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## ON THE PICK INVARIANT

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In [4], I have shown that the bounded elliptic surfaces  $M$  in the equiaffine 3-space satisfying the relation  $\Phi(H, K) = 0$  with  $\Phi_H^2 + 4H\Phi_H\Phi_K + 4K\Phi_K^2 > 0$  and umbilical boundary are affine spheres. Here, I am going to study analogous problems.

1. Let  $M \subset A^3$  be an elliptic surface in the equiaffine 3-dimensional space. With each point  $m \in M$ , let us associate a frame  $\{m; v_1, v_2, v_3\}$ ; we have the fundamental equations

$$(1) \quad dm = \omega^1 v_1 + \omega^2 v_2, \quad dv_i = \omega_i^j v_j$$

with  $\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$  and the usual integrability conditions. It is possible, see [4], to choose the frames in such a way that

$$(2) \quad \omega_1^3 = \omega^1, \quad \omega_2^3 = \omega^2;$$

$$(3) \quad \omega_1^1 = -\frac{1}{2}c\omega^1 + \frac{1}{2}b\omega^2, \quad \omega_2^2 = \frac{1}{2}c\omega^1 - \frac{1}{2}b\omega^2, \quad \omega_3^3 = 0, \\ \omega_1^2 + \omega_2^1 = b\omega^1 + c\omega^2.$$

The form  $\omega$  being defined by

$$(4) \quad \omega := \frac{1}{2}(\omega_1^2 - \omega_2^1),$$

we have

$$(5) \quad \omega_1^2 = \frac{1}{2}b\omega^1 + \frac{1}{2}c\omega^2 + \omega, \quad \omega_2^1 = \frac{1}{2}b\omega^1 + \frac{1}{2}c\omega^2 - \omega$$

and

$$(6) \quad d\omega^1 = -\omega^2 \wedge \omega, \quad d\omega^2 = \omega^1 \wedge \omega.$$

The integrability conditions of (3) are

$$(7) \quad (dc + 3b\omega - 2\omega_3^1) \wedge \omega^1 - (db - 3c\omega) \wedge \omega^2 = 0, \\ (dc + 3b\omega) \wedge \omega^1 - (db - 3c\omega - 2\omega_3^2) \wedge \omega^2 = 0, \\ (db - 3c\omega + \omega_3^2) \wedge \omega^1 + (dc + 3b\omega + \omega_3^1) \wedge \omega^2 = 0.$$

From (7<sub>1,2</sub>),  $\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0$  follows, and we get the existence of functions  $\alpha, \beta, \gamma$  such that

$$(8) \quad \omega_3^1 = \alpha\omega^1 + \beta\omega^2, \quad \omega_3^2 = \beta\omega^1 + \gamma\omega^2.$$

We get the following invariant forms (see [3]): the *metric form*

$$(9) \quad ds^2 = (\omega^1)^2 + (\omega^2)^2,$$

and the form of the lines of the affine curvature

$$(10) \quad \mathcal{C} = \beta(\omega^1)^2 + (\alpha - \gamma)\omega^1\omega^2 - \beta(\omega^2)^2.$$

Further, we consider the following invariants

$$(11) \quad J = \frac{1}{2}(b^2 + c^2), \quad H = -\frac{1}{2}(\alpha + \gamma), \quad K = \alpha\gamma - \beta^2, \quad \varkappa = J + H,$$

i.e., the *Pick invariant*, the *mean* and *affine curvature* and the *Gauss curvature* resp.

**Theorem 1.** *Let  $M \subset A^3$  be an elliptic real analytic surface which is a bounded simply connected domain. Let  $D \subset \mathbb{R}^2$  be a domain such that, for each  $m \in M$ ,  $(H(m), K(m)) \in D$ . On  $D$ , let a real analytic function  $F(u, v)$  be given such that*

$$(12) \quad \left(\frac{\partial F}{\partial u}\right)^2 + 4u \frac{\partial F}{\partial u} \frac{\partial F}{\partial v} + 4v \left(\frac{\partial F}{\partial v}\right)^2 > 0 \quad \text{in } D.$$

Let us suppose: (i) We have

$$(13) \quad J(m) = F(H(m), K(m)) \quad \text{for each } m \in M.$$

(ii) There is an arc  $\gamma \subset M$  such that

$$(14) \quad J(m) = 0, \quad \mathcal{C}(m) \equiv 0 \quad \text{for each } m \in \gamma.$$

Then  $M$  is a part of an elliptic quadratic surface.

