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A_r-CONDITION FOR TWO WEIGHT FUNCTIONS AND COMPACT IMBEDDINGS OF WEIGHTED SOBOLEV SPACES

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1. INTRODUCTION

In our paper we will establish some sufficient conditions on p, q and the weight functions v_0, v_1, w under which the compact imbedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{1.1}$$

takes place.

Similar problems have been studied by various authors in several papers. In [5] P. I. Lizorkin and M. Otelbaev gave some necessary and sufficient conditions on the weight functions v_0, v_1, w for the imbedding (1.1) to hold with Ω a bounded domain in $\mathbb{R}^N$ and $1 < p \leqq q < \infty$. Unfortunately, their conditions are rather difficult to verify. Similar conditions were found by U. K. Korenev [3] who studied the imbedding

$$W_0^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w)\,.$$

Other results for Ω an unbounded domain in $\mathbb{R}^N$ and $p = q$ were given by B. Opic in [7], and general sufficient and necessary conditions for weight functions v_0, v_1, w under which the imbedding (1.1) takes place were given by A. Avantaggiati in [1] and by B. Opic in [8].

Throughout the paper we will suppose that Ω is a domain in $\mathbb{R}^N$, $1 < r < p < Nr$, $1/q = 1/p - 1/Nr$, $r' = r/(r-1)$. By $\mathscr{W}(\Omega)$ we denote the set of weight functions, i.e. the set of all measurable, a.e. on Ω positive and finite functions.

For $u \in C^\infty(\Omega)$, $v_0, v_1 \in \mathscr{W}(\Omega)$ we denote

$$\|u\|_{\Omega;p,v_0,v_1} = \left(\int_\Omega |u(x)|^p\, v_0(x)\,\mathrm{d}x + \int_\Omega |\nabla u(x)|^p\, v_1(x)\,\mathrm{d}x\right)^{1/p} \tag{1.2}$$

$$\left(|\nabla u(x)|^p = \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}(x)\right|^p\right).$$

Let us define the weighted Sobolev space $W^{1,p}(\Omega; v_0, v_1)$ as the completion with respect to the norm (1.2) of the set $\tilde{C}^\infty(\Omega)$, which is the set of all functions $u \in C^\infty(\Omega)$ with a finite norm (1.2).

If $w \in \mathscr{W}(\Omega)$ we define the space $L^q(\Omega; w)$ as the set of all measurable functions u

defined on Ω with a finite norm

$$\|u\|_{\Omega;q,w} = \left(\int_\Omega |u(x)|^q\, w(x)\, dx\right)^{1/q}. \tag{1.3}$$

For $x \in \mathbb{R}^N$ we define $|x|_\infty = \max_{i=1,2,\dots,N} |x_i|$ and for $R > 0$ we put

$$Q_R(x) = \left\{ y \in \mathbb{R}^N;\ |x - y|_\infty < \frac{R}{2} \right\}.$$

(Sometimes we write shortly Q_R.) We will use the notation $w(Q) = \int_Q w(x)\, dx$, and $|Q|$ for the Lebesgue measure of the set Q. We wirte

$$(w, v) \in A_r(\Omega) \tag{1.4}$$

if $w, v \in \mathscr{W}(\Omega)$ and

$$\left(\frac{1}{|Q|} \int_{Q\cap\Omega} w(x)\, dx\right)\left(\frac{1}{|Q|} \int_{Q\cap\Omega} v^{1-r'}(x)\, dx\right)^{r-1} \leqq c < \infty \tag{1.5}$$

for all cubes Q in $\mathbb{R}^N$ (with c independent of Q). Let us present some results by P. Gurka and A. Kufner [2] and B. Opic [8] that we use in the next section.

Lemma 1.1. *Let $(w, v) \in A_r(Q_R)$ and $u \in C_0^\infty(Q_R)$. Then the inequality*

$$\left(\frac{1}{w(Q_R)} \int_{Q_R} |u(x)|^q\, w(x)\, dx\right)^{1/q} \leqq KR \left(\frac{1}{w(Q_R)} \int_{Q_R} |\nabla u(x)|^p\, v(x)\, dx\right)^{1/p} \tag{1.6}$$

holds with a constant $K > 0$ independent of u.

For the proof see [2].

