

Jaromír Šimša

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SOME CONVERSE THEOREMS IN THE ASYMPTOTIC THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

JAROMÍR ŠIMŠA, Brno

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1. ASYMPTOTIC THEOREMS OF TRENCH

Trench recently gave sufficient conditions for a scalar differential equation

$$(1) \quad x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = 0$$

to have a solution which behaves for  $t \rightarrow \infty$  like a given polynomial of degree  $< n$  (see [5]), and for an equation

$$(2) \quad x^{(n)} + [a_1 + p_1(t)]x^{(n-1)} + \dots + [a_n + p_n(t)]x = 0$$

to have a solution like  $\exp(\lambda_0 t)$  asymptotically, where  $\lambda_0$  is a root of the polynomial equation

$$(3) \quad \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

with constant coefficients  $a_k$  (see [6]). Trench's integrability conditions on  $p_k$  are stated largely in terms of ordinary integral convergence. This presents a significant weakening of the classical conditions that require the absolute convergence ([1, Chapter X]). The aim of the present paper is to show that Trench's sufficient conditions are close to necessary.

Throughout the paper, all functions considered are complex- or real-valued and continuous on  $[T, \infty)$ , for some real  $T$ . In all hypotheses (conclusions), the improper integrals are assumed (concluded) to converge. The symbols "o" and "O" refer to the behavior for  $t \rightarrow \infty$ .

The above mentioned results of Trench imply the following two assertions on the existence of fundamental systems of solutions of (1) and (2), with prescribed asymptotic behavior.

**Theorem A.** *Assume that  $\varphi$  is positive and nonincreasing on  $[T, \infty)$ ,  $\varrho$  is a non-negative constant and*

$$(4) \quad \int_t^\infty \varphi^2(s) ds = O(\varphi(t)) \quad \text{if} \quad \varrho = 0.$$

*Further, assume that (3) has  $n$  distinct roots  $\lambda_j$  such that  $\text{Re } \lambda_1 \geq \text{Re } \lambda_2 \geq \dots$*

...  $\geq \operatorname{Re} \lambda_n$ , and the functions  $p_k$  satisfy

$$(5) \quad \int_t^\infty p_k(s) e^{qs} ds = o(\varphi(t)) \quad (1 \leq k \leq n)$$

and

$$(6) \quad \int_t^\infty |p_1(s)| \varphi(s) ds = o(\varphi(t)).$$

If  $\operatorname{Re}(\lambda_1 - \lambda_n) > \varrho$ , assume also that  $e^{\alpha t} \varphi(t)$  is nondecreasing on  $[T, \infty)$ , for some  $\alpha$  smaller than any positive value from the set

$$\{\operatorname{Re}(\lambda_j - \lambda_m) - \varrho \mid 1 \leq j < m \leq n\}.$$

Finally, assume that

$$(7) \quad \int_t^\infty \left( \sum_{k=1}^n \lambda_j^{n-k} p_k(s) \right) e^{(\lambda_j - \lambda_m)s} ds = o(\varphi(t)),$$

whenever  $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$ . Then (2) has  $n$  solutions  $x_j$  ( $1 \leq j \leq n$ ) satisfying

$$(8) \quad x_j^{(k)}(t) = (\lambda_j^k + o(e^{-\alpha t} \varphi(t))) e^{\lambda_j t} \quad (0 \leq k \leq n-1).$$

**Theorem B.** Assume that  $\psi$  is positive and nonincreasing on  $[T, \infty)$ ,  $\nu$  is a non-negative integer and

$$(9) \quad \int_t^\infty \frac{\psi^2(s)}{s} ds = O(\psi(t)) \quad \text{if } \nu = 0.$$

