Ján Jakubík
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SEQUENTIAL CONVERGENCES IN BOOLEAN ALGEBRAS

JAN JAKUBÍK, Košice

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In this paper sequential convergences in Boolean algebras are investigated which are compatible with the Boolean operations. Analogous questions for lattice ordered groups were studied by M. Harminc [1], [2], [3] and the author [4].

Several types of sequential convergences in abelian lattice ordered groups and in Boolean algebras were dealt with by F. Papangelou [8]; for convergences in Boolean algebras cf. also H. Löwig [5]. Some questions on sequential convergences in σ-fields of sets were investigated by J. Novák a M. Novotný [7].

1. CONVERGENCES AND 0-CONVERGENCES

Let $B$ be a Boolean algebra. In this section the notion of sequential convergence in $B$ will be introduced. It will be proved that a sequential convergence is uniquely determined by the system of all sequences which converge to the zero element of $B$.

We denote by $S$ the system of all sequences of elements of $B$. Let $\mathcal{A}$ be a subset of $S \times B$. If $((x_n), x) \in \mathcal{A}$, then we shall write $x_n \to_\mathcal{A} x$. Let $N$ be the set of all positive integers. If there exists $a \in B$ such that $x_n = a$ for each $n \in N$, then we write $(x_n) = a = \text{const } a$.

1.1. Definition. A subset $\mathcal{A}$ of $S \times B$ is said to be a convergence in $B$, if the following conditions are satisfied:

(i) If $x_n \to_\mathcal{A} x$ and $(y_n)$ is a subsequence of $(x_n)$, then $y_n \to_\mathcal{A} x$.

(ii) If $(x_n) \in S$ and if for each subsequence $(y_n)$ of $(x_n)$ there exists a subsequence $(z_n)$ of $(y_n)$ such that $z_n \to_\mathcal{A} x$, then $x_n \to_\mathcal{A} x$.

(iii) For each $x \in B$, const $x \to_\mathcal{A} x$.

(iv) If $x_n \to_\mathcal{A} x$ and $x_n \to_\mathcal{A} y$, then $x = y$.

(v) If $x_n \to_\mathcal{A} x$ and $y_n \to_\mathcal{A} y$, then $x_n \wedge y_n \to_\mathcal{A} x \wedge y$, $x_n \vee y_n \to_\mathcal{A} x \vee y$ and $x_n' \to_\mathcal{A} x'$.

(vi) If $x_n \leq y_n \leq z_n$ is valid for each $n \in N$, and if $x_n \to_\mathcal{A} x$, $z_n \to_\mathcal{A} x$, then $y_n \to_\mathcal{A} x$.

The system of all convergences on $B$ will be denoted by Conv $B$. Let $\mathcal{A}$ be a fixed element of Conv $B$.

1.2. Lemma. The following conditions are equivalent:

(a) $x_n \to_\mathcal{A} x$.

(b) $x_n \wedge x \to_\mathcal{A} x$ and $x_n \vee x \to_\mathcal{A} x$.

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Proof. In view of (iii) and (v) we have (a) ⇒ (b). According to (vi), the relation (b) ⇒ (a) is valid.

1.3. Lemma. The condition (a) from 1.2 is equivalent to the following condition:
(c) \( x_n \land x' \to_a 0 \) and \( x'_n \land x \to_a 0 \).

Proof. Let (a) be valid. Then in view of (iii) and (v) the condition (c) holds. Conversely, let (c) be satisfied. Then we have
\[(x_n \land x') \lor x \to_a x \quad \text{and} \quad (x'_n \land x) \lor x' \to_a x',\]
hence \( x_n \lor x \to_a x \) and \( x'_n \lor x' \to_a x' \). In view of (v), \( x_n \land x = (x'_n \lor x')' \to_a x \).
Thus by applying 1.2 we obtain that (a) holds.

Let us denote by \( \mathcal{Z}_0 \) the set of all \( (x_n) \in S \) such that \( x_n \to_a 0 \). From 1.3 we infer:

1.4. Corollary. The set \( \mathcal{Z}_0 \) uniquely determines the convergence \( \mathcal{Z} \).

A natural problem arises, to characterize those subsets \( T \) of \( S \) for which there exists \( \mathcal{Z} \in \text{Conv } B \) such that \( T = \mathcal{Z}_0 \).

1.5. Lemma. Let \( T \) be a nonempty subset of \( S \). There exists \( \mathcal{Z} \in \text{Conv } B \) with \( T = \mathcal{Z}_0 \) if and only if the following conditions are satisfied:

(i) If \( (x_n) \in T \), then each subsequence of \( (x_n) \) belongs to \( T \).
(ii) If \( (x_n) \in S \) and if for each subsequence \( (y_n) \) of \( (x_n) \) there exists a subsequence \( (z_n) \) of \( (y_n) \) such that \( (z_n) \in T \), then \( (x_n) \in T \).
(iii) For \( a \in B \) we have \( \text{const } a \in T \) if and only if \( a = 0 \).
(iv) If \( (x_n) \) and \( (y_n) \) belong to \( T \), then \( (x_n \lor y_n) \) also belongs to \( T \).
(v) If \( (x_n) \) belongs to \( T \) and if \( (y_n) \in S \), \( y_n \leq x_n \) for all \( n \in \mathbb{N} \), then \( (y_n) \in T \).

