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THE REAL K-RING OF SOME CW-COMPLEXES
OF SMALL DIMENSION

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In 1981, L. M. Woodward in [8] classified the stable classes of orientable vector bundles over CW-complexes of small dimension. Using his results and some algebraic arguments, we describe the real K-ring of some CW-complexes of a dimension ≤ 7 in terms of cohomology and characteristic classes. We also show that our description can be really used for explicit calculation of K-rings.

1. PRELIMINARY NOTES

In this section we introduce the special symbols needed in the sequel. Let $\varrho_m: Z \rightarrow Z_m$ denote the reduction mod m , $\varrho_{42}: Z_4 \rightarrow Z_2$ the reduction mod 2 and $i: Z_2 \rightarrow Z_4$ the injection. The same symbols will abbreviate the induced homomorphisms in cohomology. The symbols δ and Δ are used to denote the Bockstein coboundary homomorphisms of the sequences $0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0$ and $0 \rightarrow Z_4 \rightarrow Z_8 \rightarrow Z_2 \rightarrow 0$, respectively.

The Pontrjagin square $P: H^2(X; Z_2) \rightarrow H^4(X; Z_4)$ (see [5; Chapt. 2, exercises] or [7; 10]) is the cohomology operation satisfying

$$(1.1) \quad \begin{aligned} \varrho_{42}(P(a)) &= i(a^2), \quad a \in H^2(X; Z_2), \\ P(a + b) &= P(a) + P(b) + i(ab), \quad a, b \in H^2(X; Z_2), \\ P(xy) &= \Delta(x) \Delta(y), \quad x, y \in H^1(X; Z_2). \end{aligned}$$

Here the first two equations are precisely [7; 10.2, 10.3] while the third follows from [7; 10.5] and from the fact that $i(x^2) = i(Sq^1x) = 0$ for $x \in H^1(X; Z_2)$.

We will also use the following relations between the Stiefel-Whitney and the Pontrjagin classes of a fibration ξ ([4], [7]):

$$(1.2) \quad \begin{aligned} \varrho_2(p_1(\xi)) &= (w_2(\xi))^2, \\ \varrho_4(p_1(\xi)) &= P(w_2(\xi)) + i(w_4(\xi) + w_1(\xi) \cdot Sq^1(w_2(\xi))), \\ w_3(\xi) &= w_1(\xi) w_2(\xi) + Sq^1(w_2(\xi)). \end{aligned}$$

Note that the first equation can be obtained by applying ϱ_{42} to the second. The characteristic classes obviously define the maps $w_i: BO \rightarrow K(\mathbb{Z}_2; i)$ and $p_j: BO \rightarrow K(\mathbb{Z}; 4j)$ of the classifying spaces.

2. RESULTS

We shall deal with the representable K-theory, i.e., with the set $KO\tilde{\sim}(X) = [X; BO]$. This set is endowed with the natural structure of a ring (see [6; 13]). Of course, if X is a finite-dimensional CW-complex, the elements of $KO\tilde{\sim}(X)$ can be viewed as stable equivalence classes of vector fibrations over X . In this case, addition is defined by the Whitney sum while multiplication is given by the tensor product as in [3; 2].

From now on, we shall write for brevity $H^*(X)$ instead of $H^*(X; \mathbb{Z})$ and $\bar{H}^*(X)$ instead of $H^*(X; \mathbb{Z}_2)$. Let us define

$$F(X) = \{(a, b, c) \in \bar{H}^1(X) \oplus \bar{H}^2(X) \oplus H^4(X); \varrho_2(c) = b^2\},$$

and define operations \boxplus and $*$ on the set $F(X)$ by

$$(a, b, c) \boxplus (x, y, z) = (a + x, b + ax + y, c + \delta a \delta x + z),$$

$$(a, b, c) * (x, y, z) = (0, ax, \delta a \delta x).$$

Finally, denote by $U = (w_1, w_2, p_1): KO\tilde{\sim}(X) \rightarrow F(X)$ the map induced by w_1, w_2 and p_1 . In the next section we prove (compare [8; Theorem 1]):

2.1. Theorem. *The system $(F(X), \boxplus, *)$ forms a commutative ring with the zero element $(0, 0, 0)$. The map $U: KO\tilde{\sim}(X) \rightarrow F(X)$ is a homomorphism of rings for each CW-complex X .*

If X is a CW-complex of a dimension ≤ 7 , the map U is an epimorphism. If, in addition, $H^4(X)$ has no 2-torsion, the map U is an isomorphism.