*Proof.* We may substitute (8) into (7) and differentiate (8); we get

$$(15) \quad \begin{aligned} (dc + 3b\omega) \wedge \omega^1 - (db - 3c\omega) \wedge \omega^2 &= -2\beta\omega^1 \wedge \omega^2, \\ (db - 3c\omega) \wedge \omega^1 + (dc + 3b\omega) \wedge \omega^2 &= (\gamma - \alpha)\omega^1 \wedge \omega^2, \\ (d\alpha - 2\beta\omega) \wedge \omega^1 + (d\beta + (\alpha - \gamma)\omega) \wedge \omega^2 &= \left\{\frac{1}{2}b(\alpha - \gamma) + c\beta\right\}\omega^1 \wedge \omega^2, \\ (d\beta + (\alpha - \gamma)\omega) \wedge \omega^1 + (d\gamma + 2\beta\omega) \wedge \omega^2 &= \left\{\frac{1}{2}c(\alpha - \gamma) - b\beta\right\}\omega^1 \wedge \omega^2. \end{aligned}$$

Thus there are functions  $b_1, \dots, \gamma_2$  such that

$$(16) \quad \begin{aligned} db - 3c\omega &= b_1\omega^1 + b_2\omega^2, \quad dc + 3b\omega = c_1\omega^1 + c_2\omega^2; \\ d\alpha - 2\beta\omega &= \alpha_1\omega^1 + \alpha_2\omega^2, \quad d\beta + (\alpha - \gamma)\omega = \beta_1\omega^1 + \beta_2\omega^2, \\ d\gamma + 2\beta\omega &= \gamma_1\omega^1 + \gamma_2\omega^2 \end{aligned}$$

with

$$(17) \quad \begin{aligned} b_1 + c_2 &= 2\beta, \quad c_1 - b_2 = \gamma - \alpha; \\ \beta_1 - \alpha_2 &= \frac{1}{2}b(\alpha - \gamma) + c\beta, \quad \gamma_1 - \beta_2 = \frac{1}{2}c(\alpha - \gamma) - b\beta. \end{aligned}$$

On  $M$ , introduce coordinates  $(x, y)$  such that

$$(18) \quad \omega^1 = r dx, \quad \omega^2 = r dy; \quad r = r(x, y) > 0.$$

Then, from (6),

$$(19) \quad \omega = -r^{-1}r_y dx + r^{-1}r_x dy;$$

here,  $r_x = \partial r(x, y)/\partial x$ , etc. Substituting into (16), we get

$$(20) \quad \begin{aligned} rb_1 &= b_x + 3r^{-1}r_y c, & rb_2 &= b_y - 3r^{-1}r_x c, & rc_1 &= c_x - 3r^{-1}r_y b, \\ & & & & rc_2 &= c_y + 3r^{-1}r_x b; \\ r\alpha_1 &= \alpha_x + 2r^{-1}r_y \beta, & r\alpha_2 &= \alpha_y - 2r^{-1}r_x \beta, \\ r\beta_1 &= \beta_x - r^{-1}r_y(\alpha - \gamma), & r\beta_2 &= \beta_y + r^{-1}r_x(\alpha - \gamma), \\ r\gamma_1 &= \gamma_x - 2r^{-1}r_y \beta, & r\gamma_2 &= \gamma_y + 2r^{-1}r_x \beta. \end{aligned}$$

The conditions (17) may be rewritten as

$$(21) \quad \begin{aligned} b_x + c_y + 3r^{-1}r_x b + 3r^{-1}r_y c &= 2r\beta, \\ c_x - b_y - 3r^{-1}r_y b + 3r^{-1}r_x c &= r(\gamma - \alpha); \end{aligned}$$

$$(22) \quad \begin{aligned} \beta_x - \alpha_y - r^{-1}r_y(\alpha - \gamma) + 2r^{-1}r_x \beta &= \frac{1}{2}rb(\alpha - \gamma) + rc\beta, \\ \gamma_x - \beta_y - 2r^{-1}r_y \beta - r^{-1}r_x(\alpha - \gamma) &= \frac{1}{2}rc(\alpha - \gamma) - rb\beta. \end{aligned}$$

