EQUIVALENCE PROBLEM FOR LAGRANGIANS

ALOIS ŠVEC, Brno

(Received June 9, 1986)

The equivalence problem for Lagrangians was solved by E. Cartan by his own methods. Recently, a solution in the same spirit was presented by R. B. Gardner, see [1]. Here, I present another approach to the same problem which seems to me to be simpler and more effective. I restrict myself just to the cases of order one and two; in order two, I get as the special case Lagrangians of the form (2.17).

I am grateful to Prof. I. Kolář for several interesting discussions.

1. FIRST ORDER LAGRANGIANS

Let a fibred manifold $\tilde{\pi}: \tilde{M} \to \tilde{N}$ be given such that $\dim \tilde{N} = 1$, $\dim \tilde{M} = 2$; let $J^1(\tilde{M})$ be its first jet prolongation and $\tilde{m} \in J^1(\tilde{M})$. On $J^1(\tilde{M})$, let us choose local fibre coordinates (ξ, η, ζ) such that $\tilde{\pi}_{1,0}(\xi, \eta, \zeta) = (\xi, \eta)$, $\tilde{\pi}(\xi, \eta) = \xi$ and $\tilde{m} = (0, 0, 0)$.

Further, let a fibred manifold $\pi: M \to N$ be given such that $\dim N = 1$, $\dim M = 2$; let $J^1(M)$ be its first jet prolongation. On $J^1(M)$, let us introduce fibre coordinates $(x, y, z) \equiv (x, y, \dot{y})$ in a similar way. Let $f_0: \tilde{M} \to M$ be a (local) bundle isomorphism, $f \equiv j^1(f_0): J^1(\tilde{M}) \to J^1(M)$ its prolongation, and let $m \equiv f(\tilde{m}) = (x_0, y_0, z_0) \in J^1(M)$. The (local) isomorphism f is given by

$$x = x(\xi), \quad y = y(\xi, \eta), \quad z = \left(\frac{dx}{d\xi}\right)^{-1} \left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} \zeta\right), \tag{1.1}$$

and we have

$$x_0 = x(0), \quad y_0 = y(0, 0), \quad z_0 = \left(\frac{dx(0)}{d\xi}\right)^{-1} \frac{\partial y(0, 0)}{\partial \xi}. \tag{1.2}$$

It is easy to check that

$$df\left(\frac{\partial}{\partial \zeta}\bigg|_{\tilde{m}}\right) = \left(\frac{dx(0)}{d\xi}\right)^{-1} \frac{\partial y(0, 0)}{\partial \eta} \frac{\partial}{\partial z}\bigg|_m, \tag{1.3}$$

$$df\left(\frac{\partial}{\partial \eta}\bigg|_{\tilde{m}}\right) = \frac{\partial y(0, 0)}{\partial \eta} \frac{\partial}{\partial y}\bigg|_m + \left(\frac{dx(0)}{d\xi}\right)^{-1} \frac{\partial^2 y(0, 0)}{\partial \xi\, \partial \eta} \frac{\partial}{\partial z}\bigg|_m,$$

$$\mathrm{d}f\left(\left.\frac{\partial}{\partial\xi}\right|_{\tilde m}\right) = \frac{\mathrm{d}x(0)}{\mathrm{d}\xi}\left.\frac{\partial}{\partial x}\right|_m + \frac{\partial y(0,0)}{\partial\xi}\left.\frac{\partial}{\partial y}\right|_m +$$

$$+ \left\{-\left(\frac{\mathrm{d}x(0)}{\mathrm{d}\xi}\right)^{-2}\frac{\mathrm{d}^2x(0)}{\mathrm{d}\xi^2}\frac{\partial y(0,0)}{\partial\xi} + \left(\frac{\mathrm{d}x(0)}{\mathrm{d}\xi}\right)^{-1}\frac{\partial^2 y(0,0)}{\partial\xi^2}\right\}\left.\frac{\partial}{\partial z}\right|_m .$$

