Szymon Plewik

On closed mappings

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 2, 313–318

Persistent URL: http://dml.cz/dmlcz/102226

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

ON CLOSED MAPPINGS

SZYMON PLEWIK, Katowice

(Received August 25, 1986, in revised form July 13, 1987)

1. Introduction. Back in early stages of the general topology M. Frèchet [3] introduced dimensional types of topological spaces. If a space X is homeomorphic with some subspace of a space Y, then X has a smaller or equal dimensional type than Y, in symbols $dX \leqq dY$. Dimensional types of metric separable spaces were investigated by S. Banach [1], K. Kuratowski [4], [6], W. Sierpiński [6], [9] and others. One can find some information about this in Sierpiński's *General Topology* [10] p. 130 or in Kuratowski's *Topology* I [5] p. 112 and 433.

We generalize dimensional types in the following manner. If a space X is the image of some subspace of a space Y under a closed mapping, then X has a smaller or equal closed-map type than Y, in symbols $ctX \leqq ctY$. If $ctX \leqq ctY$ and doesn't hold $ctY \leqq ctX$, then X has a smaller closed-map type than Y, in symbols $ctX < ctY$. If there doesn't hold $ctX \leqq ctY$ or $ctY \leqq ctX$, then spaces X and Y are imcomparable.

Obviously $dX \leqq dY$ implies $ctX \leqq ctY$. The inverse implication is not true. It is a well-known fact that the Cantor set C can be mapped continuously onto a non-void closed interval J. So $ctC \leqq ctJ$, $ctJ \leqq ctC$ and $dC \leqq dJ$, but $dJ \leqq dC$ doesn't hold.

In this paper we investigate closed-map types of first countable spaces, exactly hereditarily normal and hereditarily separable ones. Nice non-metrizable examples are the Songenfrey line and its uncountable subspaces. Theorem 1 is a main tool for proving other results. Other theorems are applications of Theorem 1 and in few cases their proofs are similar to those of S. Banach [1], K. Kuratowski [4], [6] and W. Sierpiński [6], [9] on dimensional types.

All undefined notations are as in Engelking's *General Topology* [2]. The cardinality of a set X is denoted by $|X|$. The cardinal number assigned to the set of real numbers is denoted by $\mathfrak{c}$ and is called *continuum*. By $\mathfrak{c}^+$ we denote the least cardinal number greater than $\mathfrak{c}$. If $|X| = \lambda$, then 2^λ denotes the cardinality of the family of all subsets of X. By $f \restriction M$ we denote the restriction of a function f to the set M. A mapping means a continuous function. A mapping $f: X \to Y$ is a closed mapping if it is closed as a function from X onto $f(X)$. If X is a topological space and $A \subset X$, then $\bar{A}$ denotes the closure of A.

2. The main theorem. The proof of the below lemma is straightforward, so we omit it.

Lemma 1. *Suppose X and Y are first countable spaces with X a T_1-space. A mapping $f\colon X \to Y$ is closed iff each sequence $\{x_n\colon n = 1, 2, \ldots\} \subset X$, such that $\lim f(x_n) = y$ and $y \neq f(x_n)$ for all n, has a limit point in $f^{-1}(y)$.* □

Lemma 2. *Let X and Y be first countable Hausdorff spaces. If $f\colon A \to f(A) \subset Y$ and $g\colon B \to g(B) \subset Y$ are closed mappings, $D \subset A \subset \bar{D} \subset X$, $D \subset B \subset \bar{D} \subset X$, $f \restriction D = g \restriction D$, $t \in f(A) \cap g(B)$ and $f^{-1}(t)$ is a nowhere dense, relative $\bar{D}$, subset, then $f^{-1}(t) = g^{-1}(t)$.*

Proof. Suppose $t \in f(A) \cap g(B)$ and $f^{-1}(t)$ is a nowhere dense, relative $\bar{D}$, subset. Assume to the contrary that $b \in g^{-1}(t) \setminus f^{-1}(t)$. There exists a sequence $E = \{a_1, a_2, \ldots\} \subset D \setminus f^{-1}(t)$ such that $\lim a_n = b$, since spaces X and Y are Hausdorff $b \notin A$ and the set E is closed relative A. Whence the set $f(E)$ is closed relative $f(A)$ and $t \notin f(E)$. On the other hand $t \in \overline{g(E)}$, since $f(E) = g(E)$, $\lim g(a_n) = g(b) = t$ and $t \in f(A)$. So we have $t \in f(A) \cap \overline{f(E)} = f(E)$, a contradiction. Thus we have $g^{-1}(t) \subset f^{-1}(t)$. The inclusion $f^{-1}(t) \subset g^{-1}(t)$ one can prove in a similar way. □

Let X and Y be topological spaces and $f\colon A \to Y$ be a function. A function $h\colon B \to Y$ is a X, Y-expansion of f if $h \restriction A = f$, $A \subset B \subset \bar{A} \subset X$ and $t \in f(A)$ implies $f^{-1}(t) = h^{-1}(t)$. Any X, Y-expansion of f is a closed X, Y-expansion of f if it is a closed mapping.