If  $\nu < n-1$ , assume also that  $t^\alpha \psi(t)$  is nondecreasing on  $[T, \infty)$  for some constant  $\alpha < 1$ . Finally, assume that the functions  $p_k$  satisfy

$$(10) \quad \int_t^\infty |p_1(t)| dt < \infty,$$

$$(11) \quad \int_t^\infty p_k(s) s^{k-1} ds = o(t^{-\nu} \psi(t)) \quad (1 \leq k \leq n)$$

and, if  $\nu < n$ ,

$$(12) \quad \int_t^\infty g_j(s) s^{n-j+\nu-1} ds = o(\psi(t)) \quad (\nu \leq j \leq n-1),$$

where the functions  $g_j$  are given by

$$(13) \quad g_j(t) = \sum_{k=1}^n p_k(t) (t^j)^{(n-k)} \quad (\nu \leq j \leq n-1).$$

Then (1) has  $n$  solutions  $x_j$  ( $0 \leq j \leq n-1$ ) satisfying

$$(14) \quad x_j^{(k)}(t) = (t^j)^{(k)} + o(t^{j-k-\nu} \psi(t)) \quad (0 \leq k \leq n-1).$$

Remark 1. Theorem A with  $\varphi(t) = t^{-q}$  ( $q = \text{const.} \geq 0$ ) was essentially proved in [2]. Then (4) means that  $q \geq 1$  if  $\varrho = 0$ . As shown in [3], Theorem A becomes false without this restriction on  $\varrho$  and  $q$ . The case  $\varrho = 0$  and  $q < 1$  was discussed in [4].

Remark 2. In an unpublished work the author observed that Theorem A holds with (5) replaced by the weaker assumption

$$(15) \quad \int_t^\infty p_k(s) ds = o(e^{-\alpha t} \varphi(t)) \quad (1 \leq k \leq n).$$

(Integration by parts shows that (5) implies (15); the converse implication is false.)

## 2. THE FIRST CONVERSE THEOREM

**Theorem 1.** *Let  $\varrho$  and  $\varrho$  be as in the first sentence of Theorem A, including (4). Assume that (3) has  $n$  distinct roots  $\lambda_j$ , (2) has  $n$  solutions  $x_j$  satisfying (8), and (6) holds. Then the functions  $p_k$  satisfy (15). Moreover, (7) holds whenever  $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$ .*

It is convenient to state two preparatory lemmas separately from the proof of Theorem 1.

**Lemma 1.** *Let  $x$  be a function in  $C^{(n)}[T, \infty)$  satisfying*

$$(16) \quad x^{(k)}(t) = (\lambda^k + o(e^{-\varrho t} \varphi(t))) e^{\lambda t} \quad (0 \leq k \leq n-1),$$

where  $\varrho$  and  $\varphi$  are as in the first sentence of Theorem A and  $\lambda$  is a constant. Then the functions

$$(17) \quad h_k(t) = \frac{x^{(k)}(t)}{x(t)} - \lambda^k \quad (1 \leq k \leq n)$$

satisfy

$$(18) \quad h_k(t) = o(e^{-\varrho t} \varphi(t)) \quad (1 \leq k \leq n-1),$$

$$(19) \quad h'_k(t) = o(e^{-\varrho t} \varphi(t)) \quad (1 \leq k \leq n-2)$$

and

$$(20) \quad \int_t^\infty h_k(s) ds = o(e^{-\varrho t} \varphi(t)) \quad (1 \leq k \leq n).$$

**Lemma 2.** *Suppose that the equation*

$$(21) \quad u^{(n)} + a_1 u^{(n-1)} + \dots + a_n u = f(t)$$

has a solution  $u = u(t)$  satisfying

$$(22) \quad u^{(k)}(t) = o(e^{\beta t} \varphi(t)) \quad (0 \leq k \leq n-1),$$

where  $\beta$  is a real constant and  $\varphi$  is positive and nonincreasing on  $[T, \infty)$ . If  $\lambda_m$  is a root of (3) with  $\operatorname{Re} \lambda_m = \beta$ , then

$$(23) \quad \int_t^\infty f(s) e^{-\lambda_m s} ds = o(\varphi(t)).$$

We leave the proofs of Lemmas 1 and 2 for the appendix.