Lemma 1.2. *For every domain G from a countable system of domains $\{G_n\}_{n=1}^\infty$ such that $G_n \subset G_{n+1} \subsetneqq \Omega$, $\Omega = \bigcup_{n=1}^\infty G_n$, let the imbedding*

$$W^{1,p}(G; v_0, v_1) \hookrightarrow\hookrightarrow L^q(G; w) \tag{1.7}$$

hold. Then

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{1.8}$$

if and only if

$$\lim_{n\to\infty} \left(\sup_{\|u\|_{\Omega;p,v_0,v_1} \leqq 1} \|u\|_{\Omega\setminus G_n;q,w} \right) = 0. \tag{1.9}$$

For the proof see [8].

2. MAIN THEOREMS

By $C^{0,1}$ we denote the class of all bounded domains in $\mathbb{R}^N$ with a Lipschitz boundary (in the sense of [4], Definition 5.5.6). First of all (Theorem 2.1) we will study the case when $\Omega = \mathbb{R}^N$ or $\Omega = \operatorname{int}(\mathbb{R}^N \setminus \tilde{\Omega})$ with $\tilde{\Omega} \in C^{0,1}$, and the weight functions may have singularities or degenerations only at infinity (that is, on any bounded domain $G \subset \Omega$ the weight functions are bounded from above and from below by positive constants, and thus we can use the fact that classical Sobolev imbedding theorems take place on G). Further (Theorem 2.2), we will consider the imbedding (1.1) on a domain $\Omega \in C^{0,1}$ containing zero, with weight functions which may have singularities or degenerations only at zero.

Theorem 2.1. *Let the following conditions be fulfilled*:

(2.1) $$\Omega = \operatorname{int}(\mathbb{R}^N \setminus \tilde{\Omega}) \quad with \quad \tilde{\Omega} \in C^{0,1} \quad or \quad \tilde{\Omega} = \emptyset\,;$$

(2.2) $$w, v \in \mathscr{W}(\Omega) \quad and \quad (w, v) \in A_r(\mathbb{R}^N \setminus Q_m(0)) \quad for\ some \quad m \in \mathbb{N}$$ $$such\ that \quad \tilde{\Omega} \subset Q_m(0)\,;$$

(2.3) $$\lim_{n\to\infty} \mathscr{A}_n = 0\,, \quad where \quad \mathscr{A}_n = \sup_{Q_1 \subset \mathbb{R}^N \setminus Q_n(0)} w(Q_1)^{(p-q)/p}\,;$$

(2.4) $$W^{1,p}(Q_{2n}(0) \setminus \tilde{\Omega};\, v, v) \hookrightarrow\hookrightarrow L^q(Q_{2n}(0) \setminus \tilde{\Omega};\, w) \quad for \quad n = m, m+1, \ldots.$$

Then we have the imbedding

(2.5) $$W^{1,p}(\Omega;\, v, v) \hookrightarrow\hookrightarrow L^q(\Omega;\, w)\,.$$

Proof. Let us fix n, $n > m$, and put

$$J_n = \{(\tfrac{3}{4}k_1, \tfrac{3}{4}k_2, \ldots, \tfrac{3}{4}k_n);\ k_1, \ldots, k_N \text{ integers},\ |(k_1, \ldots, k_N)|_\infty > n\}$$

(it is easy to see that $J_n \cap Q_{(3/2)n}(0) = \emptyset$). The set $\mathbb{R}^N \setminus Q_{(3/2)n}(0)$ is covered by cubes $Q^{n,j} = Q_1(j)$, $j \in J_n$. This covering has a finite multiplicity c_N (i.e. $\sum_{j\in J_n} \chi_{Q^{n,j}} \leqq c_N$, where $\chi_{Q^{n,j}}$ is the characteristic function of $Q^{n,j}$). By a standard method we find a partition of unity $\{\phi_j^n\}_{j\in J_n \cup\{0\}}$ submitted to the covering of $\mathbb{R}^N$ by the open sets $Q_{(3/2)n}(0)$, $Q^{n,j}$, $j \in J_n$, with the following properties:

(i) $\phi_0^n \in C_0^\infty(Q_{(3/2)n}(0))$, $\phi_j^n \in C_0^\infty(Q^{n,j})$, $j \in J_n$;
(ii) $\sum_{j\in J_n\cup\{0\}} \phi_j^n(x) = 1$, $x \in \mathbb{R}^N$;
(iii) $0 \leqq \phi_j^n(x) \leqq 1$, $j \in J_n \cup \{0\}$, $x \in \mathbb{R}^N$;
(iv) there exists a constant $M > 1$ such that $|\partial\phi_j^n/\partial x_i| \leqq M$, $i = 1, 2, \ldots, N$, $j \in J_n$.