Let us write

$$D := \frac{\mathrm{d}x(0)}{\mathrm{d}\xi}, \quad C := \frac{\partial y(0,0)}{\partial\eta}, \quad A := D^{-1}\frac{\partial^2 y(0,0)}{\partial\xi\,\partial\eta}, \tag{1.4}$$

$$B := -D^{-2}\frac{\mathrm{d}^2x(0)}{\mathrm{d}\xi^2}\frac{\partial y(0,0)}{\partial\xi} + D^{-1}\frac{\partial^2 y(0,0)}{\partial\xi^2}.$$

Notice that

$$DC \neq 0, \tag{1.5}$$

f being a (local) isomorphism; from (1.2_3),

$$\frac{\partial y(0,0)}{\partial\xi} = z_0 D. \tag{1.6}$$

Thus the differential $\mathrm{d}f\colon T_{\tilde m}(J^1(\tilde M)) \to T_m(J^1(M))$ maps the vectors $\partial/\partial\zeta$, $\partial/\partial\eta$, $\partial/\partial\xi$ at $\tilde m$ to the vectors

$$v_1 = CD^{-1}\frac{\partial}{\partial z}, \quad v_2 = C\frac{\partial}{\partial y} + A\frac{\partial}{\partial z}, \quad v_3 = D\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) + B\frac{\partial}{\partial z} \tag{1.7}$$

at m. Each triple of vectors $\{v_1, v_2, v_3\}$, $v_i \in T_m(J^1(M))$, of the form (1.7) is called a *frame* of $J^1(M)$ at m.

On $J^1(M)$, let a Lagrangian

$$\lambda = f(x, y, z)\,\mathrm{d}x, \quad ff_z \neq 0 \tag{1.8}$$

be given; here, $f_z = \partial f/\partial z$ etc. A frame $\{v_1, v_2, v_3\}$ of $J^1(M)$ at m will be called a *λ-frame* if

$$\lambda(v_3) = 1 \tag{1.9}$$

at m; from (1.7_3), we get $f(x_0, y_0, z_0)\,D = 1$. Now, let us consider, on $J^1(M)$, a field of λ-frames

$$v_1 = Cf\frac{\partial}{\partial z}, \quad v_2 = C\frac{\partial}{\partial y} + A\frac{\partial}{\partial z}, \quad v_3 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) + B\frac{\partial}{\partial z}, \tag{1.10}$$

A, B and C being functions on $J^1(M)$. Let functions $c^i_{jk}\colon J^1(M) \to \mathbb{R}$ be defined by

$$[v_i, v_j] = \sum_{k=1}^{3} c^k_{ij} v_k; \quad i, j = 1, 2, 3; \tag{1.11}$$

$[v_i, v_j]$ being the Lie bracket of the vector fields v_i, v_j. Let us try to exhibit special fields of λ-frames by imposing suitable conditions on the functions c^k_{ij}.

We get $[v_1, v_3] = c^1_{13}v_1 + v_2 - f_zCv_3$, and the condition

$$c^3_{13} = -1 \tag{1.12}$$

implies

(1.13) $$C = f_z^{-1}\,.$$

Further,

$$[v_2, v_3] = c_{23}^1 v_1 + \{f^{-1}f_z A + f^{-1}f_z^{-1}(f_{xz} + zf_{yz}) + f_z^{-1}f_{zz}B\}\, v_2 - \\ - f^{-1}(f_y f_z^{-1} + f_z A)\, v_3\,.$$

The condition

(1.14) $$c_{23}^3 = 0$$

yields

(1.15) $$A = -f_y f_z^{-2}$$

and

(1.16) $$c_{23}^2 = f^{-1}f_z^{-1}(f_{xz} + zf_{yz} - f_y) + f_z^{-1}f_{zz}B\,.$$

Proposition 1.1. *Let the Lagrangian λ (1.8) satisfy $f_{zz} \neq 0$. Then there is, on $J^1(M)$, exactly one field of λ-frames $\{v_1, v_2, v_3\}$ such that we have, in (1.11), $c_{13}^3 = -1$, $c_{23}^3 = c_{23}^2 = 0$ (this implying $c_{13}^2 = 1$). This field is given by*