**Proof of Theorem 1.** We proceed by induction with respect to  $n$ , the order of (2). In the case  $n = 1$ , any solution  $x$  of (2) satisfies

$$(24) \quad x(t) = C \exp \left[ -a_1 t - \int_T^t p_1(s) ds \right],$$

where  $C$  is a constant and  $t \geq T$ . If  $x_1$  is a solution of (2) as in (8), with  $\lambda_1 = -a_1$ , then (15) follows from (8) and (24) with  $x = x_1$ . Obviously, if  $n = 1$ , then  $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$  holds only if  $j = m = 1$  and  $\varrho = 0$ , which reduces (7) to (15).

Assume now that (2) satisfies the hypotheses of Theorem 1 with  $n > 1$ . We use reduction of order. Given  $n$  solutions  $x_j$  of (2) as in (8), we introduce constants  $b_k$

and functions  $h_k, q_k$  and  $z_j$  by

$$(25) \quad b_k = \sum_{j=0}^k \binom{n-j}{k-j} a_j \lambda_1^{k-j}, \quad h_k(t) = \frac{x_1^{(k)}(t)}{x_1(t)} - \lambda_1^k, \quad (1 \leq k \leq n, a_0 = 1),$$

$$(26) \quad q_k(t) = p_k(t) + \sum_{j=0}^{k-1} \binom{n-j}{k-j} a_j h_{k-j}(t) + \sum_{j=1}^{k-1} \binom{n-j}{k-j} p_j(t) h_{k-j}(t) + \sum_{j=1}^{k-1} \binom{n-j}{k-j} p_j(t) \lambda_1^{k-j}, \quad (1 \leq k \leq n)$$

and

$$(27) \quad z_j(t) = (\lambda_j - \lambda_1)^{-1} [x_j(t)/x_1(t)]' \quad (2 \leq j \leq n).$$

Since  $x_1$  is as in (8), the functions in (25)–(27) are defined on  $[T_1, \infty)$  for some real  $T_1 \geq T$ . Moreover, Lemma 1 with  $x = x_1$  and  $\lambda = \lambda_1$  implies that (18)–(20) hold for our functions  $h_k$  in (25). The constants  $b_k$  are chosen in (25) so that the polynomial  $\lambda^n + b_1 \lambda^{n-1} + \dots + b_{n-1} \lambda + b_n$  has  $n$  distinct zeros  $\lambda_j - \lambda_1, 1 \leq j \leq n$ .

The following assertion makes the meaning of the definitions (26) and (27) clear: the equation

$$(28) \quad z^{(n-1)} + [b_1 + q_1(t)] z^{(n-2)} + \dots + [b_{n-1} + q_{n-1}(t)] z = 0$$

has  $(n-1)$  solutions (27) that satisfy

$$(29) \quad z_j^{(k)}(t) = [(\lambda_j - \lambda_1)^k + o(e^{-\alpha t} \varphi(t))] e^{(\lambda_j - \lambda_1)t}, \quad (0 \leq k \leq n-2).$$

To see this, we first put  $x = x_1(t) y$ . A routine computation shows that (2) is transformed into

$$(30) \quad y^{(n)} + [b_1 + q_1(t)] y^{(n-1)} + \dots + [b_n + q_n(t)] y = 0,$$

with  $b_k$  and  $q_k$  as in (25) and (26). Since  $x_1$  is a solution of (2) and  $\lambda_1$  is a root of (3), we have

$$(31) \quad b_n = 0 \quad \text{and} \quad q_n(t) = 0 \quad (t \geq T).$$

Consequently, we may put  $z = y' = (x/x_1(t))'$  to obtain the equation (28) with  $(n-1)$  solutions (27). To prove (29), we need to show that the functions  $y_j$  in

