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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FOURTH ORDER
LINEAR DIFFERENTIAL EQUATION

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1. Introduction. The asymptotic behavior of solutions of the second order differential equation

$$(1) \quad y'' + p(t)y = 0$$

assuming either

$$(2) \quad p \in C^1[a, \infty), \quad p'(t) \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} p(t) = \infty$$

or

$$(3) \quad p \in C^1[a, \infty), \quad p'(t) \leq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} p(t) = 0$$

has been widely studied. (Reference [2] gives a history of the study of (1) with condition (2)). It is known, for example, that assuming condition (2), (1) has at least one non-trivial solution which tends to zero as t tends to infinity [3]. It need not be the case, however, that all solutions of (1) tend to zero [3]. Similarly, assuming condition (3), (1) has at least one non-trivial solution y such that $\limsup |y(t)| = \infty$. Again, however, it need not be the case that $\limsup |y(t)| = \infty$ for every non-trivial solution of (1).

We call a non-trivial solution $y(t)$ of the fourth order differential equation

$$(4_i) \quad y^{IV} + (-1)^i p(t)y = 0 \quad i = 1, 2$$

oscillatory if the set of zeros of $y(t)$ is not bounded above.

Assuming condition (2), Hastings and Lazer [1] show that unlike (1), every oscillatory solution of (4₁) tends to zero.

The purpose of this note is to study asymptotic behavior of solutions of (4₁) under condition (3) and (4₂) under conditions (2) and (3). In each case we show that stronger conclusions can be made for (4_i) than for (1).

2. Preliminary results. In this section we will give some simple results for (4_i) $i = 1, 2$ that will be used to prove our main theorems.

We define, for $y \in C^3[a, \infty)$ and $i = 1, 2$

$$(5_i) \quad G_i[y(t)] \equiv (y'''(t))^2 / (-1)^{i+1} p(t) - 2y(t)y''(t) + y'^2(t),$$

$$(6) \quad H_i[y(t)] \equiv (-1)^{i+1} p(t) y^2(t) - 2 y'(t) y'''(t) + y''^2(t),$$

and

$$(7) \quad F[y(t)] \equiv y'(t) y''(t) - y(t) y'''(t).$$

Lemma. Let $y_i(t)$ be a solution of (4_i) for $i = 1, 2$.

a) Assuming (3)

1. $G_1[y_1(t)]$ is increasing and $H_1[y_1(t)]$ is decreasing, while
2. $G_2[y_2(t)]$ is decreasing and $H_2[y_2(t)]$ is increasing.

b) Assuming (2), $G_2[y_2(t)]$ is increasing, $H_2[y_2(t)]$ is decreasing and $F[y_2(t)]$ is increasing.

Proof. The proof of each statement of the Lemma follows from the facts that for $i = 1, 2$

$$(8_i) \quad G_i[y_i(t)] = (-1)^i p'(t) (y_i'''(t)/p(t))^2,$$

$$(9_i) \quad H_i[y_i(t)] = (-1)^{i+1} p'(t) y_i^2(t)$$

and

$$(10) \quad F[y_2(t)] = (y_2''(t))^2 + p(t) y_2^2(t).$$

3. In this section we consider (4₁) assuming (3).

Theorem 1. If $p \in C^1[a, \infty)$, $p'(t) \leq 0$ and $\lim_{t \rightarrow \infty} p(t) = 0$, then every oscillatory solution of $y^{IV} - p(t) y = 0$ is unbounded.

Proof. Suppose $y(t)$ is an oscillatory solution of (4₁) that is bounded. Let $\{b_n\}$ be the divergent sequence along which y'' assumes its relative maximum or relative minimum values. Then by the Lemma and (6₁)

$$(11) \quad (y''(b_n))^2 \leq p(b_n) y^2(b_n) + (y'''(b_n))^2 = H[y(b_n)] \leq H[y(a)].$$

Hence y'' is bounded. Let $\{c_n\}$ be the divergent sequence along which y''' assumes its relative maximum or a relative minimum values. If n is such that $b_n > c_1$, then by (5₁) and the Lemma

$$(12) \quad 0 \leq G_1[y(c_1)] < G_1[y(b_n)] = -2 y(b_n) y''(b_n) + (y'(b_n))^2 \leq \\ \leq |2 y(b_n) y''(b_n)| + (y'(b_n))^2.$$