Proof. If there is \( \mathcal{Z} \in \text{Conv } G \) such that \( T = \mathcal{Z}_0 \), then from 1.1 we immediately obtain that the conditions (i)–(v) are satisfied.

Conversely, suppose that \( T \subseteq S \) is such that (i)–(v) hold. For \( (x_n) \in S \) and \( x \in B \) we put \( x_n \to_a x \) if \( (x_n \land x') \in T \) and \( (x'_n \land x) \in T \).

First we observe that the relation
\[(x_n) \in T \iff x_n \to_a 0\]
is valid for each \( (x_n) \in S \).

Indeed, let \( (x_n) \in T \). We have \( (x_n) = (x_n \land 0') \in T \) and \( \text{const } 0 = (x'_n \land 0) \in T \), whence \( x_n \to_a 0 \). Conversely, let \( x_n \to_a 0 \). Then \( (x_n \land 0') \in T \), whence \( (x_n) \in T \).

Now we have to verify that the conditions (i)–(vi) from 1.1 are satisfied.

The conditions (i), (ii) and (iii) are consequences of (i_1), (i_i) and (i_ii), respectively.

(v): Let \( x_n \to_a x \) and \( y_n \to_a y \). In view of the first relation we have \( (x_n \land x') \in T \) and \( (x'_n \land x) \in T \), whence \( x'_n \to_a x' \). Denote \( z_n = x_n \lor y_n, z = x \lor y \). Then
\[
z_n \land z' = (x_n \lor y_n) \land (x \lor y)' = (x_n \lor y_n) \land (x' \lor y') = [x_n \land (x' \lor y') \lor [y_n \land (x' \lor y')].
\]
According to (v_1), both \( (x_n \land (x' \lor y')) \) and \( (y_n \land (x' \lor y')) \) belong to \( T \); hence
in view of (iv), \((z_n \wedge z')\) belongs to \(T\). Similarly we obtain that \((z_n' \wedge z)\) belongs to \(T\). Thus \(z_n \rightarrow_a z\).

Next, let \(v_n = x_n \wedge y_n, v = x \wedge y\). Then
\[
v_n \wedge v' = (x_n \wedge y_n) \wedge (x \wedge y)' = (x_n \wedge y_n) \wedge (x' \vee y') = \]
\[
= [(x_n \wedge y_n) \wedge x'] \vee [(x_n \wedge y_n) \wedge y'] .
\]
By applying (v) and (iv) we obtain that \((v_n \wedge v') \in T\). Similarly, \((v_n' \wedge v) \in T\).
Thus \(v_n \rightarrow_a v\).

(vi): Let \(x_n \rightarrow_a x, z_n \rightarrow_a x\) and suppose that \(x_n \leq y_n \leq z_n\) is valid for each \(n \in N\).
Hence the sequences \((x'_n \wedge x)\) and \((z'_n \wedge x')\) belong to \(T\). Then according to (v)
we have \(y'_n \wedge x \in T\) and \(y'_n \wedge x' \in T\); therefore \(y_n \rightarrow_a x\).

(iv): First we shall verify that if \((a_n) = \text{const} 0\) and if \(a_n \rightarrow_a a\) then \(a = 0\). In
fact, in view of the assumption we have \(a'_n \wedge a \in T\), hence \(\text{const} a \in T\). Thus according
to (iii), \(a = 0\). Now assume that \(x_n \rightarrow_a x\) and \(x_n \rightarrow_a y\). Hence \(x'_n \rightarrow_a y'\) and therefore
\(x_n \wedge x'_n \rightarrow_a x \wedge y'\). Since \((x_n \wedge x'_n) = \text{const} 0\), we infer that \(x \wedge y' = 0\) and hence
\(x \leq y\). Similarly we obtain that \(y \leq x\). Hence \(x = y\). The proof is complete.

Denote \(\text{Conv}_0 B = \{x_0; x \in \text{Conv} B\}\). The elements of \(\text{Conv}_0 B\) are said to be
0-convergences in \(B\). For \(x, y \in \text{Conv} B\) we put \(x \leq y\) if, whenever \((x_n) \in S, x \in B\)
and \(x_n \rightarrow_a x\), then \(x_n \rightarrow fry\). Further, we put \(x_0 \leq y_0\) if \(x_0\) is a subset of \(y_0\). Then we have
\[
x \leq y \iff x_0 \leq y_0 .
\]
Let \((x_n) \in S, x \in B\). We put \(x_n \rightarrow_d x\) if there is \(m \in N\) such that \(x_n = x\) for each \(n \in N\)
with \(n > m\). The following assertion is easy to verify.