$$(32) \quad x_j(t) = x_1(t) y_j(t) \quad (2 \leq j \leq n)$$

satisfy

$$(33) \quad y_j^{(k)}(t) = [(\lambda_j - \lambda_1)^k + o(e^{-\alpha t} \varphi(t))] e^{(\lambda_j - \lambda_1)t}$$

for  $k = 1, 2, \dots, n-1$ . First we note that (33) with  $k = 0$  follows from (8) with  $k = 0$ . Further, assume that (33) holds with any  $k \leq m-1$  for some  $m, 1 \leq m \leq n-1$ . If we differentiate (32)  $m$  times, we obtain

$$x_j^{(m)}(t) = x_1(t) y_j^{(m)}(t) + \sum_{k=1}^m \binom{m}{k} x_1^{(k)}(t) y_j^{(m-k)}(t)$$

and, therefore,

$$(34) \quad y_j^{(m)}(t) = x_j^{(m)}(t)/x_1(t) - \sum_{k=1}^m \binom{m}{k} (\lambda_1^k + h_k(t)) y_j^{(m-k)}(t)$$

(see the definition of  $h_k$  in (25)). Now (8), (18), (33) with  $k \leq m - 1$  and (34) imply that

$$\begin{aligned} e^{(\lambda_1 - \lambda_j)t} y_j^{(m)}(t) &= \frac{\lambda_j^m + o}{1 + o} - \sum_{k=1}^m \binom{m}{k} (\lambda_1^k + o) [(\lambda_j - \lambda_1)^{m-k} + o] = \\ &= \lambda_j^m - \sum_{k=1}^m \binom{m}{k} \lambda_1^k (\lambda_j - \lambda_1)^{m-k} + o = (\lambda_j - \lambda_1)^m + o, \end{aligned}$$

where, for brevity, “ $o$ ” stands for “ $o(e^{-\epsilon t} \varphi(t))$ ”. Thus (33) with  $k = m$  holds, which proves (29).

Assuming now that Theorem 1 holds if (2) is of order  $n - 1$ , we conclude from (29) that

$$(35) \quad \int_t^\infty q_k(s) ds = o(e^{-\epsilon t} \varphi(t)) \quad (1 \leq k \leq n - 1),$$

because, as we now verify,

$$(36) \quad \int_t^\infty |q_1(s)| \varphi(s) ds = o(\varphi(t)).$$

Indeed, we see from (18) and (26) that

$$(37) \quad q_1(t) - p_1(t) = n h_1(t) = o(e^{-\epsilon t} \varphi(t)),$$

hence (36) follows from (6), (37) and the fact that

$$(38) \quad \int_t^\infty o(e^{-\epsilon s} \varphi^2(s)) ds = o(\varphi(t)).$$

The last relation follows either from (4), or from

$$\int_t^\infty e^{-\epsilon s} \varphi^2(s) ds \leq \varphi^2(t) \int_t^\infty e^{-\epsilon s} ds = o(\varphi(t)) \quad \text{if } \varrho > 0.$$

The next step of our proof is to show that

$$(39) \quad \int_t^\infty p_k(s) ds = o(e^{-\epsilon t} \varphi(t))$$

holds for  $k = 1, 2, \dots, n$ . If  $k = 1$ , then (39) follows from (18), (35) and (37). Assuming now that (39) holds with  $k \leq m - 1$  for some  $m$ ,  $1 < m \leq n$ , we obtain from (20), (26), (35) and (39) with  $k \leq m - 1$  that

$$\int_t^\infty p_m(s) ds = o(e^{-\epsilon t} \varphi(t)) - \sum_{j=1}^{m-1} \binom{n-j}{m-j} \int_t^\infty p_j(s) h_{m-j}(s) ds,$$

provided the integrals on the right hand side converge. (Note that (35) holds also with  $k = n$  because of (31).) Consequently, (39) with  $k = m$  holds if

$$(40) \quad \int_t^\infty p_j(s) h_{m-j}(s) ds = o(e^{-\epsilon t} \varphi(t))$$

is valid for  $j = 1, 2, \dots, m - 1$ . If  $j = 1$ , (40) follows from (6) and (18), because

$$\int_t^\infty |p_1(s) h_{m-1}(s)| ds = \int_t^\infty |p_1(s)| \varphi(s) o(e^{-\epsilon s}) ds = o(e^{-\epsilon t} \varphi(t)).$$

