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*Czechoslovak Mathematical Journal*, Vol. 38 (1988), No. 3, 473–478

Persistent URL: <http://dml.cz/dmlcz/102243>

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ON CERTAIN INFINITESIMAL ISOMETRIES OF SURFACES

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(Received June 9, 1986)

In the following, I am going to characterize the surfaces of the Euclidean 3-space which admit non-trivial infinitesimal isometries preserving the mean curvature. In this sense, the paper is an infinitesimal version of the recent paper [1] of S. S. Chern; nevertheless, the results are quite different and many problems remain open.

We consider, in the Euclidean space  $E^3$ , a piece of a surface  $M$ . With each point  $m \in M$ , let us associate an orthonormal frame  $\{m; v_1, v_2, v_3\}$  such that  $v_3$  is a unit normal vector at  $m$ . Then we may write

$$(1) \quad \begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2, & dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3, & dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3, \\ & & dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \end{aligned}$$

with the usual integrability conditions.

Further, let us consider a 1-parametric family of surfaces  $M(t)$ ,  $t \in (-\varepsilon, \varepsilon)$ , such that  $M(0) = M$ ; for each  $t$ , let an isometry  $\iota_t: M \rightarrow M(t)$  be given. On  $M(t)$ , take the field of orthonormal frames  $\{m(t); v_1(t), v_2(t), v_3(t)\}$  with  $m(t) = \iota_t(m)$ ,  $v_1(t) = d\iota_t(v_1)$ ,  $v_2 = d\iota_t(v_2)$ . Then

$$(2) \quad \begin{aligned} dm(t) &= \omega^1 v_1(t) + \omega^2 v_2(t), & dv_1(t) &= \omega_1^2(t) v_2(t) + \omega_1^3(t) v_3(t), \\ dv_2(t) &= -\omega_1^2(t) v_1(t) + \omega_2^3(t) v_3(t), & dv_3(t) &= -\omega_1^3(t) v_1(t) - \omega_2^3(t) v_2(t) \end{aligned}$$

with  $\omega_i^j(0) = \omega_i^j$  and the integrability conditions

$$(3) \quad \begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega_1^2(t), \\ d\omega^2 &= \omega^1 \wedge \omega_1^2(t), & \omega^1 \wedge \omega_1^3(t) + \omega^2 \wedge \omega_2^3(t) &= 0, \\ d\omega_1^2(t) &= -\omega_1^3(t) \wedge \omega_2^3(t), & d\omega_1^3(t) &= \omega_1^2(t) \wedge \omega_2^3(t), \\ d\omega_2^3(t) &= -\omega_1^2(t) \wedge \omega_1^3(t). \end{aligned}$$

Let

$$(4) \quad \varphi_i^j := (d\omega_i^j(t)/dt)_{t=0}.$$

From  $(3_{1,2})$ , we get  $\omega^2 \wedge \varphi_1^2 = \omega^1 \wedge \varphi_1^2 = 0$ , i.e.,

$$(5) \quad \varphi_1^2 = 0.$$

This together with (3<sub>3-6</sub>) yield

$$(6) \quad \omega^1 \wedge \varphi_1^3 + \omega^2 \wedge \varphi_2^3 = 0, \quad \omega_1^3 \wedge \varphi_2^3 + \varphi_1^3 \wedge \omega_2^3 = 0,$$

$$(7) \quad d\varphi_1^3 = \omega_2^1 \wedge \varphi_2^3, \quad d\varphi_2^3 = -\omega_1^2 \wedge \varphi_1^3.$$

Given a surface  $M$ , the couple  $\{\varphi_1^3, \varphi_2^3\}$  of 1-forms on  $M$  satisfying (6) + (7) is called the *infinitesimal isometry*  $\Phi$  of  $M$ .

