coefficients from $Z \setminus \{0\}$, converges to p.

Throughout the paper we shall use the following fact proved in [4]. Let $\mathfrak{G}$ be a $FLUSH$-convergence for a group G. Then there is a coarse convergence for G larger than $\mathfrak{G}$. For the sake of simplicity we shall denote by $\mathfrak{G}_c$ any coarse convergence for G larger than $\mathfrak{G}$.

As a rule, by an FLS-convergence we understand a convergence satisfying axioms (F), (L) and (S), and we use the same convention for other sets of axioms.

1. FINITE PRODUCTS

Let G_1 and G_2 be groups and let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be coarse convergences for G_1 and G_2, respectively. Let $G_1 \times G_2$ be their product and let $\mathfrak{G}_1 \times \mathfrak{G}_2$ be the product $FLUSH$-convergence. If $S \in G_1^N$ and $T \in G_2^N$, then $S \otimes T$ denotes the sequence in $G_1 \times G_2$ the n-th term of which is $(S(n), T(n))$. Recall that a sequence $S \otimes T$ $\mathfrak{G}_1 \times \mathfrak{G}_2$-converges to (x, y) iff S $\mathfrak{G}_1$-converges to x and T $\mathfrak{G}_2$-converges to y.

For a nonempty set $\mathfrak{H} \subset (G_1 \times G_2)^N \times (G_1 \times G_2)$ define $\mathrm{proj}_1(\mathfrak{H})$ to be the set of all $(S, x) \in G_1^N \times G_1$ such that for some $T \in G_2^N$ and $y \in G_2$ we have $(S \otimes T, (x, y)) \in \mathfrak{H}$. Similarly, define $\mathrm{proj}_2(\mathfrak{H})$ to be the set of all $(T, y) \in G_2^N \times G_2$ such that

$(S \otimes T, (x, y)) \in \mathfrak{H}$ for some $S \in G_1^N$ and $x \in G_1$. Clearly, if $\mathfrak{H} = (G_1 \times G_2)_c$ is a coarse convergence for $G_1 \times G_2$ such that $\mathfrak{G}_1 \times \mathfrak{G}_2 \subset (G_1 \times G_2)_c$, then $\mathfrak{G}_1 \subset \mathfrak{G}_1 \subset \mathrm{proj}_1(\mathfrak{H})$ and $\mathfrak{G}_2 \subset \mathrm{proj}_2(\mathfrak{H})$.

Lemma 1.1. (i) *If* $\mathfrak{H}$ *satisfies axiom* (F), *then both* $\mathrm{proj}_1(\mathfrak{H})$ *and* $\mathrm{proj}_2(\mathfrak{H})$ *satisfy axiom* (F).

(ii) *If* $\mathfrak{H}$ *satisfies axiom* (L), *then both* $\mathrm{proj}_1(\mathfrak{H})$ *and* $\mathrm{proj}_2(\mathfrak{H})$ *satisfy axiom* (L).

(iii) *If* $\mathfrak{H}$ *satisfies axiom* (S), *then both* $\mathrm{proj}_1(\mathfrak{H})$ *and* $\mathrm{proj}_2(\mathfrak{H})$ *satisfy axiom* (S).

Proof. The assertions follow easily from the definition of $\mathrm{proj}_i(\mathfrak{H})$, $i = 1, 2$.

Lemma 1.2. *Let* $\mathfrak{H}$ *be an FLS-convergence for* $G_1 \times G_2$. *Assume that* $(\langle e \rangle \otimes T, (x, y)) \in \mathfrak{H}$ *implies* $x = e$. *Then* $\mathrm{proj}_1(\mathfrak{H})$ *satisfies axiom* (H). *Similarly, if* $(S \otimes \langle e \rangle, (x, y)) \in \mathfrak{H}$ *implies* $y = e$, *then* $\mathrm{proj}_2(\mathfrak{H})$ *satisfies axiom* (H).