(1.17) $$v_1 = ff_z^{-1}\frac{\partial}{\partial z}, \quad v_2 = f_z^{-1}\frac{\partial}{\partial y} - f_y f_z^{-2}\frac{\partial}{\partial z},$$

$$v_3 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) - f^{-1}f_{zz}^{-1}(f_{xz} + zf_{yz} - f_y)\frac{\partial}{\partial z}\,.$$

Now, let $f_{zz} = 0$, i.e.,

(1.18) $$f(x, y, z) = a(x, y) + b(x, y)\, z\,.$$

Let us suppose (1.12) and (1.14), i.e., (1.13) and (1.15). Then

(1.19) $$[v_2, v_3] = c_{23}^1 v_1 + Fv_2 \quad \text{with}$$

$$F = f^{-1}f_z^{-1}(f_{xz} + zf_{yz} - f_y) = f^{-1}b^{-1}(b_x - a_y)\,.$$

Further, $[v_1, v_3] = c_{13}^1 v_1 + v_2 - v_3$, i.e.,

(1.20) $$[v_3, [v_2, v_3]] = c_{323}^1 v_1 + (v_3 F - F^2 - c_{23}^1)\, v_2 + c_{23}^1 v_3\,,$$

the functions $c_{jkl}^i\colon J^1(M) \to \mathbb{R}$ being given by

(1.21) $$[v_j, [v_k, v_l]] = \sum_{i=1}^{3} c_{jkl}^i v_i\,.$$

The condition

(1.22) $$c_{23}^1 + c_{323}^2 = 0$$

reads

$$v_3 F - F^2 = f^{-1}(F_x + zF_y) + BF_z - F^2 = 0\,,$$

i.e.,

(1.23) $$(b_x - a_y)\, B = f\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right)(f^{-1}b^{-1}(b_x - a_y)) - f^{-1}b^{-2}(b_x - a_y)^2\,.$$

Proposition 1.2. *Let the Lagrangian λ be of the form (1.18) with $b_x \neq a_y$. Then* *there is, on* $J^1(M)$, *exactly one field of* λ*-frames* $\{v_1, v_2, v_3\}$ *such that we have* $c^3_{13} = -1$, $c^3_{23} = c^1_{23} + c^2_{323} = 0$; *these* λ*-frames are given by* (1.10) *with* (1.13), (1.15) *and* (1.23).

Finally, let us suppose (1.18) with $b_x = a_y$. Then there is a function c such that we may write

$$(1.24)\qquad f(x, y, z) = c_x + c_y z\,, \quad c = c(x, y)\,.$$

On $J^1(M)$, introduce new fibre coordinates (x, Y, Z) with $Y = c(x, y)$. Then $Z = c_x + c_y z$, and we have the following

Proposition 1.3. *Let the Lagrangian* λ *be of the form* (1.18) *with* $b_x = a_y$. *Then there are, on* $J^1(M)$, *new bundle coordinates* (x, Y, Z) *such that* $\lambda = Z\,\mathrm{d}x$.

Let us remark that the coframe dual to the frame

$$(1.25)\ v_1 = ff_z^{-1}\frac{\partial}{\partial z}, \quad v_2 = f_z^{-1}\frac{\partial}{\partial y} - f_y f_z^{-2}\frac{\partial}{\partial z}, \quad v_3 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) + B\frac{\partial}{\partial z}$$

is given by the 1-forms

$$(1.26)\qquad \omega^1 = -f^{-1}f_z B\,\mathrm{d}x + f^{-1}f_y(\mathrm{d}y - z\,\mathrm{d}x) + f^{-1}f_z\,\mathrm{d}z\,,$$
$$\omega^2 = f_z(\mathrm{d}y - z\,\mathrm{d}x)\,, \quad \omega^3 = \lambda = f\,\mathrm{d}x\,.$$

On $J^1(M)$, let a Lagrangian λ (1.8) with $f_{zz} \neq 0$ be given. The corresponding Euler equation is

$$(1.27)\qquad E(\lambda) \equiv f_y - \frac{\mathrm{d}}{\mathrm{d}x} f_z = f_y - f_{xz} - f_{yz}y' - f_{zz}y'' = 0\,.$$

Let γ be a critical section of λ given by $y = y(x)$. Then $j^1\gamma$ is given by $y = y(x)$, $z = y'(x)$, and its tangent vector at each point is

$$(1.28)\qquad \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f_{zz}^{-1}(f_y - f_{xz} - f_{yz}z)\frac{\partial}{\partial z} = fv_3\,,$$

v_3 being exactly the vector (1.17_3).