Lemma 3. *Suppose X is a first countable hereditarily normal space, and Y is a first countable regular space. If A is a dense subset of X and $f\colon A \to Y$ is a closed mapping, then there exists the maximal X, Y-expansion of f.*

Proof. Take $F(x) = f(x)$ for $x \in A$ and $F(x) = h(x)$ whenever $x \in h^{-1}(t)$, $t \notin f(A)$ and h is a closed X, Y-expansion of f. The function F is well defined because of Lemma 2.

Suppose to the contrary that F is not continuous. Thus there has to exist a sequence $x_1, x_2, \ldots$ such that $\lim x_n = b$ and the sequence $F(x_1), F(x_2), \ldots$ isn't convergent to $F(b)$. Since the space Y is regular we can assume that there is an open neighbourhood W of the point $F(b)$ such that $F(x_n) \notin \bar{W}$ for all n, if need be we can take a suitable subsequence. Let h_n be a closed X, Y-expansion of f such that $h_n(x_n)$ exists and let $x_n^m \in h_n^{-1}(Y \setminus \bar{W}) \cap A$ be such that $\lim x_n^m = x_n$. Take $E = \{x_n^m\colon n, m = 1, 2, \ldots\}$. Since $b \in \bar{E}$ there exists a sequence $\{y_1, y_2, \ldots\} \subset E$ such that $\lim y_n = b$. Let h be a closed X, Y-expansion of f such that $h(b)$ exists. We have $h(y_n) = f(y_n) = F(y_n) \notin \bar{W}$ and $\lim h(y_n) = h(b) = F(b) \in W$ because of h is continuous, a contradiction. Thus the function F is continuous.

Suppose $F^{-1}(b) \neq \emptyset$, $\lim a_n = b$ and $F(y_n) = a_n \neq b$ for all n. Assume to the contrary that the sequence $y_1, y_2, \ldots$ has no limit points in $F^{-1}(Y)$, if a limit point of this sequence exists, then it has to belong to $F^{-1}(b)$ because of F is continuous, but this doesn't contrary the fact that F is a closed mapping. Sets $F^{-1}(b)$ and $\{y_1, y_2, \ldots\}$ are disjoint and closed relative $F^{-1}(Y)$. Since X is hereditarily normal there exists an open set V such that $F^{-1}(b) \subset V$ and $\{y_1, y_2, \ldots\} \cap \bar{V} = \emptyset$. Let h be a closed X, Y-expansion of f such that $h^{-1}(b)$ is non-empty and let $y_n^m \in A \setminus \bar{V}$

be such that $\lim y_n^m = y_n$. The set $D = \{y_n^m : n, m = 1, 2, \ldots\}$ has no limit points in the set $F^{-1}(b)$. On the other hand $b \in \overline{h(D)}$ and therefore there exists a sequence $\{q_1, q_2, \ldots\} \subset h(D) = F(D)$ such that $\lim q_n = b$. Let $p_n \in F^{-1}(q_n) \cap D$ for all n. Since Lemma 1 the set $\{p_1, p_2, \ldots\} \subset D$ has a limit point in $h^{-1}(b) = F^{-1}(b)$ because of h is a closed mapping, a contradiction. Thus F is a closed mapping. □

Theorem 1. *Suppose X is a first countable hereditarily normal and hereditarily separable space, Y is a regular space and $|Y| \leqq \mathfrak{c}$. There exists a family S satisfying the following:*

(i) *S consists of closed mappings from X into Y,*

(ii) $|S| \leqq \mathfrak{c}$,

(iii) *for each $B \subset X$ and for each closed mapping $h: B \to h(B) \subset Y$ there exists $f \in S$ and a countable set $E \subset h(B)$ such that $f \restriction B \setminus h^{-1}(E) = h \restriction B \setminus h^{-1}(E)$ and $f^{-1}(t) = h^{-1}(t)$ for each $t \in h(B) \setminus E$.*

Proof. Let P be a countable subset of X and let $f: P \to Y$ be a continuous function. If there exists a closed mapping $h: A \to Y$ such that $P \subset A \subset \bar{P} \subset X$ and $h \restriction P = f$, then let F be the maximal closed X, Y-expansion of $h \restriction h^{-1}(h(P))$, it exists because of Lemma 3. Let us fix a particular F for a function f. Let S be the family of all such mappings F. Obviously the family S has the cardinality of at most continuum, i.e. $|S| \leqq \mathfrak{c}$.