Assuming that y' is bounded, then from the monotone property of $G[y(t)]$ we conclude that $G[y(t)]$ is also bounded. Now

$$(y'''(c_n))^2/p(c_n) \leq (y'''(c_n))^2/p(c_n) + (y'(c_n))^2 = G_1[y(c_n)].$$

Since $G[y(t)]$ is bounded and

$$(13) \quad \lim_{t \rightarrow \infty} p(t) = 0$$

we conclude that

$$(14) \quad \lim_{t \rightarrow \infty} y'''(t) = 0.$$

Let $\{d_n\}$ be the divergent sequence along which y' assumes its relative maximum or

relative minimum values. Then

$$H_1[y(d_n)] = p(d_n) y^2(d_n) - 2 y'(d_n) y'''(d_n).$$

From the monotone property of $H_1[y(t)]$, the assumptions that y and y' are bounded, (13) and (14), we conclude

$$(15) \quad \lim_{t \rightarrow \infty} H[y(t)] = 0.$$

It now follows from (11) and (15) that

$$(16) \quad \lim_{t \rightarrow \infty} y''(t) = 0.$$

If n is large enough so that $b_n > c_2$ by the Lemma we have

$$(17) \quad 0 \leq G_1[y(c_1)] < G_1[y(c_2)] < G_1[y(b_n)] = -2 y(b_n) y''(b_n) + y'^2(b_n).$$

Hence from (17) and (16) we have

$$(18) \quad \limsup |y'(t)| = A \neq 0.$$

Suppose, without loss of generality, that $\limsup y'(t) = A$. Let $\{x_n\}$ be a divergent sequence such that $y'(x_n) = A/2$ and $\{t_n\}$ be a divergent sequence such that on $[t_n, x_n]$, $y'(x) \geq A/4$ with $y'(t_n) = A/4$. Then by the Mean Value Theorem there is an $s_n \in [t_n, x_n]$ so that

$$(19) \quad (A/4)/(x_n - t_n) = [y'(x_n) - y'(t_n)]/(x_n - t_n) = y''(s_n).$$

Because of (16), it follows from (19) that

$$(20) \quad \lim_{n \rightarrow \infty} (x_n - t_n) = \infty.$$

Hence, since

$$y(x_n) - y(t_n) = \int_{t_n}^{x_n} y'(t) dt \geq (A/4)(x_n - t_n)$$

either y is not bounded or y' is not bounded. Assume y' is not bounded and without loss of generality that $\limsup y' = \infty$. Let $\{s_n\}$ be a divergent sequence on which y' assumes a relative maximum and where

$$(21) \quad \lim_{n \rightarrow \infty} y'(s_n) = \infty$$

and $y'(s_n) > 1$ for all n . Let $\{t_n\}$ be a divergent sequence so that $y'(t_n) = 1$, $t_n < s_n$ and $y'(t) \geq 1$ for $t \in [t_n, s_n]$. By (11) y'' is bounded. Let $B > 0$ be such that $|y''(t)| < B$. Then

$$\begin{aligned} |y'(s_n) - 1| &= |y'(s_n) - y'(t_n)| = \\ &= \left| \int_{t_n}^{s_n} y''(t) dt \right| \leq \int_{t_n}^{s_n} |y''(t)| dt \leq B(s_n - t_n). \end{aligned}$$

Thus by (21)

$$(22) \quad \lim_{n \rightarrow \infty} (s_n - t_n) = \infty.$$

Hence

$$|y(s_n) - y(t_n)| = \left| \int_{t_n}^{s_n} y'(t) dt \right| \geq \int_{t_n}^{s_n} dt = s_n - t_n.$$

As a consequence of (22), y is not bounded.

4. It is known that either all or none of the solutions of (4₂) oscillate. Assuming (2) or (3) we get information about pairs of oscillatory solutions.

Theorem 2. *If $p \in C^1[a, \infty)$, $p'(t) \leq 0$, $\lim_{t \rightarrow \infty} p(t) = 0$ and $y^{IV} + py = 0$ is oscillatory, then there is a pair of linearly independent solutions that are unbounded.*

Proof. Let y be a solution of (4₂) such that $y(a) = y'(a) = 0$. From the Lemma, (5₂) and (6₂) $H_2[y(t)]$ is positive and $G_2[y(t)]$ is negative. If y is bounded then

$$(23) \quad \lim_{t \rightarrow \infty} p(t) y^2(t) = 0.$$

Integrating (9₂) shows $H_2[y(t)]$ to be bounded. As in Theorem 1, we let $\{b_n\}$ be the divergent sequence along which y'' assumes its relative maximum or relative minimum values. Then

$$(24) \quad H_2[y(b_n)] = -p(b_n) y^2(b_n) + y''^2(b_n).$$

Since $H_2[y(t)]$ is bounded and (23),

$$(25) \quad 0 < \limsup |y''| = A < \infty.$$

Letting $\{d_n\}$ be the divergent sequence along which y' assumes its relative maximum or relative minimum values, then

$$(26) \quad H_2[y(d_n)] = -p(d_n) y^2(d_n) - 2 y'(d_n) y'''(d_n).$$

Again using the boundedness of $H_2[y(t)]$ and (23),

$$(27) \quad 0 < \limsup |y' y'''| = B < \infty.$$

Suppose y''' does not go to zero. If

$$(28) \quad \lim_{t \rightarrow \infty} G_2[y(t)] = -\infty$$

then $G_2[y(b_n)] = -2 y(b_n) y''(b_n) + y'^2(b_n)$ yields

$$(29) \quad \limsup |y y''| = \infty.$$

Since we are assuming y is bounded (29) implies y'' is not bounded contrary to (25).