If  $1 < j \leq m - 1$ , then integration by parts yields

$$(41) \quad \int_t^{t_1} p_j(s) h_{m-j}(s) ds = -P_j(s) h_{m-j}(s) \Big|_t^{t_1} + \int_t^{t_1} P_j(s) h'_{m-j}(s) ds,$$

where  $P_j(t) = o(e^{-\alpha t} \varphi(t))$  is the integral (39) with  $k = j$ . Since both  $P_j(s) h_{m-j}(s)$  and  $P_j(s) h'_{m-j}(s)$  are  $o(e^{-2\alpha s} \varphi^2(s))$  (see (18) and (19)), we can let  $t_1 \rightarrow \infty$  in (41) and use (38) to obtain (40). Thus (15) is proved.

To complete the proof of Theorem 1, we need to show that (7) holds if  $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$ . We put  $u_j(t) = x_j(t) - \exp(\lambda_j t)$ , where  $x_j$  is the solution of (2) that satisfies (8). Then

$$(42) \quad u_j^{(k)}(t) = o(e^{(\operatorname{Re} \lambda_j - \alpha)t} \varphi(t)) \quad (0 \leq k \leq n - 1).$$

Moreover,  $u_j$  is a solution of (21) with  $f$  given by

$$(43) \quad f(t) = - \sum_{k=1}^n p_k(t) \lambda_j^{n-k} e^{\lambda_j t} - \sum_{k=1}^n p_k(t) u_j^{(n-k)}(t).$$

Since  $\operatorname{Re}(\lambda_j - \lambda_m) = \varrho$ , Lemma 2 with  $\beta = \operatorname{Re} \lambda_j - \varrho$  and (42) imply (23) with  $f$  as in (43). Now (23) and (43) imply (7) provided

$$(44) \quad \int_t^\infty p_k(s) u_j^{(n-k)}(s) e^{-\lambda_m s} ds = o(\varphi(t))$$

for  $k = 1, 2, \dots, n$ . If  $k = 1$ , then (44) follows from (6) and (42):

$$\int_t^\infty |p_1(s) u_j^{(n-1)}(s) e^{-\lambda_m s}| ds = \int_t^\infty |p_1(s)| o(\varphi(s)) ds = o(\varphi(t)).$$

If  $1 < k \leq n$ , then integration by parts yields

$$(45) \quad \int_t^{t_1} p_k(s) u_j^{(n-k)}(s) e^{-\lambda_m s} ds = -P_k(s) u_j^{(n-k)}(s) e^{-\lambda_m s} \Big|_t^{t_1} + \int_t^{t_1} P_k(s) [u_j^{(n-k+1)}(s) - \lambda_m u_j^{(n-k)}(s)] e^{-\lambda_m s} ds,$$

where  $P_k(t) = o(e^{-\alpha t} \varphi(t))$  is the integral (39). By virtue of (39) and (42), the integrand and the outintegral function on the right hand side of (45) are  $o(e^{-\alpha t} \varphi^2(t))$ . In view of (38), we can let  $t_1 \rightarrow \infty$  in (45) to obtain (44). This completes the proof of Theorem 1.