Suppose $B \subset X$ and $f: B \to f(B) \subset Y$ is a closed mapping. Let $P \subset B$ be a dense in B and countable subset. Take $F \in S$, which is assigned to $f \restriction P$, and let $E = f(P)$. Thus we have $f \restriction B \setminus f^{-1}(E) = F \restriction B \setminus f^{-1}(E)$ and if $t \in f(B) \setminus E$, then $F^{-1}(t) = f^{-1}(t)$ because of Lemma 2. □

3. A generalization of Kuratowski's theorem [4].

Lemma 4. ([5] p. 425). *If X is an infinite set of the cardinality λ and S is a family of the cardinality of at most λ of functions from subsets of X onto subsets of the cardinality λ of X, then there exists a family F of the cardinality 2^λ of subsets of X such that conditions $Y, Z \in F$ and $Y \neq Z$ imply $|f(Z) \setminus Y| = \lambda$, for each function $f \in S$.* □

Theorem 2. *In any hereditarily separable, hereditarily normal and first countable space of the cardinality continuum there is a family of $2^\mathfrak{c}$ subspaces whose closed-map types are imcomparable.*

Proof. Let S be the family of closed mappings obtained by virtue of Theorem 1 in the case $X = Y$, where X is as in hypotheses. If $f \in S$, then let f_* be some fixed one-to-one function such that $f(f_*(x)) = x$, for each $x \in f(X)$. Take the family $S_* = \{f_* : f \in S \text{ and } |f(X)| = \mathfrak{c}\}$ and make use of Lemma 4. The family F of subspaces of X, which we obtain, is that we required. Indeed, let $A, B \in F$, $A \neq B$ and $D \subset A$. Suppose to the contrary that $h: D \to B$ is a closed mapping onto B. Let $P \subset D$ be a countable set such that $D \subset \bar{P}$ and let $f \in S$ be the closed mapping assigned to $h \upharpoonright P$. We have $|f_*(B) \setminus A| = \mathfrak{c}$ and $f_*(B) \setminus A \subset f_*(B) \setminus D$, a contradiction, because the set $f_*(B) \setminus D$ is countable. □

Theorem 2 is a generalization of the similar result on dimensional types, see K. Kuratowski [4] and [5] p. 433.

4. A generalization of Banach's result [1]. The below lemma is a modification of Banach's Lemme from [1]. One can prove it by making some changes in the proof of Banach's Lemme.

Lemma 5. *Suppose λ is an infinite cardinal number and $\tau < \lambda$. Let E be a set of the cardinality λ and let S be a family of functions from subsets of E onto subsets of E. If $|S| \leqq \lambda$ and $|f^{-1}(t)| \leqq \tau$ for each $f \in S$ and $t \in E$, then there exists a family $\{H_\alpha : \alpha < \lambda\}$ of subsets of E such that the following holds:*

(i) *$\{H_\alpha : \alpha < \lambda\}$ is a partition of E,*

(ii) *if $\gamma < \lambda$, then $|\bigcup\{H_\alpha : \alpha < \gamma\}| < \lambda$,*

(iii) *if $f \in S$, then there is $\gamma < \lambda$ such that $f(H_\alpha) \subset H_\alpha$ for each $\alpha > \gamma$.* □

A space X is *unperfect* if its compact subspaces are countable, e.g. a metric space is unperfect if it lacks any copy of the Cantor set. We will need the following Michael's result [8]. If X is a paracompact space, Y is a first countable space and $f: X \to Y$ is a closed mapping, then the boundary of $f^{-1}(t)$ is compact for each $t \in Y$.