1.6. Lemma. \(d \in \text{Conv} B\) and for each \(x \in \text{Conv} B\) we have \(d \leq x\).

1.7. Corollary. \(d_0\) is the least element of \(\text{Conv}_0 B\).

2. REGULAR SETS OF SEQUENCES

A nonempty subset \(A\) of \(S \times B\) will be called regular if there exists \(x \in \text{Conv} B\)
such that \(A \subseteq a\). A set \(A\) is regular if and only if \(A \cup \{(\text{const} 0, 0)\}\) is regular.
Analogously, a nonempty subset \(T\) of \(S\) will be said to be regular if there exists
\(x_0 \in \text{Conv}_0 B\) such that \(T \subseteq x_0\). The set \(C\) is regular if and only if \(C \cup \{(\text{const} 0)\}\)
is regular.

Let \(\emptyset \neq A \subseteq S \times B\). Denote
\[
A_1 = \{(x_n \wedge x') : (x_n, x) \in A\} ,
A_2 = \{(x'_n \wedge x) : (x_n, x) \in A\} ,
A_3 = A_1 \cup A_2 .
\]
Let \(\emptyset \neq C \subseteq S\). We put
\[
C_1 = \{(x_n) : (x_n \wedge x') \in C\} \text{ and } (x'_n \wedge x) \in C\} .
\]

In view of the results of Section 1 we have

2.1. Lemma. (i) Let $(\text{const } 0, 0) \in A \subseteq S \times B$. Then $A$ is regular if and only if $A_3$ is regular.

(ii) Let $\text{const } 0 \in C \subseteq S$. Then $C$ is regular if and only if $C_1$ is regular.

Thus it suffices to investigate the regularity of subsets $C$ of $S$ such that $\text{const } 0 \in C$. Let $(x_n)$ and $(y_n)$ be elements of $S$. We put $(x_n) \land (y_n) = (x_n \land y_n), (x_n) \lor (z_n) = (x_n \lor z_n), (x_n)' = (x_n^*)$. Then $S$ turns out to be a Boolean algebra.

Let $A$ be a nonempty subset of $S$. We denote by

- $\delta A$ — the set of all subsequences of sequences belonging to $A$;
- $A^*$ — the set of all $(x_n) \in S$ such that for each subsequence $(y_n)$ of $(x_n)$ there exists a subsequence $(z_n)$ of $(y_n)$ which belongs to $A$;
- $[A]$ — the ideal of the Boolean algebra $S$ generated by the set $A$.

The following assertions 2.2—2.4 are easy to verify; the proofs will be omitted.

2.2. Lemma. Let $b \in B$. Then $\text{const } b \in A$ if and only if $\text{const } b \in A^*$.

2.3. Lemma. $\delta [\delta A] = [\delta A]$.

2.4. Lemma. $\delta (A^*) \subseteq (\delta A)^*$ and $[A^*] \subseteq [\delta A]^*$.


From 1.5 and 2.5 we infer:

2.6. Corollary. $[\delta A]^*$ belongs to Conv$_0$ $B$ if and only if for each nonzero element $b$ of $B$ we have $\text{const } b \notin [\delta A]^*$.

2.7. Proposition. A nonempty subset $A$ of $S$ is regular if and only if for each nonzero element $b$ of $B$ we have $\text{const } b \notin [\delta A]$.

Proof. This is a consequence of 2.6 and 2.2.

2.8. Proposition. Let $A$ be a regular subset of $S$. Let $\alpha \in \text{Conv}_0 B$, $A \subseteq \alpha$. Then $[\delta A]^* \subseteq \alpha$.

Proof. This is an immediate consequence of 1.5.

In view of 2.5 and 2.8, for a regular subset $A$ of $S$ the 0-convergence $[\delta A]^*$ will be said to be generated by the set $A$. If $A = \{(x_n)\}$ and $A$ is regular, then $A$ is said to be generated by $(x_n)$; in such a case $[\delta A]^*$ is called principal.

If $\emptyset \neq A \subseteq S$, then $[A]$ is the set of all $(x_n) \in S$ which have the following property: there exist $(y_1^1, y_2^1, \ldots, y_n^1)$ in $A$ such that $(x_n) \leq (y_1^1) \lor (y_2^1) \lor \ldots \lor (y_n^1)$. From 2.7 we obtain:

2.9. Proposition. Let $\emptyset \neq A \subseteq S$. Then the following conditions are equivalent:

(i) $A$ is regular.

(ii) If $(y_1^1, y_2^1, \ldots, y_n^1)$ are elements of $\delta A$ and if $b$ is an element of $B$ such that $b \leq y_1^1 \lor y_2^1 \lor \ldots \lor y_n^1$ is valid for each $n \in N$, then $b = 0$.
2.10. Lemma. Let $I \neq \emptyset$ and for each $i \in I$ let $\alpha_i^0 \in \text{Conv}_0 B$. Put $A = \bigcup_{i \in I} \alpha_i^0$. Then the following conditions are equivalent:

(i) $A$ is regular.
(ii) If $i_1, i_2, \ldots, i_n$ are distinct elements of $I$ and if $(y^k_n) \in \alpha_k^0$ for each $k \in \{i_1, i_2, \ldots, i_n\}$, $b \in B$, $b \leq y^1_n \lor y^2_n \lor \ldots \lor y^m_n$ for each $n \in N$, then $b = 0$.

Proof. This follows from 2.9 and from the fact that $\delta \alpha_i^0 = \alpha_i^0 = [\alpha_i^0]$ for each $i \in I$.