2. SECOND ORDER LAGRANGIANS

Let $\tilde{\pi}: \tilde{M} \to \tilde{N}$ be a fibred manifold with $\dim \tilde{N} = 1$, $\dim \tilde{M} = 2$. On $J^2(\tilde{M})$, let a point $\tilde{m}$ and fibre coordinates (ξ, η, ζ, τ) be given such that $\tilde{m} = (0, 0, 0, 0)$. Analogously, let $\pi: M \to N$ be a fibred manifold with fibre coordinates $(x, y, z, t) \equiv (x, y, \dot{y}, \ddot{y})$ on $J^2(M)$. The prolongation $f := j^2(f_0)$ of a (local) bundle isomorphism $f_0: \tilde{M} \to M$ is given by (1.1) and

$$(2.1)\quad t = -\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3}\frac{\mathrm{d}^2x}{\mathrm{d}\xi^2}\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\zeta\right) + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2}\left(\frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi\,\partial \eta}\zeta + \frac{\partial^2 y}{\partial \eta^2}\zeta^2 + \frac{\partial y}{\partial \eta}\tau\right).$$

A set of vectors $v_1, v_2, v_3, v_4 \in T_m(J^2(M))$ is called a frame of $J^2(M)$ at $m =$

$= (x, y, z, t)$ if they are images of the vectors $\partial/\partial\tau$, $\partial/\partial\zeta$, $\partial/\partial\eta$, $\partial/\partial\xi \in T_{\tilde{m}}(J^2(\tilde{M}))$ under the mapping $\mathrm{d}f\colon T_{\tilde{m}}(J^2(\tilde{M})) \to T_m(J^2(M))$, $f_0\colon \tilde{M} \to M$ being an arbitrary (local) bundle isomorphism such that $f(\tilde{m}) = m$. It is easy to check

$$v_1 = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\partial y}{\partial \eta} \frac{\partial}{\partial t}, \tag{2.2}$$

$$v_2 = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1} \frac{\partial y}{\partial \eta}\frac{\partial}{\partial z} + \left\{-\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial y}{\partial \eta} + 2\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\partial^2 y}{\partial\xi\,\partial\eta}\right\}\frac{\partial}{\partial t},$$

$$v_3 = \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1} \frac{\partial^2 y}{\partial\xi\,\partial\eta}\frac{\partial}{\partial z} + \left\{-\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial^2 y}{\partial\xi\,\partial\eta} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\partial^3 y}{\partial\xi^2\,\partial\eta}\right\}\frac{\partial}{\partial t},$$

$$v_4 = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} + \left\{-\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial y}{\partial \xi} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1} \frac{\partial^2 y}{\partial\xi^2}\right\}\frac{\partial}{\partial z} +$$

$$+ \left\{3\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-4} \left(\frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\right)^2 \frac{\partial y}{\partial \xi} - \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^3 x}{\mathrm{d}\xi^3}\frac{\partial y}{\partial \xi} - \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial^2 y}{\partial \xi^2} - \right.$$

$$\left. - 2\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial^2 y}{\partial \xi^2} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\partial^3 y}{\partial \xi^3}\right\}\frac{\partial}{\partial t},$$

all the derivatives being calculated at $\tilde{m} = (0,0,0,0)$. From (1.1) and (2.1), we obtain

$$z = \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-1} \frac{\partial y}{\partial \xi}, \quad t = -\left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-3} \frac{\mathrm{d}^2 x}{\mathrm{d}\xi^2}\frac{\partial y}{\partial \xi} + \left(\frac{\mathrm{d}x}{\mathrm{d}\xi}\right)^{-2} \frac{\partial^2 y}{\partial \xi^2}, \tag{2.3}$$

again at $\tilde{m}$. From (2.3), we calculate $\partial y(0,0)/\partial\xi$, $\partial^2 y(0,0)/\partial\xi^2$ and substitute them into (2.2). Thus we see that the most general frame at $m \in J^2(M)$ is given by

$$v_1 = F^{-2}A\frac{\partial}{\partial t}, \quad v_2 = F^{-1}A\frac{\partial}{\partial z} + B\frac{\partial}{\partial t}, \quad v_3 = A\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D\frac{\partial}{\partial t}, \tag{2.4}$$

$$v_4 = F\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right) + E\frac{\partial}{\partial t},$$

$A, \ldots, F$ being arbitrary real numbers.