Proof. Assume that $(S, x_1'), (S, x_2'') \in \mathrm{proj}_1(\mathfrak{H})$. Then for some $T_1, T_2 \in G_2^N$ and $y_1, y_2 \in G_2$ we have $(S \otimes T_1, (x_1, y_1))$, $(S \otimes T_2, (x_2, y_2)) \in \mathfrak{H}$. Consequently, $((S\, S^{-1}) \otimes (T_1\, T_2^{-1}), (x_1\, x_2^{-1}, y_1\, y_2^{-1})) \in \mathfrak{H}$ implies $x_1 = x_2$, i.e., $\mathrm{proj}_1(\mathfrak{H})$ satisfies axiom (H). The second assertion can be proved similarly.

Definition. An *FLS*-convergence $\mathfrak{H}$ for $G_1 \times G_2$ is said to be *orthogonal* if $(\langle e \rangle \otimes \otimes T, (x, y)) \in \mathfrak{H}$ implies $x = e$ and $(S \otimes \langle e \rangle, (x, y)) \in \mathfrak{H}$ implies $y = e$.

Remark. Observe that an orthogonal *FLS*-convergence satisfies axiom (H). Further, let $\mathfrak{H}$ be an *FLS*-convergence for $G_1 \times G_2$. Then $\mathfrak{H}$ is orthogonal iff $(\langle x_1 \rangle \otimes T, (x_2, y)) \in \mathfrak{H}$ implies $x_1 = x_2$ and $(S \otimes \langle y_1 \rangle, (x, y_2)) \in \mathfrak{H}$ implies $y_1 = y_2$. Also $\mathrm{proj}_1(\mathfrak{H})$ and $\mathrm{proj}_2(\mathfrak{H})$ satisfy axiom (H) iff $\mathfrak{H}$ is orthogonal.

Lemma 1.3. *Let* G *be a group equipped with a FLUSH-convergence* $\mathfrak{G}$. *Let* $\mathfrak{G}_c$ *be a coarse convergence for* G *such that* $\mathfrak{G} \subset \mathfrak{G}_c$. *If* $(S, e) \in \mathfrak{G}_c \setminus \mathfrak{G}$, *then for some* $s \in MON$ *the subsequence* $S \circ s$ *of* S *does not have the property* (C) *with respect to* $\mathfrak{G}$.

Proof. Since $(S, e) \notin \mathfrak{G}$ and $\mathfrak{G}$ satisfies the Urysohn axiom (U), there exists $s \in MON$ such that for each $t \in MON$ we have $(S \circ s \circ t, e) \notin \mathfrak{G}$. Consequently, $S \circ s$ does not have the property (C_1) with respect to $\mathfrak{G}$. But $S \circ s$ does not have the property (C_2) with respect to $\mathfrak{G}$ either. For, otherwise there exists $p \in G$, $p \neq e$, such that the constant sequence $\langle p \rangle$ can be written as a finite product of sequences of the form $\langle x \rangle\, T \langle x^{-1} \rangle$, where $x \in G$ and T is either a sequence $\mathfrak{G}$-converging to e or a subsequence of $S \circ s$ (hence $\mathfrak{G}_c$-converging to e) – a contradiction with $(\langle p \rangle, e) \notin \notin \mathfrak{G}_c$.

Lemma 1.4. *Let* S *be a sequence in* $\mathfrak{G}_1$ *and* T *a sequence in* $\mathfrak{G}_2$. *If one of the sequences* S *and* T *is constant, then the sequence* $S \otimes T$ *does have the property* (C) *with respect to* $\mathfrak{G}_1 \times \mathfrak{G}_2$.

Proof. Assume, e.g., that S is a constant sequence. Since $\mathfrak{G}_2$ is coarse, it follows by Criterion 1 that T has the property (C) with respect to $\mathfrak{G}_2$. Now, it is easy to see that $S \otimes T$ has the property (C) with respect to $\mathfrak{G}_1 \times \mathfrak{G}_2$. We proceed similarly if T is a constant sequence.