The *second form* of  $M(t)$  is given by

$$(8) \quad \Pi(t) = \omega^1 \omega_1^3(t) + \omega^2 \omega_2^3(t);$$

its *Gauss curvature*  $K(t)$  and its *mean curvature*  $H(t)$  by

$$(9) \quad K(t) \omega^1 \wedge \omega^2 = \omega_1^3(t) \wedge \omega_2^3(t),$$

$$2H(t) \omega^1 \wedge \omega^2 = \omega_1^3(t) \wedge \omega^2 + \omega^1 \wedge \omega_2^3(t),$$

resp. Let us define the *variations*

$$(10) \quad \delta\Pi := (d\Pi(t)/dt)_{t=0}, \quad \delta K := (dK(t)/dt)_{t=0}, \quad \delta H := (dH(t)/dt)_{t=0};$$

we get

$$(11) \quad \delta\Pi = \omega^1 \varphi_1^3 + \omega^2 \wedge \varphi_2^3,$$

$$(12) \quad \delta K = 0, \quad 2\delta H \cdot \omega^1 \wedge \omega^2 = \varphi_1^3 \wedge \omega^2 + \omega^1 \wedge \varphi_2^3.$$

The equation (12<sub>1</sub>) is the consequence of (6<sub>2</sub>); it is the infinitesimal version of the *theorema egregium*.

Consider the surface  $M$ . The equation (3<sub>3</sub>) for  $t = 0$  yields the existence of functions  $a, b, c$  such that

$$(13) \quad \omega_1^3 = a\omega^1 + b\omega^2, \quad \omega_2^3 = b\omega^1 + c\omega^2;$$

we have, again on  $M$ ,

$$(14) \quad ds^2 = (\omega^1)^2 + (\omega^2)^2, \quad \Pi = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2;$$

$$(15) \quad 2H = a + c, \quad K = ac - b^2.$$

It is easy to see that the *lines of curvature* of  $M$  are given by

$$(16) \quad b(\omega^1)^2 + (c - a)\omega^1\omega^2 - b(\omega^2)^2 = 0.$$

The *Euler function*  $E$  on  $M$  be defined by  $E := H^2 - K$ , i.e.,

$$(17) \quad 4E = (a - c)^2 + 4b^2.$$

A point  $m \in M$  is *umbilical* if and only if  $E(m) = 0$ .

Let  $f: M \rightarrow \mathbb{R}$  be a function. Its first *covariant derivatives*  $f_i$  with respect to the coframes  $\{\omega^1, \omega^2\}$  are defined by

$$(18) \quad df = f_1\omega^1 + f_2\omega^2.$$

From this,

$$(19) \quad (df_1 - f_2\omega_1^2) \wedge \omega^1 + (df_2 + f_1\omega_1^2) \wedge \omega^2 = 0,$$

and we get the existence of the second covariant derivatives  $f_{ij} = f_{ji}$  such that

$$(20) \quad df_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2, \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2.$$

**Theorem.** Let  $M \subset E^3$  be a surface without umbilical points;  $M$  admits a non-trivial infinitesimal isometry  $\Phi$  with

$$(21) \quad \delta H = 0$$

if and only if

$$(22) \quad 2b(H_1E_1 - H_{11}E) + (c - a)(H_2E_1 + H_1E_2 - 2H_{12}E) - 2b(H_2E_2 - H_{22}E) = 0.$$

If a general surface  $M \subset E^3$  admits a non-trivial infinitesimal isometry  $\Phi$  satisfying (21), we have (22). The surfaces admitting non-trivial infinitesimal isometries  $\Phi$  with (21) depend on 4 functions of 1 variable in the sense of E. Cartan.