On $J^2(M)$, let a Lagrangian

$$\lambda = f(x, y, z, t)\,\mathrm{d}x\,, \quad ff_t \neq 0 \tag{2.5}$$

be given. A frame $\{v_1, v_2, v_3, v_4\}$ is called a λ-frame if $\lambda(v_4) = 1$. The most general λ-frame is

$$v_1 = f^2 A\frac{\partial}{\partial t}, \quad v_2 = fA\frac{\partial}{\partial z} + B\frac{\partial}{\partial t}, \quad v_3 = A\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} + D\frac{\partial}{\partial t}, \tag{2.6}$$

$$v_4 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right) + E\frac{\partial}{\partial t}; \quad A, \ldots, E \in \mathbb{R}\,.$$

Consider a field of λ-frames (2.6) on $J^2(M)$, $A, \ldots, E$ being real-valued functions on $J^2(M)$. We have

(2.7) $$\mathrm{d}\lambda = f_y\,\mathrm{d}y \wedge \mathrm{d}x + f_z\,\mathrm{d}z \wedge \mathrm{d}x + f_t\,\mathrm{d}t \wedge \mathrm{d}x$$

and

(2.8) $$\mathrm{d}\lambda(v_1, v_4) = ff_tA\,,\quad \mathrm{d}\lambda(v_2, v_4) = f_zA + f^{-1}f_tB\,,$$
$$\mathrm{d}\lambda(v_3, v_4) = f^{-1}(f_yA + f_zC + f_tD)\,.$$

Lemma. *There are fields of λ-frames satisfying*

(2.9) $$\mathrm{d}\lambda(v_1, v_4) = 1\,,\quad \mathrm{d}\lambda(v_2, v_4) = \mathrm{d}\lambda(v_3, v_4) = 0\,.$$

A general field of λ-frames satisfying (2.9) *is given by*

(2.10) $$v_1 = ff_t^{-1}\frac{\partial}{\partial t},\quad v_2 = f_t^{-1}\frac{\partial}{\partial z} - f_zf_t^{-2}\frac{\partial}{\partial t},$$
$$v_3 = f^{-1}f_t^{-1}\frac{\partial}{\partial y} - f^{-1}f_yf_t^{-2}\frac{\partial}{\partial t} + Cf_t\left(f_t^{-1}\frac{\partial}{\partial z} - f_zf_t^{-2}\frac{\partial}{\partial t}\right),$$
$$v_4 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right) + E\frac{\partial}{\partial t}\,.$$

Consider the vector fields

(2.11) $$w_1 = ff_t^{-1}\frac{\partial}{\partial t},\quad w_2 = f_t^{-1}\frac{\partial}{\partial z} - f_zf_t^{-2}\frac{\partial}{\partial t},$$
$$w_3 = f^{-1}\left(f_t^{-1}\frac{\partial}{\partial y} - f_yf_t^{-2}\frac{\partial}{\partial t}\right),\quad w_4 = f^{-1}\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right);$$

thus the general λ-frames satisfying (2.9) are

(2.12) $$v_1 = w_1\,,\quad v_2 = w_2\,,\quad v_3 = w_3 + Rw_2\,,\quad v_4 = w_4 + Sw_1\,,$$

$R, S\colon J^2(M) \to \mathbb{R}$ being arbitrary functions. By a direct calculation,