2.11. Lemma. Let $I$, $\alpha_i^0$ and $A$ be as in 2.10. Assume that $A$ is regular. Put $\alpha = [A]^*$. Then

(i) $\alpha \in \text{Conv}_0 B$;
(ii) $\alpha_i^0 \preceq \alpha$ for each $i \in I$;
(iii) if $\beta^0 \in \text{Conv}_0 B$ and $\alpha_i^0 \preceq \beta^0$ for each $i \in I$, then $\alpha \preceq \beta^0$.

Proof. Because $\alpha_i^0 \in \text{Conv}_0 B$ for each $i \in I$, we have $\delta \alpha_i^0 = \alpha_i^0$, whence $\delta A = A$. Hence $[\delta A]^* = \alpha$. According to 2.6 and 2.8, $\alpha \in \text{Conv}_0 B$. The assertions (ii) and (iii) are obvious.

A sequence $(x_n)$ in $S$ is said to be decreasing if $x_n \geq x_{n+1}$ for each $n \in N$.

2.12. Lemma. Let $(x_n)$ be a decreasing sequence in $B$ and let $A = \{(x_n)\}$. Then $A$ is regular if and only if $\bigwedge x_n = 0$.

Proof. If $A$ is regular, then in view of 2.7 we must have $\bigwedge x_n = 0$. Conversely, assume that $\bigwedge x_n = 0$. Let $(y^1_n), (y^2_n), \ldots, (y^m_n)$ be subsequences of $(x_n)$. Let $b \in B$ and suppose that $b \leq y^1_n \lor y^2_n \lor \ldots \lor y^m_n$ is valid for each $n \in N$. We have $y^k_n \leq x_n$ for $k = 1, 2, \ldots, m$, whence $b \leq x_n$ for each $n \in N$. Therefore $b = 0$. Thus according to 2.9, $A$ is regular.

3. THE PARTIALLY ORDERED SET $\text{Conv}_0 B$

As we already remarked in Section 1, the set $\text{Conv}_0 B$ is considered to be partially ordered by inclusion. Each nonempty subset of $\text{Conv}_0 B$ is partially ordered by the induced partial order. Let $\text{Conv}_0 B$ be the set of all principal elements of $\text{Conv}_0 B$.

Let $I \neq \emptyset$ and for each $i \in I$ let $\alpha_i^0 \in \text{Conv}_0 B$. If the set $\{\alpha_i^0\}_{i \in I}$ has the infimum or the supremum in $\text{Conv}_0 B$, then these elements will be denoted by $\bigwedge_{i \in I} \alpha_i^0$ or $\bigvee_{i \in I} \alpha_i^0$, respectively.

3.1. Lemma. Let $\{\alpha_i^0\}_{i \in I}$ be a nonempty subset of $\text{Conv}_0 B$. Then $\bigwedge_{i \in I} \alpha_i^0 = \bigcap_{i \in I} \alpha_i^0$.

Proof. This is a consequence of the fact that $\bigcap_{i \in I} \alpha_i^0$ satisfies the conditions from 1.5.

3.2. Corollary. Let $\alpha^0 \in \text{Conv}_0 B$. Then the interval $[d, \alpha^0]$ of $\text{Conv}_0 B$ is a complete lattice. $\text{Conv}_0 B$ is a $\bigwedge$-semilattice.

In Section 4 it will be shown that $\text{Conv}_0 B$ need not be a lattice.
3.3. Lemma. Let \( \{ x_i^0 \}_{i \in I} \) be a nonempty subset of \( \text{Conv}_0 B \). Put \( A = \bigcup_{i \in I} x_i^0 \). Then the following conditions are equivalent:

(i) \( A \) is regular.
(ii) \( [A]^* = \bigvee_{i \in I} x_i^0 \).

Proof. The implication (i) \( \Rightarrow \) (ii) is a consequence of 2.11. The implication (ii) \( \Rightarrow \) (i) is obvious.

3.4. Lemma. The following conditions are equivalent:

(i) \( \text{Conv}_0 B \) has no greatest element.
(ii) There are \( \beta_1^0, \beta_2^0 \in \text{Conv}_p B \) such that the set \( \{ \beta_1^0, \beta_2^0 \} \) is not upper bounded in \( \text{Conv}_0 B \).

Proof. The implication (ii) \( \Rightarrow \) (i) is trivial. Assume that (i) holds. Let \( \text{Conv}_0 B = \{ x_j^0 \}_{j \in J} \). Put \( A = \bigcup_{j \in J} x_j^0 \). In view of 3.3, \( A \) fails to be regular. Hence according to 2.10 there exists a positive integer \( m \), elements \( j_1, j_2, \ldots, j_m \in J \), sequences \( (y_{n_i}^1) \in x_{j_1}^0, \ldots, (y_{n_i}^m) \in x_{j_m}^0 \) and an element \( b \neq 0 \) in \( B \) such that \( b \leq y_{n_1}^1 \vee y_{n_2}^2 \vee \ldots \vee y_{n_m}^m \) is valid for each \( n \in N \).