Lemma 1.5. *The convergence $(\mathfrak{G}_1 \times \mathfrak{G}_2)_c$ is orthogonal.*

Proof. Let $(\langle e\rangle \otimes T, (x, y)) \in (\mathfrak{G}_1 \times \mathfrak{G}_2)_c$. Then $(\langle x^{-1}\rangle \otimes T\langle y^{-1}\rangle, (e, e)) \in$ $\in (\mathfrak{G}_1 \times \mathfrak{G}_2)_c$. According to Lemma 1.4, for each $s \in MON$ the sequence $\langle x^{-1}\rangle \otimes$ $\otimes T \circ s\langle y^{-1}\rangle$ has the property (C) with respect to $\mathfrak{G}_1 \times \mathfrak{G}_2$. By Lemma 1.3, we have $(\langle x^{-1}\rangle \otimes T\langle y^{-1}\rangle, (e, e)) \in \mathfrak{G}_1 \times \mathfrak{G}_2$ and hence $(\langle x^{-1}\rangle, e) \in \mathfrak{G}_1$. Thus $x = e$. In the same way we can prove that $(S \otimes \langle e\rangle, (x, y)) \in (\mathfrak{G}_1 \times \mathfrak{G}_2)_c$ implies $y = e$. This completes the proof.

Theorem 1. $\mathfrak{G}_1 \times \mathfrak{G}_2$ *is a coarse convergence for* $G_1 \times G_2$.

Proof. On the contrary, assume that $\mathfrak{G}_1 \times \mathfrak{G}_2$ is not coarse. Let $(\mathfrak{G}_1 \times \mathfrak{G}_2)_c$ be a coarse convergence for $G_1 \times G_2$ such that $\mathfrak{G}_1 \times \mathfrak{G}_2 \subset (\mathfrak{G}_1 \times \mathfrak{G}_2)_c$. Choose $(S \otimes T, (x, y)) \in (\mathfrak{G}_1 \times \mathfrak{G}_2) \setminus (\mathfrak{G}_1 \times \mathfrak{G}_2)_c$. Then either $(S, x) \notin \mathfrak{G}_1$ or $(T, y) \notin \mathfrak{G}_2$. According to Lemma 1.5, $(\mathfrak{G}_1 \times \mathfrak{G}_2)_c$ is an orthogonal (*FLUSH*-) convergence. Put $\mathfrak{H}_1 = \mathrm{proj}_1((\mathfrak{G}_1 \times \mathfrak{G}_2)_c)$ and $\mathfrak{H}_2 = \mathrm{proj}_2((\mathfrak{G}_1 \times \mathfrak{G}_2)_c)$. By Lemma 1.1 and Lemma 1.2, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are *FLUSH*-convergences for G_1 and G_2, respectively. Clearly $\mathfrak{G}_1 \subset \mathfrak{H}_1$, $(S, x) \in \mathfrak{H}_1$, $\mathfrak{G}_2 \subset \mathfrak{H}_2$, $(T, y) \in \mathfrak{H}_2$. Since $\mathfrak{H}_1$ and $\mathfrak{H}_2$ can be enlarged (cf. [5]) to *FLUSH*-converges $\mathfrak{H}_1^*$ and $\mathfrak{H}_2^*$, respectively, we have a contradiction with the fact that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are coarse. This completes the proof.

Corollary 1. *The product of finitely many coarse convergence group is coarse.*

2. COUNTABLE PRODUCTS

In [2], D. Dikranjan gave an example of a sequence of coarse groups the (convergence) product of which fails to be coarse. In this section we present another such example. Our construction is based on the properties of *FLUSH*-convergences for free groups described in [4].