The supposition (13) reads

$$(23) \quad \frac{1}{2}(b^2 + c^2) = F(-\frac{1}{2}(\alpha + \gamma), \alpha\gamma - \beta^2)$$

with the differential consequences

$$(24) \quad \begin{aligned} 2bb_x + 2cc_x + (F_u - 2\gamma F_v)\alpha_x + (F_u - 2\alpha F_v)\gamma_x + 4\beta F_v \beta_x &= 0, \\ 2bb_y + 2cc_y + (F_u - 2\gamma F_v)\alpha_y + (F_u - 2\alpha F_v)\gamma_y + 4\beta F_v \beta_y &= 0. \end{aligned}$$

Let us use the following notation:  $A \equiv B$  means  $A = B +$  linear combination of  $b, c, \alpha - \gamma, \beta$ . Then, using (22), (24) may be rewritten as

$$(25) \quad \begin{aligned} 2bb_x + 2cc_x + (F_u - 2\gamma F_v)(\alpha - \gamma)_x + \\ + 2(F_u - (\alpha + \gamma)F_v)\beta_y + 4\beta F_v \beta_x &\equiv 0, \\ 2bb_y + 2cc_y - (F_u - 2\alpha F_v)(\alpha - \gamma)_y + \\ + 2(F_u - (\alpha + \gamma)F_v)\beta_x + 4\beta F_v \beta_y &\equiv 0. \end{aligned}$$

Consider the  $\mathbb{R}^4$ -valued function

$$(26) \quad u = (b, c, \alpha - \gamma, \beta)^T;$$

we get, from (21) and (25),

$$(27) \quad Au_x + Bu_y + Cu = 0$$

with

$$(28) \quad \begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2b & 2c & F_u - 2\gamma F_v & 4\beta F_v \\ 0 & 0 & 0 & 2(F_u - (\alpha + \gamma)F_v) \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(F_u - (\alpha + \gamma)F_v) \\ 2b & 2c & -(F_u - 2\alpha F_v) & 4\beta F_v \end{pmatrix}. \end{aligned}$$

The symbol of the system (27) is defined as

$$(29) \quad \sigma(\xi, \eta) = A\xi + B\eta; \quad (\xi, \eta) \in \mathbb{R}^2.$$

It is easy to see that

$$(30) \quad \det \sigma(\xi, \eta) = 2(F_u + 2HF_v)(\xi^2 + \eta^2) \Phi(\xi, \eta) \quad \text{with} \\ \Phi(\xi, \eta) = (F_u - 2\gamma F_v) \xi^2 + 4\beta F_v \xi \eta + (F_u - 2\alpha F_v) \eta^2.$$

The discriminant  $\Delta$  of the form  $\Phi(\xi, \eta)$  being equal to  $F_u^2 + 4HF_u F_v + 4KF_v^2$ , the supposition (12) implies that  $\Phi(\xi, \eta)$  is definite. Further,  $F_u + 2HF_v \neq 0$ . Indeed,  $F_u = -2HF_v$  implies  $\Delta = -4(H^2 - K)F_v^2 \leq 0$ , a contradiction to (12). Thus the symbol  $\sigma(\xi, \eta)$  is an invertible matrix for each real pair  $(\xi, \eta) \neq (0, 0)$ , and the system (27) is elliptic; see [6], p. 76. Now, let us apply Theorem 5.4.1 of [6] which claims that the zeros of any non-trivial solution of an elliptic system with real analytic coefficients are isolated and of finite order. This provides the uniqueness of the Cauchy problem, and we have  $u = 0$  on  $M$ . But this means  $J = 0$  on  $M$ , and we are finished.

**Remark.** For  $F(u, v) = -u + \text{const.}$ , the condition (13) turns out to be  $\kappa = \text{const.}$ , and we get a version of the  $\kappa$ -Theorem; compare with [2] and [3].