Let \( m \) be the least positive integer having the just mentioned property. We must have \( m \geq 2 \). Assume that \( m > 2 \). In view of this assumption, the set \( \{ (y_{n_1}^1 \vee y_{n_2}^2 \vee \ldots \vee y_{n_m}^m) \} = \{(z_n)\} \) is regular as well. Since \( b \leq y_{n_1}^1 \vee z_n \) holds for each \( n \in N \), by virtue of the relation \( m > 2 \) we have \( b = 0 \), which is a contradiction. Hence we have \( m = 2 \). Both the sets \( A_1 = \{(y_n^1)\}, A_2 = \{(z_n)\} \) are regular, hence \( \beta_1^0 = [\delta A_1]^* \) and \( \beta_2^0 = [\delta A_2]^* \) belong to \( \text{Conv}_0 B \). But the set \( \{ (y_n^1), (z_n) \} \) is not regular. Thus the set \( \{ \beta_1^0, \beta_2^0 \} \) fails to be upper bounded in \( \text{Conv}_0 B \).

3.5. Lemma. Let \( x_1^0 \) and \( x_2^0 \) be principal elements of \( \text{Conv}_0 B \) generated by the sequences \( (x_n^1) \) and \( (x_n^2) \), respectively. Assume that the set \( \{ x_1^0, x_2^0 \} \) is upper bounded in \( \text{Conv}_0 B \). Then \( x_1^0 \vee x_2^0 \) is principal and it is generated by \( (x_n^1 \vee x_n^2) \).

Proof. In view of 3.2, \( x_1^0 \vee x_2^0 \) does exist in \( \text{Conv}_B \). Hence the one-element set \( A = \{ (x_n^1 \vee x_n^2) \} \) is regular. Thus there exists \( \beta^0 \in \text{Conv}_0 G \) such that \( \beta^0 \) is generated by \( A \). Clearly \( \beta^0 \leq x_1^0 \vee x_2^0 \) since \( x_1^0 \vee x_2^0 \rightarrow \gamma \), where \( \gamma = x_1^0 \vee x_2^0 \). On the other hand, from \( [\delta(x_n^1)] \subseteq [\delta((x_n^1 \vee x_n^2))] \) we obtain that \( x_1^0 \leq \beta^0 \); similarly we have \( h_2^0 \leq \beta^0 \). Thus \( \beta^0 = x_1^0 \vee x_2^0 \).

From 3.2, 3.4 and 3.5 we infer:

3.6. Theorem. Let \( B \) be a Boolean algebra. The following conditions are equivalent:

(i) \( \text{Conv}_0 B \) has the greatest element.
(ii) \( \text{Conv}_0 B \) is a \( \vee \)-semilattice.
(iii) \( \text{Conv}_0 B \) is a lattice.
(iv) \( \text{Conv}_0 B \) is a complete lattice.

For a related result concerning lattice ordered groups cf. [3].
Let us remark that if $x_1^0$ and $x_2^0$ are as in 3.5, then the element $x_1^0 \wedge x_2^0 = x_1^0 \wedge x_2^0$ of Conv$_0$ $B$ need not be generated by the sequence $(x_n \wedge y_n)$. Also, if $\beta \in$ Conv$_0$ $B$ such that $\beta < \beta_1^0$, then $\beta$ need not be principal.

4. COMPLETE DISTRIBUTIVITY

In this section the following result will be proved:

4.1. **Theorem.** Let $B$ be a Boolean algebra. Assume that $B$ is completely distributive. Then Conv$_0$ $B$ has the greatest element.

Next it will be shown that there exists a Boolean algebra $B$ such that Conv$_0$ $B$ has no greatest element.

**Proof** of 4.1. Since $B$ is completely distributive, there exists a set $I$ such that there is an isomorphism $\varphi$ of $B$ into a complete field $C$ of subsets of $I$ such that, whenever $\bigwedge_{i \in I} x_i = x$ is valid in $B$, then $\bigcap_{i \in I} \varphi(x_i) = \varphi(x)$ is valid (and dually). Without loss of generality we can assume that $\varphi(0) = 0$ and $\varphi(1) = 1$. Let $A$ be the set of all $(x_n) \in S$ which have the following property: for each $i \in I$ there exists a positive integer $n(i)$ such that $i \notin \varphi(x_n)$ whenever $n \geq n(i)$. Then we clearly have $[\mathcal{A}]^* = A$. Let $(y_1^1, y_2^1, \ldots, y_n^m) \in A$, $b \in B$ and suppose that $b \leq y_1^1 \vee y_2^1 \vee \ldots \vee y_m^m$ is valid for each $n \in N$. Assume that $b > 0$. Then there exists $i \in I$ such that $i \in \varphi(b)$. On the other hand, there exists $n_0 \in N$ such that for each $n \geq n_0$ and each $k \in \{1, 2, \ldots, m\}$ we have $i \notin \varphi(y_k^n)$. Thus $i \notin \varphi(y_1^1 \vee y_2^1 \vee \ldots \vee y_m^m)$ for $n \geq n_0$, which is a contradiction. Therefore in view of 2.9, $A$ is regular. Hence $A \in$ Conv$_0$ $B$.