### 3. THE SECOND CONVERSE THEOREM

**Theorem 2.** *Let  $\psi$  and  $\nu$  be as in the first sentence of Theorem B, including (9). If (1) has  $n$  solutions  $x_j$  ( $0 \leq j \leq n - 1$ ) satisfying (14) and (10) holds, then the functions  $p_k$  satisfy (11) and, if  $\nu < n$ , also (12).*

**Proof of Theorem 2.** We will show that Theorem 2 is a consequence of Theorem 1. We introduce new variables  $y$  and  $\tau$  by

$$(46) \quad \tau = \log t, \quad y(\tau) = x(t).$$

Then

$$(47) \quad x^{(k)}(t) = e^{-k\tau} Q_k(D) y(\tau) \quad \left( D = \frac{d}{d\tau}, \quad Q_k(\lambda) = \prod_{j=0}^{k-1} (\lambda - j), \quad 1 \leq k \leq n \right)$$

and therefore, (1) is transformed into

$$(48) \quad Q_n(D)y + \sum_{k=1}^n q_k(\tau) D^{n-k}y = 0,$$

where

$$(49) \quad q_k(\tau) = \frac{1}{(n-k)!} \sum_{m=1}^k Q_{n-m}^{(n-k)}(0) p_m(e^\tau) e^{m\tau}, \quad 1 \leq k \leq n.$$

Now we verify that (48) satisfies the conditions of Theorem 1 with  $t$  replaced by  $\tau$ ,  $\varrho = \nu$ ,  $\lambda_j = j - 1$  ( $1 \leq j \leq n$ ) and  $\varphi(\tau) = \psi(e^\tau)$ . Namely, we show that (48) has  $n$  solutions  $y_j$  ( $0 \leq j \leq n - 1$ ) satisfying

$$(50) \quad D^k y_j(\tau) = [j^k + o(e^{-\nu\tau} \varphi(\tau))] e^{j\tau} \quad (0 \leq k \leq n - 1),$$

$$(51) \quad \int_{\tau}^{\infty} |q_1(r)| \varphi(r) dr = o(\varphi(\tau)),$$

and that  $\varphi$  obeys

$$(52) \quad \int_{\tau}^{\infty} \varphi^2(r) dr = O(\varphi(\tau)) \quad \text{if } \nu = 0.$$

Indeed, if we put  $x = x_j$  in (46), where  $x_j$  are solutions of (1) as in (14), we obtain  $n$  solutions  $y_j$  of (48) satisfying

$$(53) \quad F(D)y_j(\tau) = [F(j) + o(e^{-\nu\tau} \psi(e^\tau))] e^{j\tau}$$

for any polynomial  $F$  of degree  $< n$ . The last relation holds, because (14), (46) and (47) imply (53) with  $F = 1, Q_1, \dots, Q_{n-1}$  (note that  $(t^j)^{(k)} = Q_k(j) t^{j-k}$ ). Thus (50) is proved. To verify (51) and (52), we substitute  $s = e^r$  in the integrals on their left hand sides. Since  $q_1(\tau) = e^\tau p_1(e^\tau)$  (see (49)), we obtain

$$\int_{\tau}^{\infty} |q_1(r)| \varphi(r) dr \leq \varphi(\tau) \int_{\tau}^{\infty} |p_1(s)| ds$$

and

$$\int_{\tau}^{\infty} \varphi^2(r) dr = \int_t^{\infty} \frac{\psi^2(s)}{s} ds,$$

where  $t = e^\tau$  (see (46)). Consequently, (51) and (52) follow from (10) and (9), respectively.