Let $\{c_n\}$ be the divergent sequence along which y''' assumes its relative maximum or relative minimum values. Then

$$(30) \quad G_2[y(c_n)] = y'''^2(c_n)/(-p(c_n)) + y'^2(c_n).$$

Since we are assuming y''' does not go zero, $\limsup y'''^2(c_n)/p(c_n) = \infty$. Thus if $G_2[y(t)]$ is bounded (30) implies $\limsup y'^2(c_n) = \infty$. Hence

$$(31) \quad \limsup |y'(c_n) y'''(c_n)| = \infty.$$

Now

$$(32) \quad H_2[y(c_n)] = -2 y'(c_n) y'''(c_n) + y''^2(c_n).$$

Thus by (31) and (25), $H_2[y(c_n)]$ is unbounded which is a contradiction. Hence

either

$$(33) \quad \lim_{t \rightarrow \infty} y'''(t) = 0$$

or y is unbounded. Assuming (33), since $H_2[y(t)]$ is increasing, it follows from (26) and (23) that $-2 y'(d_n) y'''(d_n)$ is bounded away from zero. Hence from (33)

$$(34) \quad \limsup |y'(t)| = +\infty.$$

Let $\{a_n\}$ be a divergent sequence such that

$$(35) \quad \lim_{n \rightarrow \infty} y'(a_n) = +\infty.$$

By (25) y'' is bounded. Assume $|y''(t)| < C$. Then for $x \in [a_n, a_n + 1]$

$$y'(x) - y'(a_n) = \int_{a_n}^x y''(t) dt > -C$$

or

$$(36) \quad y'(x) > y'(a_n) - C.$$

By The Mean Value Theorem

$$(37) \quad y(a_n + 1) - y(a_n) = y'(\varepsilon_n) \quad \text{for } a_n < \varepsilon_n < a_n + 1.$$

From (37), (36) and (35) it follows that y is not bounded. Since y_1 and y_2 satisfying $y_1(a) = y_1'(a) = y_1''(a) = 0$, $y_1'''(a) = 1$ and $y_2(a) = y_2'(a) = y_2''(a) = 0$ and $y_2'''(a) = 1$ are independent, the theorem follows.

Theorem 3. *If $p \in C[a, \infty)$, $p'(t) \geq 0$ and $\lim_{t \rightarrow \infty} p(t) = \infty$, then $y^{IV} + py = 0$ has a pair of oscillatory solutions that go to zero.*

Proof. We first show that the conclusion of the Theorem holds for any solution y of (4₂) for which

$$(38) \quad G_2[y(t)] < 0 \quad t \in [a, \infty),$$

$$(39) \quad H_2[y(t)] > 0 \quad t \in [a, \infty) \quad \text{and}$$

$$(40) \quad F[y(t)] < 0 \quad t \in [a, \infty).$$

Later we will show the existence of two such solutions which are linearly independent.

Suppose y is a solution of (4₂) that satisfies (38), (39) and (40). Suppose

$$(41) \quad \lim_{t \rightarrow \infty} y(t) \neq 0.$$

Let $\{t_n\}$ be the divergent sequence along which y assumes its relative maximum or relative minimum values. Then from (41) follows

$$(42) \quad \limsup p(t_n) y^2(t_n) = \infty.$$

From (6₂)

$$H_2[y(t_n)] = -p(t_n) y^2(t_n) + y''^2(t_n).$$

Hence since $H_2[y(t)]$ is positive and decreasing, (42) implies

$$(43) \quad \limsup y''^2(t_n) = +\infty.$$

Hence

$$(44) \quad \limsup |y(t_n) y''(t_n)| = +\infty .$$

But

$$(45) \quad G_2[y(t_n)] = -y'''(t_n)/p(t_n) - 2 y(t_n) y''(t_n) .$$

Since $G_2[y(t_n)]$ is negative and increasing, (44) implies

$$(46) \quad \limsup y'''(t_n)/p(t_n) = \infty ,$$

which in turn implies

$$(47) \quad \limsup |y'''(t_n)| = \infty .$$

But

$$(48) \quad F[y(t_n)] = -y(t_n) y'''(t_n) .$$

Since $F[y(t)]$ is negative and increasing, $F[y(t_n)]$ is bounded, while (47) implies

$$(49) \quad \limsup |y(t_n) y'''(t_n)| = \infty .$$

Hence $\lim_{t \rightarrow \infty} y(t) = 0$.