Remark. Improving slightly [8; Theorem 1], it is possible to prove that the kernel of U is isomorphic with $\text{Tor}(H^4(X); \mathbb{Z}_2)$ for each CW-complex of a dimension ≤ 7 .

In the following example, $C_k(a)$ and $C(a)$ abbreviate the cyclic group of the order k and the infinite cyclic group generated by a , respectively.

Example. The orthogonal K-ring of the Grassmann manifold $G(2, 2)$ of all two-dimensional linear subspaces in R^4 is isomorphic with the direct sum $C_4(f) \oplus C(g)$ for some $f, g \in KO\tilde{\sim}(G(2, 2))$ satisfying $f^2 = 2f$ and $g^2 = fg = 0$.

Indeed, $G(2, 2)$ is known to be a four-dimensional oriented compact manifold, hence $H^4(G(2, 2)) \cong C(s)$ for some $s \in H^4(G(2, 2))$. It can be easily deduced from the description of the ring $\bar{H}^*(G(2, 2))$, given in [4; exercise 7.B] or [2], that there are unique $w_i \in \bar{H}^i(G(2, 2))$, $i = 1, 2$, with $\bar{H}^i(G(2, 2)) = C_2(w_i)$. It is not hard to compute that $w_2^2 = \varrho_2(s)$ and $w_1^4 = 0$. Now, it is clear that $F(G(2, 2)) \cong C_4((w_1, 0, 0)) \oplus C((0, 0, s))$ as Abelian groups. By definition, $(w_1, 0, 0) * (w_1, 0, 0) = (0, w_1^2, 0) = 2(w_1, 0, 0)$ and $(0, 0, s)^2 = (w_1, 0, 0) * (0, 0, s) = 0$.

Since $H^4(G(2, 2))$ has no 2-torsion, our statement follows from Theorem 2.1 (with $f = U^{-1}((w_1, 0, 0))$ and $g = U^{-1}((0, 0, s))$).

The next theorem, as well as the previous one, compares $KO\sim(X)$ with some ring created from the cohomology of the space X , but the very restrictive assumption on the non-existence of a 2-torsion is replaced by a weaker one. We introduce the following notation:

$$G(X) = \{(a, b, c, d) \in \bar{H}^1(X) \oplus \bar{H}^2(X) \oplus \bar{H}^4(X) \oplus H^4(X) ; \\ \varrho_4(d) = P(b) + i(c + a \cdot Sq^1(b))\}.$$

The operations \boxplus and $*$ are defined by

$$(a, b, c, d) \boxplus (x, y, z, w) = (a + x, b + ax + y, c + (Sq^1 b + ab)x + by + \\ + a(Sq^1 y + xy) + z, d + \delta a \delta x + w), \\ (a, b, c, d) * (x, y, z, w) = (0, ax, a^2(x^2 + y) + a(x^3 + xy) + \\ + (a^3 + ab)x + (a^2 + b)x^2, \delta a \delta x).$$

Finally, denote by V the map $(w_1, w_2, w_4, p_1): KO\sim(X) \rightarrow G(X)$. We prove the following theorem (compare [8; p. 178]):

2.2. Theorem. *The system $(G(X), \boxplus, *)$ forms a commutative ring with $(0, 0, 0, 0)$ as the zero element. The map $V: KO\sim(X) \rightarrow G(X)$ is a homomorphism of rings for each CW-complex X .*

If X is a CW-complex of a dimension ≤ 7 , the map V is an epimorphism. If, in addition, $H^4(X)$ has no 4-torsion, the map V is an isomorphism.

Remark. Using an improved form of the results of [8] we can establish the existence of the natural exact sequence

$$0 \rightarrow \text{Tor}(H^4(X); Z_2) \rightarrow \text{Tor}(H^4(X); Z_4) \rightarrow KO\sim(X) \xrightarrow{V} G(X) \rightarrow 0.$$

In the next example we compute anew the real K-ring of the real projective spaces $P^k = P^k(\mathbb{R})$ for $k \leq 7$ (see [1] or [3; 4.6]).