Theorem 3. *If X is an unperfect, hereditarily separable, hereditarily paracompact and first countable space with $|X| = \mathfrak{c}$ and Y is a regular and first countable space with $|Y| \leqq \mathfrak{c}$, then there exists a family $\{A_\alpha : \alpha < \mathfrak{c}\}$ such that the following holds:*

(i) *for each $\alpha < \mathfrak{c}$, there is $A_\alpha \subset X$ such that $|A_\alpha| = \mathfrak{c}$,*

(ii) *for each $B \subset Y$, if there are distinct $\alpha, \beta < \mathfrak{c}$ with $ctB \leqq ctA$ and $ctB \leqq ctA$, then $|B| < \mathfrak{c}$.*

Proof. Let S be a family of closed mappings as in Theorem 1. If $f \in S$, then let $Z = \bigcup\{f^{-1}(t) : t \in Y$ and $f^{-1}(t)$ is nowhere dense relative $f^{-1}(Y)\}$. Take $f_* = f \upharpoonright Z$ and $S_* = \{f_* : f \in S\}$. Now we make use of Lemma 5. Let us observe that $f^{-1}(t)$ is always countable because of Michael's result [8]. We obtain a family $\{H_\alpha : \alpha < \mathfrak{c}\}$ as in Lemma 5. Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a family of pairwise disjoint subsets of $\{\alpha : \alpha < \mathfrak{c}\}$ such that $|P_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. Sets $A_\alpha = \bigcap\{H_\beta : \beta \in P_\alpha\}$ are that we required. □

Theorem 3 is a generalization of the result on dimensional types obtained in S. Banach [1].

5. Metric cases. It is well-known facts that any metric separable space is embeddable in the Hilbert cube Q and that any metric compact space is the image of the Cantor set C under a closed mapping, among others Q is the image of C under some closed mapping. Therefore any metric separable space is the image of some subspace of C under a closed mapping. Thus if we consider the closed-map type of a space X, then an interesting case is, when X lacks any copy of the Cantor set, i.e. X is unperfect.

Lemma 6. *Suppose X is a metric separable space. If there is a closed mapping f from X onto the Cantor set C, then X contains some copy of C.*

Proof. Let $A \subset C$ be a copy of the Cantor set such that if $t \in A$, then $f^{-1}(t)$ is a nowhere dense subset of X, whence $f^{-1}(t)$ is compact by virtue of Vaĭnšteĭn's Lemma [2] p. 356. The set $f^{-1}(A)$ being an uncountable compact metric space, because of Lubben's result [2] p. 236, contains some copy of the Cantor set. $\square$

From the above lemma we obtain immediately the following.

Theorem 4. *If a metric separable space X is unperfect, then its closed-map type is smaller than the closed-map type of the Cantor set C, i.e. $ctX < ctC$.* $\square$

If we use Theorem 1 and make appropriate changes in proofs in [6], then we obtain the following:

Theorem 5. *If X is a metric separable space and the closed-map type of X is smaller than the closed-map type of the Cantor set C, then there exists a metric space Y such that $ctX < ctY < ctC$.* $\square$

Theorem 6. *There exists a family $\{A_\alpha : \alpha < \mathfrak{c}^+\}$ of metric separable spaces such that $\alpha < \beta < \mathfrak{c}^+$ implies $ctA_\alpha < ctA_\beta$.* $\square$

We don't know for which metric separable space X one can assume that the family as in Theorem 6 is contained in X. Can it be any metric separable space X with $|X| = \mathfrak{c}$?

Theorem 7. *Suppose X is a metric separable space with $|X| = \mathfrak{c}$. There exists a family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subspaces of X such that $|A_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$ and if $ctB \leqq ctA_\alpha$, $ctB \leqq ctA_\beta$ and $\alpha \neq \beta$, then $|B| < \mathfrak{c}$.*

Proof. We will need the following Lanšev's result [7]. If X is a metric space and $f: X \to Y$ is a closed mapping, then $Y = Y_0 \cup Y_1 \cup Y_2 \cup \ldots$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_n are discrete for all n. If X is, additionally, separable, then Y_0 is a metrizable and separable subspace, cf. the Hanai-Morita-Stone theorem [2] p. 356, and $\bigcup\{Y_n : n = 1, 2, \ldots\}$ is a coutable subspace.