Applying Theorem 1 to (48), we conclude that

$$(54) \quad \int_{\tau}^{\infty} q_k(r) dr = o(e^{-\nu\tau} \varphi(\tau)) \quad (1 \leq k \leq n)$$

and, if  $\nu < n$ ,

$$(55) \quad \int_{\tau}^{\infty} \sum_{k=1}^n j^{n-k} q_k(r) e^{\nu r} dr = o(\varphi(\tau)) \quad (\nu \leq j \leq n - 1).$$

Using (49) and substituting  $s = e^r$ , we find that

$$(56) \quad \int_{\tau}^{\infty} q_k(r) dr = \frac{1}{(n-k)!} \int_t^{\infty} \sum_{m=1}^k Q_{n-m}^{(n-k)}(0) p_m(s) s^{m-1} ds \quad (1 \leq k \leq n)$$

and

$$(57) \quad \int_{\tau}^{\infty} \sum_{k=1}^n j^{n-k} q_k(r) e^{\nu r} dr = \int_t^{\infty} \sum_{m=1}^n Q_{n-m}(j) p_m(s) s^{m+\nu-1} ds.$$



Since  $Q_{n-k}^{(n-k)}(0) = (n-k)! \neq 0$ , (11) follows from (54), (56) by induction. Finally, (55) and (57) imply (12), because

$$\sum_{m=1}^n Q_{n-m}(j) p_m(s) s^{m+\nu-1} = g_j(s) s^{n-j+\nu-1}$$

(see (13)). This completes the proof of Theorem 2.

#### 4. APPENDIX

**Proof of Lemma 1.** First we note that (18) follows immediately from (16) and (17). Routine manipulations with (17) show that

$$(58) \quad h'_k = h_{k+1} - \lambda h_k - h_k h_1 - \lambda^k h_1 \quad (1 \leq k \leq n-1).$$

Now (18) and (58) imply (19). Further, (16) and (17) imply

$$\int_t^\infty h_1(s) ds = -\log [e^{-\lambda t} x(t)].$$

Since  $\exp(-\lambda t) x(t) = 1 + o(e^{-\rho t} \varphi(t))$  (see (16)), the first relation in (20) holds. Integrating (58) we obtain

$$\int_t^{t_1} h_{k+1}(s) ds = h_k(s) \Big|_t^{t_1} + \int_t^{t_1} (\lambda h_k(s) + \lambda^k h_1(s) + h_k(s) h_1(s)) ds,$$

for  $k = 1, 2, \dots, n-1$ . Consequently, (20) is proved by induction, because, as we now verify,

$$(59) \quad \int_t^\infty |h_k(s) h_1(s)| ds = o(e^{-\rho t} \varphi(t)).$$

Indeed, if  $\rho = 0$ , then (59) follows from (4) and (18). If  $\rho > 0$ , then (59) follows from (18) and the inequality

$$\int_t^\infty e^{-2\rho s} \varphi^2(s) ds \leq \varphi^2(t) \int_t^\infty e^{-2\rho s} ds = (2\rho)^{-1} \varphi^2(t) e^{-2\rho t}.$$

This completes the proof of Lemma 1.

**Proof of Lemma 2.** We put  $u = \exp(\lambda_m t) v$  to transform (21) into

$$(60) \quad v^{(n)} + b_1 v^{(n-1)} + \dots + b_{n-1} v' + b_n v = f(t) e^{-\lambda_m t}$$

with constant coefficients  $b_k$ . Let  $u(t)$  be a solution of (21) as in (22). Then the solution  $v(t) = \exp(-\lambda_m t) u(t)$  of (60) obeys

$$(61) \quad v^{(k)}(t) = o(\varphi(t)) \quad (0 \leq k \leq n-1)$$

because  $\operatorname{Re} \lambda_m = \beta$ . Since  $\lambda_m$  is a root of (3), we have  $b_n = 0$  in (60). Consequently, integrating (60) with  $v = v(t)$ , we obtain

$$(v^{(n-1)}(s) + b_1 v^{(n-2)}(s) + \dots + b_{n-1} v(s)) \Big|_t^{t_1} = \int_t^{t_1} f(s) e^{-\lambda_m s} ds.$$

This together with  $t_1 \rightarrow \infty$  and (61) implies (23), which completes the proof of Lemma 2.

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*Author's address*: 662 95 Brno, Janáčkovo nám. 2a, Czechoslovakia. (Přírodovědecká fakulta UJEP.)