If $x \in$ Conv$_0$ $B$, $(x_n) \in x$, then $\{(x_n)\}$ is regular and therefore for each $i \in I$ there is $n_0 \in N$ such that $i \notin \varphi(x_n)$ whenever $n \geq n_0$. Hence $(x_n) \in A$ and thus $A$ is the greatest element of Conv$_0$ $B$.

An analogous result for convergences in archimedean lattice ordered groups was established in [4].

The following example shows that Conv$_0$ $B$ need not have the greatest element.

4.2. **Example.** Let $Q$ be the set of all rational numbers and let $e$ be a fixed irrational number. Put $Q_1 = \{x \in Q: e < x < e + 1\}$. Let $B$ be the set of all mappings $f$ of $Q_1$ into the set $\{0, 1\}$ having the property that there are irrational numbers $a_0 < a_1 < \ldots < a_n$ (depending on $f$), $a_0 = e$, $a_n = e + 1$ such that, whenever $j \in \{0, 1, 2, \ldots, n - 1\}$, then $f$ is a constant on the set $\{x \in Q: a_j < x < a_{j+1}\}$. The set $B$ is pointwise partially ordered; then $B$ is a Boolean algebra. Let $(S(n))$ and $(T(n))$ be as in [1], Section 5. From 2.7 and from the results of [1], Section 5 (cf. also [3], Section 7.6) it follows that the sets $(S(n))$ and $(T(n))$ are regular (with respect to $B$), but the set $\{(S(n)), (T(n))\}$ fails to be upper bounded in Conv$_0$ $B$. Hence Conv$_0$ $B$ has no greatest element.
5. DISJOINT SYSTEMS AND CHAINS IN Conv₀ B

For any partially ordered set \( P \) with the least element \( 0_p \) we define a subset \( P_1 \) of \( P \) to be disjoint if \( p > 0_p \) for each \( p \in P_1 \) and \( p \land q = 0_p \) whenever \( p \) and \( q \) are distinct elements of \( P_1 \). Denote

\[
D(P) = \sup \{ \text{card } A_i; A_i \in \mathcal{A} \},
\]

where \( \mathcal{A} \) is the system of all disjoint subsets of \( P \).

Now let \( \mathcal{A}_1 \) be the set of all linearly ordered subsets of a partially ordered set \( P \). Put

\[
L(P) = \sup \{ \text{card } A_i; A_i \in \mathcal{A}_1 \}.
\]

Let \( B \) be a Boolean algebra. The cardinals \( D(B) \) and \( L(B) \) were dealt with in several papers, cf., e.g., Pierce [9] and Monk [6].

In the present section it will be proved that for each infinite Boolean algebra \( B \) the relations

\[
D(B) \leq D(\text{Conv}_0 B), \quad D(B) \leq L(\text{Conv}_0 B)
\]

are valid. Also it will be shown that \( \text{Conv}_0 B \) has no atom.

Throughout this section we assume that \( B \) is an infinite Boolean algebra. A sequence \( (x_n) \) in \( B \) is said to be disjoint if \( x_n > 0 \) for each \( n \in N \) and \( x_n \land x_m = 0 \) whenever \( m \) and \( n \) are distinct positive integers.

5.1. Lemma. Let \( A = \{(x_n^i)\}_{i \in I} \) be a system of sequences in \( B \) such that \( x_n^{i(1)} \land x_n^{i(2)} = 0 \) whenever \( (n(1), i(1)) \) and \( (n(2), i(2)) \) are distinct elements of the set \( N \times I \). Then the set \( A \) is regular.

Proof. By way of contradiction, assume that \( A \) fails to be regular. Hence in view of 2.9 there are elements \( i(1), i(2), \ldots, i(m) \), subsequences \( (y_n^i) \) of \( (x_n^{i(t)}) \) \((t = 1, 2, \ldots, m)\) and an element \( 0 < b \in B \) such that

\[
b \leq y_n^1 \lor y_n^2 \lor \ldots \lor y_n^m
\]

is valid for each \( n \in N \).

In particular, we have

\[
b \leq y_1^1 \lor y_1^2 \lor \ldots \lor y_1^m.
\]

There exists \( n \in N \) such that for each \( t \in \{1, 2, \ldots, m\} \) and for each \( i \in I \) we have \( y_1^i \land x_n^i = 0 \). Let \( n \) have the just mentioned property. Then \( y_1^i \land y_n^{i(t)} = 0 \) for each \( t \), \( t(1) \in \{1, 2, \ldots, m\} \). Hence

\[
b = b \land (y_1^1 \land y_n^2 \land \ldots \land y_n^m) \leq
\]

\[
\leq (y_1^1 \land y_1^2 \land \ldots \land y_1^m) \land (y_n^1 \land y_n^2 \land \ldots \land y_n^m) = 0,
\]

which is a contradiction.

From 5.1, 2.6 and 2.8 we obtain:

5.2. Corollary. For each disjoint sequence \( (x_n) \) in \( B \) there exists \( \alpha(x_n) \in \text{Conv}_0 B \) such that \( \alpha(x_n) \) is generated by \( (x_n) \).