(2.13) $$[w_1, w_2] = -ff_t^{-2}f_{tt}w_2\,,\quad [w_1, w_3] = -(1 + ff_t^{-2}f_{tt})\,w_3\,,$$
$$[w_1, w_4] = \{f^{-1}f_t^{-1}(f_z + f_{xt} + zf_{yt} + tf_{zt}) -$$
$$-f^{-2}(f_x + zf_y + tf_z)\}\,w_1 + w_2 - w_4\,,$$
$$[w_2, w_3] = f^{-1}(f_t^{-2}f_{yt} - f_yf_t^{-3}f_{tt})\,w_2 + (f_zf_t^{-3}f_{tt} - f_t^{-2}f_{zt})\,w_3\,,$$
$$[w_2, w_4] = \alpha_{24}^1w_1 + \alpha_{24}^2w_2 + w_3\,,$$
$$\alpha_{24}^1 = f^{-2}f_t^{-1}(f_{xz} + zf_{yz} + tf_{zz} + f_y) - f^{-2}f_t^{-2}f_z(f_{xt} + zf_{yt} + tf_{zt} + f_z)\,,$$
$$\alpha_{24}^2 = f^{-1}f_t^{-1}(f_{xt} + zf_{yt} + tf_{zt} - f_z)\,,$$
$$[w_3, w_4] = \alpha_{34}^1w_1 + \alpha_{34}^2w_2 + \alpha_{34}^3w_3\,,$$
$$\alpha_{34}^2 = -f^{-2}f_yf_t^{-1}\,,\quad \alpha_{34}^3 = f^{-2}(f_x + zf_y + tf_z) + f^{-1}f_t^{-1}(f_{xt} + zf_{yt} + tf_{zt})\,.$$

From this,

(2.14) $$[v_2, v_4] = c_{24}^1v_1 + c_{24}^2v_2 + v_3\,,\quad [v_3, v_4] = c_{34}^1v_1 + c_{34}^2v_2 + c_{34}^3v_3$$

with

$$c_{24}^2 = f^{-1}f_t^{-1}(f_{xt} + zf_{yt} + tf_{zt} - f_z) + ff_t^{-2}f_{tt}S - R\,, \tag{2.15}$$

$$c_{34}^3 = f^{-2}(f_x + zf_y + tf_z) + f^{-1}f_t^{-1}(f_{xt} + zf_{yt} + tf_{zt}) + + (1 + ff_t^{-2}f_{tt})\,S + R\,.$$

Proposition 2.1. *On $J^2(M)$, let the Lagrangian (2.5) be given which does not satisfy*

$$1 + 2ff_t^{-2}f_{tt} = 0\,. \tag{2.16}$$

Then there is, on $J^2(M)$, a unique field of λ-frames satisfying (2.9) and $c_{24}^2 = c_{34}^3 = 0$ in (2.14).

Proof. From (2.15) with $c_{24}^2 = c_{34}^3 = 0$, we may calculate R and S, and our field of λ-frames is given by (2.12) and (2.11). QED.

The special case is thus formed by Lagrangians with (2.16), i.e., by Lagrangians

$$\lambda = \{a(x, y, z) + b(x, y, z)\,t\}^{2/3}\,dx\,. \tag{2.17}$$

The coframe dual to (2.12) is given by

$$\omega^1 = -Sf\,dx + f^{-1}f_y(dy - z\,dx) + f^{-1}f_z(dz - t\,dx) + f^{-1}f_t\,dt\,, \tag{2.18}$$

$$\omega^2 = -Rff_t(dy - z\,dx) + f_t(dz - t\,dx)\,,$$

$$\omega^3 = ff_t(dy - z\,dx)\,, \quad \omega^4 = \lambda = f\,dx\,.$$

Especially, the forms

$$\omega^3 = ff_t(dy - z\,dx)\,, \quad \omega^2 \wedge \omega^3 = ff_t^2(dz - t\,dx) \wedge (dy - z\,dx)\,, \tag{2.19}$$

$$\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 = ff_t^3\,dx \wedge dy \wedge dz \wedge dt$$

are invariant in all cases.

Reference

[1] *R. B. Gardner:* Differential Geometric Methods Interfacing Control Theory. In "Differential Geometric Control Theory", Progress in Math. *27*, Birkhauser, Boston, 1983, 117—180.

Author's address: Přehradní 10, 635 00 Brno, Czechoslovakia.