Example. The ring $KO\sim(P^k)$ is, for $k \leq 7$, isomorphic with $C_{j(k)}(\lambda)$, where $j(1) = 2$, $j(2) = j(3) = 4$ and $j(4) = \dots = j(7) = 8$. The element $\lambda \in KO\sim(P^k)$ corresponds to the canonical linear bundle over P^k and the multiplication is characterized by $\lambda^2 = -2\lambda$.

To prove the above statement, recall the existence of $w_1 \in \bar{H}^1(P^k)$ with $\bar{H}^*(P^k) \cong \cong Z_2[w_1]/(w_1^{k+1} = 0)$. Clearly, there exists $s \in H^4(P^k)$, $k \geq 4$, such that $H^4(P^k) \cong \cong C_2(s)$ and $\varrho_2(s) = w_1^4$. As $\varrho_{4,2}: H^4(P^k; Z_4) \rightarrow \bar{H}^4(P^k)$ is an isomorphism, the equation in the definition of $G(P^k)$ is equivalent simply with $\varrho_2(d) = b^2$ (see the remark following 1.2).

Now, using the above comments, we can easily verify that $G(P^k) \cong C_{j(k)}((w_1, 0, 0, 0))$ and that $(w_1, 0, 0, 0)^2 = -2(w_1, 0, 0, 0)$. Because $H^4(P^k) \cong Z_2$, the ring $G(P^k)$ is, by Theorem 2.2, isomorphic with $KO\sim(X)$.

3. PROOFS

This section contains the proof of Theorems 2.1 and 2.2. In the following lemma we verify the algebraic properties of the maps U and V .

3.1. Lemma. *The sets $F(X)$ and $G(X)$ with the operations \boxplus and $*$ form commutative rings. If X is a CW-complex, the maps $U: KO^{\sim}(X) \rightarrow F(X)$ and $V: KO^{\sim}(X) \rightarrow G(X)$ are homomorphisms of rings.*

Proof. It can be verified directly by using 1.1 and carrying out a long but elementary computation that the sets $F(X)$ and $G(X)$ really satisfy the axioms of commutative rings.

In order to prove the additivity and the multiplicativity of U and V , it is sufficient to do this for finite-dimensional CW-complexes only. Indeed, the additivity (multiplicativity) of the map U means that the natural transformation $A: KO^{\sim}(X) \times KO^{\sim}(X) \rightarrow F(X)$ defined by $A(x, y) = U(x) \boxplus U(y) - U(x + y)$ ($A(x, y) = U(x) * U(y) - U(xy)$) is zero. For each CW-complex we have the following commutative diagram (vertical maps are induced by the inclusions):

$$\begin{array}{ccc} KO^{\sim}(X) \times KO^{\sim}(X) & \xrightarrow{A} & F(X) \\ \downarrow & & \downarrow \cong \\ KO^{\sim}(X^5) \times KO^{\sim}(X^5) & \xrightarrow{A} & F(X^5) \end{array}$$

where $X^5 \rightarrow X$ is the 5-skeleton, i.e., a finite-dimensional CW-complex. By the diagram, the map A is zero if the induced map on the skeleton is. The argument for V is similar.

So, we can suppose that $\dim(X) < \infty$, hence the elements of $KO^{\sim}(X)$ can be viewed as the stable equivalence classes of vector fibrations over X . Such an equivalence class will be denoted by square brackets. The addition and multiplication are defined by

$$\begin{aligned} [\xi] + [\eta] &= [\xi \oplus \eta], \\ [\xi^m] \cdot [\eta^n] &= [\xi^m \otimes \eta^n] - [\varepsilon^m \otimes \eta^n] - [\xi^m \otimes \varepsilon^n], \end{aligned}$$

where ε^k denotes the k -dimensional trivial vector bundle over X . To verify the algebraic properties of U and V , we need only to express the characteristic classes of $\xi \oplus \eta$ and $[\xi] \cdot [\eta]$ in terms of those of ξ and η . By [4; § 4] we have

$$\begin{aligned} w_1(\xi \oplus \eta) &= w_1(\xi) + w_1(\eta), \quad w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)w_1(\eta) + w_2(\eta), \\ w_4(\xi \oplus \eta) &= w_4(\xi) + w_3(\xi)w_1(\eta) + w_2(\xi)w_2(\eta) + w_1(\xi)w_3(\eta) + w_4(\eta), \end{aligned}$$

where $w_3 = w_1w_2 + Sq^1w_2$ by 1.2.