In the sequel, the notation \( \alpha(x_n) \) from 5.2 will be applied whenever the set \( \{(x_n)\} \) will be regular.
5.3. Lemma. Let \((x_n)\) and \((y_n)\) be disjoint sequences in \(B\) such that \(x_n \land y_m = 0\) for each \(m, n \in \mathbb{N}\). Then \(\alpha(x_n) \land \alpha(y_n) = d\).

Proof. By way of contradiction, assume that there exists \((s_n) \in \alpha(x_n) \land \alpha(y_n)\) such that \((s_n) \notin d\). Then without loss of generality we can assume that \(s_n > 0\) for each \(n \in \mathbb{N}\).

From \((s_n) \in \alpha(x_n)\) we infer that there is a subsequence \((s^i_n)\) of \((s_n)\) with \((s^i_n) \in [\delta(x_n)]\). Hence there are subsequences \((x^i_n)\), \((x^2_n)\), \ldots, \((x^k_n)\) of \((x_n)\) such that

\[
\tag{1}
s^i_n \leq x^1_n \lor x^2_n \lor \ldots \lor x^k_n \quad \text{for each} \quad n \in \mathbb{N}.
\]

We have \((s^i_n) \in \alpha(y_n)\). Hence by an analogous reasoning we deduce that there are subsequences \((y^i_n)\), \((y^2_n)\), \ldots, \((y^m_n)\) of \((y_n)\) and a subsequence \((s^j_n)\) of \((s^i_n)\) such that

\[
\tag{2}
s^j_n \leq y^1_n \lor y^2_n \lor \ldots \lor y^m_n \quad \text{for each} \quad n \in \mathbb{N}.
\]

In view of (1) and (2) the relation \(s^j_n = 0\) is valid for each \(n \in \mathbb{N}\), which is a contradiction.

Since \(B\) is infinite, there exists an infinite disjoint subset of \(B\).

5.4. Theorem. Let \(B\) be a Boolean algebra. Let \(X\) be an infinite disjoint subset of \(B\), \(\operatorname{card} X = \kappa\). Then there exists a system \(S_1 = \{\alpha^0_i\}_{i \in I}\) in \(\operatorname{Conv}_0 B\) such that

(i) the system \(S_1\) is disjoint and \(\operatorname{card} S_1 = \kappa\);

(ii) for each \(i \in I\), the 0-convergence \(\alpha^0_i\) is generated by a disjoint sequence.

Proof. Without loss of generality we can assume that we have \(X = \{x_n\}_{n \in \mathbb{N}}\), \(\operatorname{card} I = \kappa\), and that \(x^{(i(1))}_n \land x^{(i(2))}_n\) whenever \((i(1)), (n(1)) \neq (i(2)), (n(2))\). For each \(i \in I\) we put \(\alpha^0_i = \alpha(x^i_n)\). In view of 5.2, \(\alpha^0_i \in \operatorname{Conv}_0 B\) for each \(i \in I\). According to 5.3, the system \(S_1\) is disjoint in \(\operatorname{Conv}_0 B\).

We clearly have \(\operatorname{card} S_1 = \kappa\). Thus we obtain:

5.5. Corollary. Let \(B\) be an infinite Boolean algebra. Then \(D(B) \leq D(\operatorname{Conv}_0 B)\).

5.6. Lemma. Let \((x_n)\) be a disjoint sequence in \(B\). Assume that \(y_n = \bigvee_{m \geq n} x_m\) is valid for each \(n \in \mathbb{N}\). Then \((y_n)\) is decreasing and \(\bigwedge y_n = 0\).

Proof. Let \(z \in B\), \(z \leq y_n\) for each \(n \in \mathbb{N}\). First suppose that there exists \(n \in \mathbb{N}\) such that \(0 < z_1 = z \land x_n\). There exists \(z_2 \in B\) such that \(z_1 \land z_2 = 0\) and \(z_1 \lor z_2 = z\). Then \(z_1 \land x_m = 0\) for each \(m \in \mathbb{N}\setminus\{n\}\) and hence \(z_1 \land y_m = 0\) for each \(m > n\). Hence for \(m > n\) we have \(z \land y_m = z_2 \land y_m < z\), which is a contradiction.

Hence \(z \land x_n = 0\) for each \(n \in \mathbb{N}\). Thus \(z = z \land y_m = z \land (\bigvee_{m \geq n} x_m) = \bigvee_{m \geq n} (z \land x_m) = 0\) and therefore \(\bigwedge y_n = 0\). It is obvious that \((y_n)\) is decreasing.

5.7. Theorem. Let \(B\) be a complete Boolean algebra. Let \(X\) be an infinite disjoint subset of \(B\), \(\operatorname{card} X = \kappa\). Then there exists a system \(S_2 = \{\beta^0_i\}_{i \in I}\) in \(\operatorname{Conv}_0 B\) such that

(i) the system \(S_2\) is disjoint and \(\operatorname{card} S_2 = \kappa\);
(ii) for each \( i \in I \), the 0-convergence \( \beta_i^0 \) is generated by a decreasing sequence.