We can assume that X is an unperfect space, if need be we can take a suitable subspace. By virtue of Theorem 3 there exists a family $\{A_\alpha : \alpha < \mathfrak{c}\}$ of subspace of X such that $|A_\alpha| = \mathfrak{c}$ and if B is a subspace of the Hilbert cube, $ctB \leqq ctA_\alpha$, $ctB \leqq ctA_\beta$ and $\alpha \neq \beta$, then $|B| < \mathfrak{c}$. This family is that we required. Suppose to the contrary that there are $\alpha \neq \beta$ and a space B such that $|B| = \mathfrak{c}$, $ctB \leqq ctA_\alpha$ and $ctB \leqq ctA_\beta$. Thus there are $H \subset A_\alpha$, $G \subset A_\beta$ and closed mappings $h: H \to B = h(H)$, $g: G \to B = g(G)$. Let $B_* = \{y \in B : h^{-1}(y)$ and $g^{-1}(y)$ are compact$\}$. The space B_* is metrizable, separable and $|B_*| = \mathfrak{c}$ because of Lašnev result, a contradiction, since B_* is embeddable in the Hilbert cube, $ctB_* \leqq ctA_\alpha$ and $ctB_* \leqq ctA_\beta$. $\square$

In next two theorems we assume that the cardinal number $\mathfrak{c}$ is regular. We don't know if this assumption can be omitted.

Theorem 8. *If X is a metric separable space with $|X| = \mathfrak{c}$, then there is a family $\{B_\alpha: \alpha < \mathfrak{c}\}$ of subspaces of X such that $\alpha < \beta$ implies $ctB_\alpha < ctB_\beta$.*

Proof. Let $\{A_\alpha: \alpha < \mathfrak{c}\}$ be a family as in Theorem 7. Take $B_\alpha = \bigcup\{A_\beta: \beta \leqq \alpha\}$. If $\alpha < \beta < \mathfrak{c}$, then $ctB_\alpha \leqq ctB_\beta$ since $B_\alpha \subset B_\beta$. We will show that $ctB_\beta \leqq ctB_\alpha$ doesn't hold. Suppose to the contrary that there are $H \subset B_\alpha$ and a closed mapping $h: H \to A_\beta = h(H) \subset B_\beta$. We have $|A_\beta \cap h(A_\gamma)| < \mathfrak{c}$ for each $\gamma \leqq \alpha$. Whence $|A_\beta| = |\bigcup\{A_\beta \cap h(A_\gamma): \gamma \leqq \alpha\}| < \mathfrak{c}$, a contradiction, because of $\mathfrak{c}$ is regular. □

Theorem 9. *If X is a metric separable space with $|X| = \mathfrak{c}$, then there exists a family $\{B_\alpha: \alpha < \lambda\}$ of subspaces of X such that if $\alpha < \beta$, then $ctB_\beta < ctB_\alpha$.*

Proof. Let $\{A_\alpha: \alpha < \mathfrak{c}\}$ be a family as in Theorem 7. Take $B_\alpha = \bigcap\{A_\beta: \alpha \leqq \beta < \lambda\}$. If $\alpha < \beta < \lambda$, then $ctB_\beta \leqq ctB_\alpha$ because of $B_\beta \subset B_\alpha$. We will show that $ctB_\alpha \leqq ctB_\beta$ doesn't hold. Suppose to the contrary that there are $H \subset B_\beta$ and a closed mapping $h: H \to A_\alpha = h(H) \subset B_\alpha$. We have $|A_\alpha \cap h(A_\gamma)| < \mathfrak{c}$ for each γ such that $\beta \leqq \gamma < \lambda$. Whence $|A_\alpha| = |\bigcup\{A_\alpha \cap h(A_\gamma): \beta \leqq \gamma < \lambda\}| < \mathfrak{c}$, a contradiction. □

References

[1] *S. Banach:* Sur les transformations biunique, Fund. Math. *19* (1932), 10—16.

[2] *R. Engelking:* General Topology, Warszawa (1977).

[3] *M. Frèchet:* Les dimensions d'un ensemble abstrait, Math. Ann. *68* (1910), 145—168.

[4] *K. Kuratowski:* Sur la puissance de l'ensemble des "nombres de dimensions" au sens de M. Frèchet, Fund. Math. *8* (1926), 201—208.

[5] *K. Kuratowski:* Topology I, New York (1966).

[6] *L. Kuratowski, W. Sierpiński:* Sor un problème de M. Frèchet concernant les dimensions des ensembles lineaires, Fund. Math. (1926), 193—200.

[7] *N. Lašnev:* Continuous decompositions and closed mappings of metric spaces; English translation, Soviet Math. Dokl. *6* (1965), 1504—1506.

[8] *E. Michael:* A note on closed maps and compact sets, Israel Journ. of Math. *2* (1964), 173—176.

[9] *W. Sierpiński:* Sur un problème concernant les types de dimensions, Fund. Math. *19* (1932), 65—71.

[10] *W. Sierpiński:* General Topology, Toronto (1952).

Author's address: Instytut Matematyki Uniwerstetu Śląskiego, ul. Bankowa 14, 40 007 Katowice, Poland.