**Proof.** From (6) and (13), we get the existence of functions  $R_1, R_2, R_3$  on  $M$  such that

$$(23) \quad \varphi_1^3 = R_1\omega^1 + R_2\omega^2, \quad \varphi_2^3 = R_2\omega^1 + R_3\omega^2,$$

$$(24) \quad cR_1 - 2bR_2 + aR_3 = 0.$$

The condition (21) is equivalent, see (12<sub>2</sub>), to

$$(25) \quad R_1 + R_3 = 0.$$

First of all, let us suppose that  $M$  contains no umbilical points, i.e.,  $E \neq 0$  on  $M$ . From (24) and (25), we get  $(c - a)R_1 = 2bR_2$ , and  $E \neq 0$  implies the existence of a function  $R$  on  $M$  such that  $R_1 = 2bR$ ,  $R_2 = (c - a)R$ . Thus (23) turn out to be

$$(26) \quad \varphi_1^3 = R\{2b\omega^1 + (c - a)\omega^2\}, \quad \varphi_2^3 = R\{(c - a)\omega^1 - 2b\omega^2\}.$$

Let us mention that, see (11),

$$(27) \quad \delta II = 2R\{b(\omega^1)^2 + (c - a)\omega^1\omega^2 - b(\omega^2)^2\}.$$

The couple  $M + \Phi$  is thus given by (13) + (26). The differential consequences are

$$(28) \quad (da - 2b\omega_1^2) \wedge \omega^1 + (db + (a - c)\omega_1^2) \wedge \omega^2 = 0,$$

$$(db + (a - c)\omega_1^2) \wedge \omega^1 + (dc + 2b\omega_1^2) \wedge \omega^2 = 0,$$

$$(29) \quad dR \wedge \{2b\omega^1 + (c - a)\omega^2\} + R\{2(db + (a - c)\omega_1^2) \wedge \omega^1 + (dc - da + 4b\omega_1^2) \wedge \omega^2\} = 0,$$

$$dR \wedge \{(c - a)\omega^1 - 2b\omega^2\} + R\{(dc - da + 4b\omega_1^2) \wedge \omega^1 - 2(db + (a - c)\omega_1^2) \wedge \omega^2\} = 0.$$

Using Cartan's lemma, we get the existence of functions  $\alpha, \beta, \gamma, \delta, r_1, r_2$  such that

$$(30) \quad da - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2, \quad db + (a - c)\omega_1^2 = \beta\omega^1 + \gamma\omega^2,$$

$$dc + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2,$$

$$(31) \quad dR = r_1\omega^1 + r_2\omega^2,$$

and satisfying

$$(32) \quad (c - a)r_1 - 2br_2 = R(\alpha + \gamma), \quad 2br_1 + (c - a)r_2 = -R(\beta + \delta).$$

It is elementary to see that the equations (28) + (29) are linearly independent. Thus the system (13) + (26) is in involution and its solutions depend on 4 functions of 1 variable.

Let us rewrite (32). From  $(9)_{t=0}$ , we see that

$$(33) \quad 2H = a + c,$$

with the notation (18), the equations  $(30_{1,3})$  imply  $2H_1 = \alpha + \gamma$ ,  $2H_2 = \beta + \delta$ , i.e., (32) may be written as

$$(34) \quad (c - a)r_1 - 2br_2 = 2H_1R, \quad 2br_1 + (c - a)r_2 = -2H_2R.$$

Because of  $E \neq 0$ , we may evaluate  $r_1, r_2$  from them, and (31) turns out to be

$$(35) \quad dR = \frac{1}{2}R\omega$$

with

$$(36) \quad \omega := E^{-1}[\{(c - a)H_1 - 2bH_2\}\omega^1 - \{2bH_1 + (c - a)H_2\}\omega^2].$$

The integrability condition of (35) being  $Rd\omega = 0$ , there exists a non-trivial function  $R$  on  $M$  if and only if  $d\omega = 0$ . By a direct calculation, this is exactly (22).