Proof. Let \( X \) be as in the proof of 5.4. For each \( i \in I \) and each \( n \in N \) put \( y_i^1 = \bigvee_{m \geq n} x_m^i \). Then for each \( i \in I \), \( \{ y_i^1 \} \) is a decreasing sequence and \( \bigwedge_n y_n^1 = 0 \) (cf. 5.6). Hence according to 2.12 there exists \( \beta_i^0 = \varepsilon(y_i^1) \) in \( \text{Conv}_0 \ B \). From the fact that \( X \) is a disjoint system and from 5.3 we infer that the system \( S_2 \) is disjoint. Clearly \( \text{card } S_2 = \kappa \).

5.8. Remark. The question whether the assumption of completeness of \( B \) can be cancelled in 5.7 remains open.

5.9. Theorem. Let \( B \) be a Boolean algebra. Let \( X \) be an infinite disjoint subset of \( B \), card \( X = \kappa \). Then there exists a system \( S_3 = \{ \beta_i^0 \}_{i \in I} \) in \( \text{Conv}_0 \ B \) such that \( S_3 \) is a chain and \( \text{card } S_3 = \kappa \).

Proof. Let \( X \) be expressed as in the proof of 5.4. Without loss of generality we may suppose that the set \( I \) is linearly ordered. For each \( i \in I \) put

\[
A_i = \{(x_j^i) : j \in I, j \leq i\}.
\]

Then for each \( i \in I \), the set \( A_i \) is regular. Moreover, if \( i(1) \) and \( i(2) \) are elements of \( I \) such that \( i(1) < i(2) \), then \( \varepsilon(A_{i(1)}) \subset \varepsilon(A_{i(2)}) \). (we denote \( \varepsilon(A_{i(1)}) = [\delta A_{i(1)}]^* \), and similarly for \( A_{i(2)} \).) Hence the system \( S_3 = \{ \varepsilon(A_i) \}_{i \in I} \) is a chain and \( \text{card } S_3 = \kappa \).

5.10. Corollary. Let \( B \) be an infinite Boolean algebra. Then \( D(B) \leq L(\text{Conv}_0 \ B) \).

5.11. Theorem. Let \( B \) be an infinite Boolean algebra. Then the partially ordered set \( \text{Conv}_0 \ B \) has no atom.

Proof. Let \( A \in \text{Conv}_0 \ B \). Then for each \((x_n) \in A\), the set \( \{(x_n)\} \) is regular, hence \( \varepsilon(x_n) \in \text{Conv}_0 \ B \) and \( \varepsilon(x_n) \leq A \). If \( \varepsilon(x_n) = d \) is valid for each \((x_n) \in A\), then \( A = d \).

Thus it suffices to verify that no principal element of \( \text{Conv}_0 B \) is an atom of \( \text{Conv}_0 B \).

To each sequence \((x_n)\) such that \( \{(x_n)\} \) is regular and \( \varepsilon(x_n) \neq d \) we shall assign in a constructive way a sequence \((z_n)\) such that \( \{(z_n)\} \) is regular and \( d < \varepsilon(z_n) < \varepsilon(x_n) \).

The construction proceeds as follows. Let \((x_n)\) have the above mentioned properties. We denote by \( n(1) \) the first positive integer \( n \) with \( x_n \neq 0 \). Since \( \{(x_n)\} \) is regular, there exists \( n \in N \) such that \( n > n(1) \), \( x_n \neq 0 \) and \( x_n \geq x_{n(1)} \); let \( n(2) \) be the least positive integer having this property. Then \( x_{n(1)} \land x_{n(2)} < x_{n(1)} \). Let \( y_1 \) be the relative complement of \( x_{n(1)} \land x_{n(2)} \) in the interval \([0, x_{n(1)}]\). We have \( 0 < y_1 \leq x_{n(1)} \) and \( y_1 \land x_{n(2)} = 0 \).

There exists \( n \in N \) such that \( n > n(2) \), \( x_n \neq 0 \) and \( x_n \neq x_{n(2)} \); let \( n(3) \) be the least \( n \) having this property. We construct \( y_2 \) by means of \( x_{n(2)} \) and \( x_{n(3)} \) in the same way as we did \( y_1 \) by means of \( x_{n(1)} \) and \( x_{n(2)} \). Then \( 0 < y_2 \leq x_{n(2)} \) and \( y_2 \land x_{n(3)} = 0 \). We have also \( y_1 \land y_2 = 0 \).

We proceed by the obvious induction, obtaining a disjoint sequence \((y_n) \) in \( B \).
such that $y_1 \leq x_{n(1)}, \ y_2 \leq x_{n(2)}, \ldots$. Hence $\alpha(y_n) \leq \alpha(x_n)$. For each $n \in N$ let $z_n = y_{2n}$ and $t_n = y_{2n + 1}$. Then $\{z_n\}$ is regular and $d < \alpha(z_n) \leq \alpha(x_n)$. Moreover, $(t_n) \in (x_n)$, but $(t_n)$ does not belong to $\alpha(z_n)$. Thus $\alpha(z_n) < \alpha(x_n)$.

References


Author's address: 040 01 Košice, Ždanovova 6, Czechoslovakia (Matematický ústav SAV, Dislokované pracovisko).