Similarly, using the formula for the Chern classes of the Whitney sum [4; § 14] and the definition of the Pontrjagin class, we obtain

$$p_1(\xi \oplus \eta) = -c_2(\xi_C \oplus \eta_C) = p_1(\xi) - c_1(\xi_C)c_1(\eta_C) + p_1(\eta),$$

where $c_1(\xi_C)c_1(\eta_C) = \delta w_1(\xi)\delta w_1(\eta)$ [4; exercise 15D]. These formulas make the

additivity obvious. Writing formally

$$w(\xi) = \prod_{1 \leq i \leq m} (1 + x_i), \quad w(\eta) = \prod_{1 \leq j \leq n} (1 + y_j)$$

and using [4; exercise 7C] we can write

$$\begin{aligned} w([\xi] \cdot [\eta]) &= \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (1 + x_i + y_j) \left\{ \prod_{1 \leq f \leq m} (1 + x_f)^n \prod_{1 \leq g \leq n} (1 + y_g)^m \right\}^{-1} = \\ &= \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} ((1 + x_i + y_j) (1 + x_i)^{-1} (1 + y_j)^{-1}). \end{aligned}$$

The last polynomial can be expressed in terms of the elementary symmetric polynomials in the variables x_1, \dots, x_n and y_1, \dots, y_n , from which we shall deduce that

$$\begin{aligned} w_1([\xi] \cdot [\eta]) &= 0, \quad w_2([\xi] \cdot [\eta]) = w_1(\xi) w_1(\eta), \\ w_4([\xi] \cdot [\eta]) &= w_1^2(\xi) w_1^2(\eta) + w_1^2(\xi) w_2(\eta) + w_2(\xi) w_1^2(\eta) + \\ &+ w_1(\xi) (w_1^3(\eta) + w_1(\eta) w_2(\eta)) + (w_1^3(\xi) + w_1(\xi) w_2(\xi)) w_1(\eta). \end{aligned}$$

Again we omit this long but elementary computation. Using a similar formula for the Chern classes we can verify that $p_1([\xi] \cdot [\eta]) = \delta w_1(\xi) \delta w_1(\eta)$ and the lemma follows.

Let us consider the rings $\tilde{F}(X) = \{(a, b, c) \in F(X); a = 0\}$ and $\tilde{G}(X) = \{(a, b, c, d) \in G(X); a = 0\}$. Clearly, the maps U and V restrict to $\tilde{U}: [X; BSO] \rightarrow \tilde{F}(X)$ and $\tilde{V}: [X; BSO] \rightarrow \tilde{G}(X)$. The results [8; Theorem 1 and the note at the top of p. 178] can be reformulated as follows:

3.2. Proposition. *The maps \tilde{U} and \tilde{V} are homomorphisms of Abelian groups. If X is a CW-complex with $\dim(X) \leq 7$, our maps are epimorphisms. If, in addition, the group $H^4(X)$ has no 2-torsion, the map \tilde{U} is an isomorphism. If $H^4(X)$ has no 4-torsion, the map \tilde{V} is an isomorphism.*

Now, we are able to complete our proofs. We can form the commutative diagram of Abelian groups:

$$\begin{array}{ccccc} \tilde{F}(X) & \hookrightarrow & F(X) & \xrightarrow{q} & F(X)/\tilde{F}(X) \\ \tilde{v} \uparrow & & v \uparrow & & B \uparrow \\ [X; BSO] & \hookrightarrow & KO^{\sim}(X) & \xrightarrow{p} & KO^{\sim}(X)/[X; BSO] \end{array}$$

where \hookrightarrow are the natural inclusions, p, q are projections and the map B is defined by $B(p([\xi])) = q(U([\xi]))$.

Notice that there are identifications $KO^{\sim}(X)/[X; BSO] \cong [X; BO(1)]$ and $F(X)/\tilde{F}(X) \cong \bar{H}^1(X)$ such that $B(p([\xi])) = w_1(\xi)$. Because $BO(1) \cong K(1; \mathbb{Z}_2)$, the map B is an isomorphism. It is not hard to deduce from the above diagram that U is an epimorphism if \tilde{U} is, and that U is an isomorphism if \tilde{U} is. This concludes the proof of Theorem 2.1.

In order to prove Theorem 2.2 we can form the diagram analogous to the above

for the rings $G(X)$ and $\tilde{G}(X)$. Then Theorem 2.2 follows by the same argument as Theorem 2.1.

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