Now, let us drop the supposition  $E \neq 0$  on  $M$ . Because of (25), (23) may be written as

$$(37) \quad \varphi_1^3 = R_1\omega^1 + R_2\omega^2, \quad \varphi_2^3 = R_2\omega^1 - R_1\omega^2;$$

the condition (24) being

$$(38) \quad (c - a)R_1 - 2bR_2 = 0.$$

Of course

$$(39) \quad \delta\Pi = R_1(\omega^1)^2 + 2R_2\omega^1\omega^2 - R_1(\omega^2)^2.$$

Our problem is thus given by (13) + (37) + (38). The differential consequences of (13) are (28), and from (37) we get

$$(40) \quad (dR_1 - 2R_2\omega_1^2) \wedge \omega^1 + (dR_2 + 2R_1\omega_1^2) \wedge \omega^2 = 0, \\ (dR_2 + 2R_1\omega_1^2) \wedge \omega^1 - (dR_1 - 2R_2\omega_1^2) \wedge \omega^2 = 0.$$

Using Cartan's lemma, we get (30) from (28) and, from (40), the existence of functions  $S_1, S_2$  such that

$$(41) \quad dR_1 - 2R_2\omega_1^2 = S_1\omega^1 + S_2\omega^2, \quad dR_2 + 2R_1\omega_1^2 = S_2\omega^1 - S_1\omega^2.$$

The differential consequences of (38) are then

$$(42) \quad (c - a)S_1 - 2bS_2 + (\gamma - \alpha)R_1 - 2\beta R_2 = 0, \\ (c - a)S_2 + 2bS_1 + (\delta - \beta)R_1 - 2\gamma R_2 = 0.$$

The exterior differentiation of (30) + (41) yields

$$(43) \quad \begin{aligned} (d\alpha - 3\beta\omega_1^2) \wedge \omega^1 + (d\beta + (\alpha - 2\gamma)\omega_1^2) \wedge \omega^2 &= 2Kb\omega^1 \wedge \omega^2, \\ (d\beta + (\alpha - 2\gamma)\omega_1^2) \wedge \omega^1 + (d\gamma + (2\beta - \delta)\omega_1^2) \wedge \omega^2 &= K(c - a)\omega^1 \wedge \omega^2, \\ (d\gamma + (2\beta - \delta)\omega_1^2) \wedge \omega^1 + (d\delta + 3\gamma\omega_1^2) \wedge \omega^2 &= -2Kb\omega^1 \wedge \omega^2, \\ (dS_1 - 3S_2\omega_1^2) \wedge \omega^1 + (dS_2 + 3S_1\omega_1^2) \wedge \omega^2 &= 2KR_2\omega^1 \wedge \omega^2, \\ (dS_2 + 3S_1\omega_1^2) \wedge \omega^1 - (dS_1 - 3S_2\omega_1^2) \wedge \omega^2 &= -2KR_1\omega^1 \wedge \omega^2, \end{aligned}$$

and we get the existence of functions  $A, \dots, E, T_1, T_2$  such that

$$(44) \quad \begin{aligned} d\alpha - 3\beta\omega_1^2 &= A\omega^1 + (B - bK)\omega^2, \\ d\beta + (\alpha - 2\gamma)\omega_1^2 &= (B + bK)\omega^1 + (C + aK)\omega^2, \\ d\gamma + (2\beta - \delta)\omega_1^2 &= (C + cK)\omega^1 + (D + bK)\omega^2, \\ d\delta + 3\gamma\omega_1^2 &= (D - bK)\omega^1 + E\omega^2, \\ dS_1 - 3S_2\omega_1^2 &= (T_1 + KR_1)\omega^1 + (T_2 - KR_2)\omega^2, \\ dS_2 + 3S_1\omega_1^2 &= (T_2 + KR_2)\omega^1 - (T_1 - KR_1)\omega^2. \end{aligned}$$