To show the existence of two linearly independent solutions of (4₂) that satisfy (38), (39) and (40), we use standard compactness arguments in the following way. Let Z_i for $i = 0, 1, 2, 3$ be solutions of (4₂) defined by the initial conditions

$$\begin{aligned} Z_i^{(j)}(a) &= \delta_{ij} = 0, \quad i \neq j, \\ &= 1, \quad i = j. \end{aligned}$$

For each integer $n > a$, let $b_{0n}, b_{2n}, b_{3n}, c_{1n}, c_{2n}, c_{3n}$ be numbers such that

$$(50) \quad b_{0n}^2 + b_{2n}^2 + b_{3n}^2 = 1, \quad c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$$

and

$$(51) \quad \begin{aligned} b_{0n} Z_0^{(i)}(n) + b_{2n} Z_2^{(i)}(n) + b_{3n} Z_3^{(i)}(n) &= 0 \quad \text{for } i = 0, 1, \\ c_{1n} Z_1^{(i)}(n) + c_{2n} Z_2^{(i)}(n) + c_{3n} Z_3^{(i)}(n) &= 0 \quad \text{for } i = 0, 1. \end{aligned}$$

Let

$$\begin{aligned} U_n(t) &= b_{0n} Z_0(t) + b_{2n} Z_2(t) + b_{3n} Z_3(t), \\ V_n(t) &= c_{1n} Z_1(t) + c_{2n} Z_2(t) + c_{3n} Z_3(t). \end{aligned}$$

By (50), there exists a sequence of integers $\{n_j\}$ such that the sequences $\{b_{in_j}\}$ and $\{c_{in_j}\}$ converge to numbers b_i and c_i respectively. Let u and v be solutions of (4₂) defined by

$$(52) \quad \begin{aligned} u(t) &= b_0 Z_0(t) + b_2 Z_2(t) + b_3 Z_3(t) \\ v(t) &= c_1 Z_1(t) + c_2 Z_2(t) + c_3 Z_3(t). \end{aligned}$$

From (50) it follows that neither $u(t)$ nor $v(t)$ are identically zero. Clearly the sequences $\{U_{n_j}(t)\}$ and $\{V_{n_j}(t)\}$ converge uniformly on compact intervals to $u(t)$ and $v(t)$ respectively. From (50) and the monotone properties of G_2, H_2 and F it follows that

$$(53) \quad G_2[y_n(x)] \leq 0, \quad H_2[y_n(x)] \geq 0 \quad \text{and} \quad F[y_n(x)] \leq 0 \quad \text{on} \quad [a, n]$$

for $y_n = u_n$ or v_n .

Thus

$$(54) \quad G_2[y(x)] \leq 0, \quad H_2[y(x)] \geq 0 \quad \text{and} \quad F[y(x)] \leq 0 \quad \text{on} \quad [a, \infty) \\ \text{for } y = u \quad \text{or} \quad v.$$

If u and v are linearly dependent then by (52)

$$u(t) = k v(t) = a_2 z_2(t) + a_3 z_3(t).$$

In that case by the Lemma and (7) $F[u(t)] > 0$ for $t > a$ contrary to (54). Hence u and v are linearly independent.

Added in proof. M. Švec, Sur le comportement asymptotique des intégrales de l'équation différentielle $y^{(4)} + Q(x)y = 0$, Czech. Math. J. 8(83) (1958), pp. 230–245, gets the conclusion of Theorem 3 assuming only that $p(x) \geq m > 0$. The author has been able to prove a theorem with the same conclusion as Theorem 2 assuming only that $0 < p(x) < M$.

References

- [1] S. P. Hastings and A. C. Lazer: On the Asymptotic Behavior of Solutions of the Differential Equation $y^{(4)} = p(t)y$, Czechoslovak Math. J., Vol. 18 (93) (1968) pp. 224–229.
- [2] J. W. Macki: Regular Growth and Zero-Tending Solutions, Lecture Notes in Mathematics, Vol. 1032, Springer, (1983) pp. 358–374.
- [3] J. W. Macki and J. S. Muldowney: The Asymptotic Behavior of Solutions to Linear Systems of Ordinary Differential Equations, Pacific J. Math., Vol. 33 (1970) pp. 693–706.

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