Example 1. For each natural number $k > 1$ let $FC(X_k)$ be the free commutative group generated by $X_k = \{x_{k,n};\ n \in N\} \cup \{x_k\}$. Put $S_k = \langle x_{k,n}\rangle$, $T_k = \langle kx_{k,n} - x_k\rangle$ and $\mathcal{A}(k) = \{T_k\}$. Let $\mathfrak{G}_{\mathcal{A}(k)}$ be the minimal *FLUS*-convergence for $FC(X_k)$ in which the sequence T_k converges to zero (see [4]).

Lemma 2.1. $\mathfrak{G}_{\mathcal{A}(k)}$ *is a FLUSH-convergence.*

Proof. We are to prove that $\mathfrak{G}_{\mathcal{A}(k)}$ satisfies axiom (H). As shown in [4], it suffices to verify that no nonzero constant sequence $\mathfrak{G}_{\mathcal{A}(k)}$-converges to zero, i.e., no nonzero constant sequence can be written in the free group $FC(X_k)$ as a finite linear combination $z_1 T_k \circ s_1 + \ldots + z_m T_k \circ s_m$ of subsequences $T_k \circ s_i$ of T_k such that $z_i \in Z \setminus \{0\}$ and $(T_k \circ s_i)(n) \neq (T_k \circ s_j)(n)$ for all $n \in N$ whenever $i \neq j$. But this is an immediate consequence of the definition of the sequence T_k and the properties of a free group.

Let $\mathfrak{G}_k$ be a coarse *FLUSH*-convergence for $FC(X_k)$ such that $\mathfrak{G}_{\mathcal{A}(k)} \subset \mathfrak{G}_k$. Denote by G_k the free group $FC(X_k)$ equipped with $\mathfrak{G}_k$.

Lemma 2.2. *Let l be a natural number such that $l \notin kN$. Then no subsequence of the sequence $lS_k = \langle lx_{k,n} \rangle$ converges in G_k.*

Proof. Contrariwise, assume that for some $l \in N \setminus kN$ a subsequence $lS_k \circ s$, $s \in MON$, converges in G_k to a point x. Then $klS_k \circ s$ converges to kx and at the same time to lx_k. By the uniqueness of limits we have $kx = lx_k$. Since x_k is a generator of the free group $FC(X_k)$, we have a contradiction.

Denote by G the convergence product group $\prod_{l=2}^{\infty} G_l$. Recall that G is equipped with the coordinatewise convergence.

Theorem 2. *The group G fails to be coarse.*

Proof. Denote by S the sequence in G the l-th projection of which to G_l, is the sequence $S_l = \langle x_{l,n} \rangle$, $l = 2, 3, \ldots$. According to Criterion 1, it suffices to show that S satisfies neither condition (C_1) nor condition (C_2). Since by Lemma 2.2 no S_1 satisfies condition (C_1), the sequence S itself fails to satisfy (C_1). Now assume that, on the contrary, S satisfies (C_2), i.e., some finite linear combination $z_1 S \circ s_1 + \ldots + z_m S \circ s_m$ of subsequences $S \circ s_i$ of S with nonzero integer coefficients z_i, $i = 1, \ldots, m$, converges in G to a nonzero point g. Denote by g_1 the l-th projection of g to G_l, $l = l = 2, 3, \ldots$. There are two possibilities. First, if $z_1 + \ldots + z_m = 0$, then for all $l = 2, 3, \ldots$ we have $lg_l = 0$. This would be a contradiction with the assumption that g is a nonzero element of G. Secondly, if $z_1 + \ldots + z_m \neq 0$, then choose a natural number k such that $k > |z_1 + \ldots + z_m|$. Since the sequence $k(z_1 S \circ s_1 + \ldots + z_m S \circ s_m)$ converges in G to kg, its k-th projection $k(z_1 S_k \circ s_1 + \ldots + z_m S_k \circ s_m)$ converges in G_k to kg_k. But $k(z_1 S_k \circ s_1 + \ldots + z_m S_k \circ s_m)$ converges in G_k also to $(z_1 + \ldots + z_m)\, x_k$ and hence $kg_k = (z_1 + \ldots + z_m)\, x_k$. This would be a contradiction with the fact that x_k is a generator of the free group $FC(X_k)$.