Using these, we get, from (42),

$$(45) \quad \begin{aligned} (c - a)T_1 - 2bT_2 + 2(\gamma - \alpha)S_1 - 4\beta S_2 + \\ + (C - A + 2cK - aK)R_1 - 2(B + 2bK)R_2 &= 0, \\ 2bT_1 + (c - a)T_2 + (\beta + \delta)S_1 - (\alpha + \gamma)S_2 + \\ + (D - B)R_1 - (2C + aK + cK)R_2 &= 0, \\ -(c - a)T_1 + 2bT_2 + 4\gamma S_1 + 2(\delta - \beta)S_2 + \\ + (E - C - 2aK + cK)R_1 - 2(D + 2bK)R_2 &= 0. \end{aligned}$$

From (45<sub>1,3</sub>),

$$(46) \quad \begin{aligned} 2(3\gamma - \alpha)S_1 + 2(\delta - 3\beta)S_2 + \\ + (E - A + 3cK - 3aK)R_1 - 2(B + D + 4bK)R_2 &= 0. \end{aligned}$$

Consider the system (46) + (42) + (38) for  $S_1, S_2, R_1, R_2$ . If  $\Phi$  is non-trivial, this system must have a non-trivial solution — see (39) — and its determinant  $\Delta$  must vanish. Let us calculate  $\Delta$  at a point  $m_0 \in M$ . Because of  $\Pi = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2$  — see (8) — we may choose the frames in such a way that  $b(m_0) = 0$ . Then, at  $m_0$ ,

$$(47) \quad \begin{aligned} \Delta(m_0) &= 2(c - a) \begin{vmatrix} 6\gamma - 2\alpha & 2\delta - 6\beta & B + D \\ c - a & 0 & \beta \\ 0 & c - a & \gamma \end{vmatrix} = \\ &= 2(c - a) \{ (c - a)^2 (B + D) - 2(c - a)(\gamma\delta - \alpha\beta) \}. \end{aligned}$$

Again at  $m_0 \in M$ , we have

$$(48) \quad \begin{aligned} 2H_1 = \alpha + \gamma, \quad 2H_2 = \beta + \delta, \quad 2H_{12} = B + D, \\ 4E = (c - a)^2, \quad 4E_1 = 2(c - a)(\gamma - \alpha), \quad 4E_2 = 2(c - a)(\delta - \beta). \end{aligned}$$

Thus

$$(49) \quad 2EH_{12} - H_2E_1 - H_1E_2 = \frac{1}{4}\{(c-a)^2(B+D) - 2(c-a)(\gamma\delta - \alpha\beta)\},$$

and we get, at  $m_0 \in M$ ,

$$(50) \quad \Delta(m_0) = 8(c-a)(2EH_{12} - H_2E_1 - H_1E_2).$$

Thus  $\Delta(m_0) = 0$  is equivalent to (22) for  $b = 0$ . The left-hand side of (22) being an invariant of our surface, we are finished.

**Remarks.** Evidently, all surfaces of revolution satisfy (22). Indeed, let  $M$  be a surface of revolution. The frames of  $M$  be chosen in such a way that  $v_1$  be tangent to the circles of  $M$ . On each of these circles,  $H = \text{const.}$  and  $E = \text{const.}$ ; further,  $b = 0$  on  $M$ . Thus  $E_1 = H_1 = 0$  on  $M$ , and (22) is satisfied.

For  $E \neq 0$ , (22) may be written as

$$(51) \quad 2b(E^{-1}H_1)_1 + (c-a)((E^{-1}H_2)_1 + (E^{-1}H_1)_2) - 2b(E^{-1}H_2)_2 = 0.$$

Using the tensor notation, we may write (22) as follows: Let  $c_{ij} dx^i dx^j = 0$  be the lines of curvature on  $M$ ; (22) is then

$$(52) \quad c^{ij}H_{;i}E_{;j} = Ec^{ij}H_{;ij}.$$

#### *Reference*

- [1] *S. S. Chern: Deformation of Surfaces Preserving Principal Curvatures.* In: *Differential Geometry and Complex Analysis*, pp. 156–163; Springer-Verlag, Berlin—Heidelberg, 1985.

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