2. The affine normals of a surface in the equiaffine 3-space  $A^3$  coincide with its projective normals if and only if the Pick invariant  $J$  of the surface is constant; see [1], p. 111. In this connection, I am going to prove the following

**Theorem 2.** *The only compact elliptic surfaces  $M \subset A^3$  without boundary with equal affine and projective normals are the ellipsoids.*

*Proof.* Let  $(M, ds^2 = \delta_{ij}\omega^i\omega^j)$  be a Riemannian manifold,  $F_{ijk}$  a tensor on  $M$ ,  $F_{ijk;l}$  its covariant derivatives with respect to the coframes  $\{\omega^i\}$ . Let the 1-form  $\varphi$  on  $M$  be defined by

$$(31) \quad \varphi = \delta^{i_1 j_1} \delta^{i_2 j_2} \delta^{k l} (F_{i_1 i_2 l} F_{j_1 j_2 k; l} - F_{i_1 i_2 k} F_{j_1 j_2 l; i}) \omega^i.$$

Theorem 1.1 of [5] says that the form  $d * \varphi$  does not contain the second covariant derivatives of  $F_{ijk}$ .

Let us apply this result to the invariant form (see [4])

$$(32) \quad F \equiv F_{ijk} \omega^i \omega^j \omega^k := -2\mathcal{A} = c(\omega^1)^3 - 3b(\omega^1)^2 \omega^2 - 3c\omega^1(\omega^2)^2 + b(\omega^2)^3.$$

The covariant derivatives  $F_{ijk;l}$  being defined by

$$(33) \quad dF_{ijk} - F_{rjk}\varphi_i^r - F_{irk}\varphi_j^r - F_{ijr}\varphi_k^r = F_{ijk;l}\omega^l; \\ \varphi_1^2 = -\varphi_2^1 := \omega, \quad \varphi_1^1 = \varphi_2^2 = 0;$$

we get, from (16<sub>1,2</sub>),

$$(34) \quad F_{111;l} = c_1, \quad F_{111;2} = c_2, \quad F_{112;l} = -b_1, \quad F_{112;2} = -b_2, \\ F_{122;l} = -c_1, \quad F_{122;2} = -c_2, \quad F_{222;l} = b_1, \quad F_{222;2} = b_2.$$

The exterior differentiation of  $(16_{1,2})$  yields

$$(35) \quad \{db_1 - (b_2 + 3c_1)\omega\} \wedge \omega^1 + \{db_2 + (b_1 - 3c_2)\omega\} \wedge \omega^2 = 3\kappa c \omega^1 \wedge \omega^2, \\ \{dc_1 - (c_2 - 3b_1)\omega\} \wedge \omega^1 + \{dc_2 + (c_1 + 3b_2)\omega\} \wedge \omega^2 = -3\kappa b \omega^1 \wedge \omega^2,$$

and we get the existence of functions  $b_{11}, \dots, c_{22}$  such that

$$(36) \quad db_1 - (b_2 + 3c_1)\omega = b_{11}\omega^1 + b_{12}\omega^2, \\ db_2 + (b_1 - 3c_2)\omega = b_{21}\omega^1 + b_{22}\omega^2, \\ dc_1 - (c_2 - 3b_1)\omega = c_{11}\omega^1 + c_{12}\omega^2, \\ dc_2 + (c_1 + 3b_2)\omega = c_{21}\omega^1 + c_{22}\omega^2; \\ (37) \quad b_{21} - b_{12} = 3\kappa c, \quad c_{21} - c_{12} = -3\kappa b.$$

Considering the form (31), we have

$$(38) \quad \psi := -\frac{1}{4} * \varphi = (cb_1 - bc_1)\omega^1 + (cb_2 - bc_2)\omega^2$$

and

$$(39) \quad d\psi = 2(b_2c_1 - b_1c_2 + 3\kappa J)\omega^1 \wedge \omega^2.$$

Now, let us suppose  $J = \text{const.} > 0$ . From (11<sub>1</sub>) and  $(16_{1,2})$ ,  $bb_1 + cc_1 = bb_2 + cc_2 = 0$ , which implies  $b_2c_1 - b_1c_2 = 0$ . Thus

$$(40) \quad 0 = \int_{\partial M} \psi = \int_M d\psi = 6J \int_M \kappa \omega^1 \wedge \omega^2,$$

i.e.,  $\int_M \kappa \omega^1 \wedge \omega^2 = 0$ . This means  $\chi(M) = 0$ , and  $M$  should be a torus. But this is impossible because of the ellipticity of  $M$ . Thus  $J = 0$  on  $M$ , and we are finished.

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