Corollary 2. *An infinite product of coarse groups need not be coarse.*

3. UNCOUNTABLE PRODUCTS

It is known that every compact topological group is minimal and every abelian minimal topological group is precompact (cf. [6]). On the other hand, every sequentially compact convergence group is coarse, but an abelian coarse convergence group need not be sequentially precompact (cf. [7]).

Consider the discrete topological group $Z(2) = Z/2Z$. Denote by G the product of $\mathfrak{c} = 2^N$ copies of $Z(2)$ equipped with the coordinatewise sequential convergence. Then G is a convergence group (cf. [5]) and it is the so-called *sequential coreflection* (*modification*) of the compact topological group $Z(2)^{\mathfrak{c}}$.

Theorem 3. *The convergence group G fails to be coarse.*

Proof. According to Criterion 1, it suffices to find a sequence S in G which does not have the property (C). Observe that, in fact, it suffices to show that for each finite linear combination $T = z_1 S \circ s_1 + \ldots + z_k S \circ s_k$, where z_i is a non-zero integer and $S \circ s_i$ are mutually disjoint subsequences of S, there is a subsequence of T which does not converge in G.

The sequence S is defined as follows. Arrange the set of all sequences ranging in $Z(2)$ into a one-to-one net $\langle S_\alpha; \alpha \in \mathfrak{c} \rangle$. Let S be the sequence in G the α-th projection of which is S_α (i.e. $S(n) = (S_\alpha(n); \alpha \in \mathfrak{c})$ for all $n \in N$). Clearly, S is a one-to-one sequence. Let $z_1 S \circ s_1 + \ldots + z_k S \circ s_k$ be a finite linear combination of subsequences of S with nonzero integer coefficients. First, in view of the algebraic properties of $Z(2)$, if we put $u_i = 1$ whenever z_i is odd and $u_i = 0$ whenever z_i is even, $i = 1, \ldots, k$, we have $z_1 S \circ s_1 + \ldots + z_k S \circ s_k = u_1 S \circ s_1 + \ldots + u_k S \circ s_k$. Second, there exists $t \in MON$ such that each two subsequences $S \circ s_i \circ t$ and $S \circ s_j \circ t$ are either disjoint (we identify each sequence with its graph, so two sequences are disjoint whenever their graphs are disjoint) or identical and, moreover, t can be chosen in such a way that the sets $\{s_i(t(n)); i = 1, \ldots, k\}$ and $\{s_i(t(m)); i = 1, \ldots, k\}$ are disjoint whenever $n, m \in N$ and $n \neq m$ (even more, $\max \{s_i(t(n)); i = 1, \ldots, k\} < \min \{s_i(t(m)); i = 1, \ldots, k\}$ whenever $n < m$) Consequently, to prove that S does not have the property (C) it suffices to show that for each $k \in N$ and for each k-tuple $(s_1, \ldots, s_k)$, where $s_i \in MON$ $(i = 1, \ldots, k)$ are mutually disjoint and such that the sets $\{s_i(n); i = 1, \ldots, k\}$ and $\{s_i(m); i = 1, \ldots, k\}$ are disjoint whenever $n, m \in N$ and $n \neq m$, there exists $\alpha \in \mathfrak{c}$ such that $S_\alpha(s_1(n)) + \ldots + S_\alpha(s_k(n)) = 0$ for infinitely many $n \in N$ and, at the same time, $S_\alpha(s_1(n)) + \ldots + S_\alpha(s_k(n)) = 1$ for infinitely many $n \in N$; observe that the sequence $S \circ s_1 + \ldots + S \circ s_k$ cannot converge in G to any $g \in G$. Since $\{S_\alpha; \alpha \in \mathfrak{c}\}$ is the set of all sequences in $Z(2)$, such α does always exist.

Corollary 3. *The sequential coreflection of a compact topological group need not be coarse.*

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