By Lemma 1.2 it is sufficient to verify condition (1.9) for $u \in \tilde{C}^\infty(\Omega)$. We set $G_n = Q_{2n}(0) \cap \Omega$. Using the fact that $\sum_{j\in J_n} \phi_j^n(x) = 1$ holds for all $x \in \mathbb{R}^N \setminus Q_{2n}(0) = \Omega \setminus G_n$ we have

(2.6) $$\|u\|^q_{\Omega\setminus G_n;q,w} = \int_{\Omega\setminus G_n} |u(x)|^q\, w(x)\, dx =$$ $$= \int_{\Omega\setminus G_n} \Big|\sum_{j\in J_n} \phi_j^n(x)\, u(x)\Big|^q w(x)\, dx \leqq c_N^{q-1} \sum_{j\in J_n} \int_{Q^{n,j}} |\phi_j^n(x)\, u(x)|^q\, w(x)\, dx\,.$$

By Lemma 1.1 and properties (iii), (iv) we obtain

(2.7) $$\int_{Q^{n,j}} |\phi_j^n(x)\, u(x)|^q\, w(x)\, dx \leqq K^q[w(Q^{n,j})]^{(p-q)/p} \times$$ $$\times \Big(\int_{Q^{n,j}} |\nabla(\phi_j^n(x)\, u(x))|^p\, v(x)\, dx\Big)^{q/p} \leqq K_0 \mathscr{A}_n \Big(\int_{Q^{n,j}} (|u|^p + |\nabla u|^p)\, v\, dx\Big)^{q/p}\,,$$

with $K_0 = K^q M^q N^{q/p}\, 2^{q(p-1)/p}$.

Substituting (2.7) into the last term in (2.6) and using the inequality $q/p > 1$ we get

(2.8) $$\|u\|^q_{\Omega\setminus G_n;q,w} \leqq c_N^{q-1} K_0 \mathscr{A}_n \sum_{j\in J_n} \left(\int_{Q^{n,j}} \left(|u|^p + |\nabla u|^p\right) v \, dx\right)^{q/p} \geqq$$
$$\leqq c_N^{q-1+q/p} K_0 \mathscr{A}_n \|u\|^q_{\Omega;p,v,v}$$

and the constant $c_N^{q-1+q/p} K_0$ does not depend on n.

The assumption (2.3) together with the inequality (2.8) imply the desired condition (1.9), which proves the theorem.

Theorem 2.2. *Let the following conditions be fulfilled:*

(2.9) $\Omega \in C^{0,1}\,, \quad 0 \in \Omega\,;$

(2.10) $w, v_0, v_1 \in \mathscr{W}(\Omega)$ *and there exist* $\eta > 0$ *and* $c > 0$ *such that*

$$v_1(x)\,|x|^{-p} \leqq c\, v_0(x) \quad \textit{for a.e.} \quad x \in \Omega\,, \quad |x| < \eta\,;$$

(2.11) $(w, v_1) \in A_r(Q_{2^{-n_0}}(0))$ *for some* $n_0 \in \mathbb{N}\,;$

(2.12) $\lim_{n\to\infty} \mathscr{B}_n = 0\,,$ *where*

$$\mathscr{B}_n = \sup\left\{|Q|^{1/N}\, w(Q)^{(p-q)/p};\ Q = Q_{|y|_\infty}(y),\ |y|_\infty = 2^{-j},\ j \geqq n\right\};$$

(2.13) $W^{1,p}(\Omega \setminus Q_{2^{-n}}(0);\ v_0, v_1) \subset\subset L^q(\Omega \setminus Q_{2^{-n}}(0);\ w)\,,\ n \geqq n_0\,.$

Then we have the imbedding

(2.14) $$W^{1,p}(\Omega; v_0, v_1) \subset\subset L^q(\Omega; w)\,.$$

Proof. For $j \in \mathbb{N}$ let $\mathscr{S}_j$ be the set of all points $k = (k_1, \ldots, k_N) \in \mathbb{R}^N$ such that $|k_i| = 2^{-j}$ for some $i \in \{1, 2, \ldots, N\}$ and $|k_l| = m \cdot 2^{-j-1}$ for $l = 1, 2, \ldots, i-1, i+1, \ldots, N$ and $m = 0, 1, 2$ (obviously $|k|_\infty = 2^{-j}$ for $k \in \mathscr{S}_j$). Let us fix $n \in \mathbb{N}$, $n > n_0$, and denote $\mathscr{S}(n) = \bigcup_{j=n}^{\infty} \mathscr{S}_j$. The system

$$\mathscr{G}_n = \{Q_{|k|_\infty}(k);\ k \in \mathscr{S}(n)\}$$

covers the set $Q_{2^{-n}}(0) \setminus \{0\}$ and has a finite multiplicity c_N, so we have a partition of unity $\{\phi^n_k\}_{k\in\mathscr{S}(n)}$ submitted to the covering $\mathscr{G}_n$ with the following properties:

(i) $\phi^n_k \in C_0^\infty(Q_{|k|_\infty}(k)\},\ k \in \mathscr{S}(n);$

(ii) $0 \leqq \phi^n_k \leqq 1,\ k \in \mathscr{S}(n);$

(iii) there exists a constant $K_1 > 0$ such that

$$\left|\frac{\partial}{\partial x_i}\, \phi^n_k(x)\right| \leqq K_1 |k|_\infty^{-1}\,, \quad k \in \mathscr{S}(n)\,;$$

(iv) $\sum_{k\in\mathscr{S}(n)} \phi^n_k(x) = 1,\ x \in Q_{2^{-n}}(0) \setminus \{0\},$

and the sum has at most c_N nonzero summands for every x.

After a standard calculation (using Lemma 1.1) we get the estimate

$$(2.15)\qquad \int_{Q_{2^{-n}}(0)} |u(x)|^q\, w(x)\,dx \leqq$$
$$\leqq c_0 \mathscr{B}_n \Big(\sum_{k\in\mathscr{S}(n)} \int_{Q_{|k|_\infty}(k)} \big[|u(x)|^p\, |k|_\infty^{-p}\, v_1(x) + |\nabla u(x)|^p\, v_1(x)\big]\,dx\Big)^{q/p}$$

(with a constant c_0 independent of n and u).

Further, for a.e. $x \in Q_{|k|_\infty}(k)$ we have

$$|k|_\infty^{-p}\, v_1(x) < (\tfrac{3}{2})^p\, |x|_\infty^{-p}\, v_1(x) < (\tfrac{3}{2})^p\, N^{p/2} |x|^{-p}\, v_1(x)$$

and this inequality together with condition (2.10) implies that

$$(2.16)\qquad |k|_\infty^{-p}\, v_1(x) < (\tfrac{3}{2})^p\, N^{p/2} c\, v_0(x)$$

for a.e. $x \in Q_{|k|_\infty}(k)$, $k \in \mathscr{S}(n)$, and n large enough $(n > -(\ln 2)^{-1} \ln (\frac{2}{3} N^{-1/2}\eta))$.

Now, using (2.15), (2.16) and condition (2.12), we easily verify condition (1.9) (where we put $G_n = \Omega \setminus Q_{2^{-n}}(0)$). The theorem is proved.

Remark 2.1. If $\Omega = \mathbb{R}^N$ and the weight functions have singularities or degenerations at infinity and zero then the validity of imbedding (2.14) can be investigated by using Lemma 1.2 where we take $G_n = Q_{2n}(0) \setminus Q_{2^{-n}}(0)$. Condition (1.9) can be verified in the following way:

$$\|u\|^q_{\mathbb{R}^N \setminus G_n; q, w} = \|u\|^q_{\mathbb{R}^N \setminus Q_{2n}(0); q, w} + \|u\|^q_{Q_{2^{-n}}(0); q, w}\,;$$

the first term on the right hand side can be estimated as in Theorem 2.1, and the second as in Theorem 2.2.

Remark 2.2. It is easy to see that the assertion of Lemma 1.2 remains valid if condition (1.7)–(1.9) are replaced by

$$(1.7)^*\qquad W^{1,p}(G; v_0, v_1) \hookrightarrow L^q(G; w)\,,$$

$$(1.8)^*\qquad W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)\,,$$

and

$$(1.9)^*\qquad \lim_{n\to\infty} \Big(\sup_{\|u\|_{\Omega; p, v_0, v_1} \leqq 1} \|u\|_{\Omega \setminus G_n; q, w} \Big) < \infty\,,$$

respectively. Using this modified „Lemma 1.2*" we can obtain theorems for continuous imbeddings similar to Theorems 2.1 and 2.2.

Example 2.1. Let α, β be real numbers and let Ω be as in Theorem 2.1, condition (2.1). For $x \in \Omega$ we put

$$w(x) = \begin{cases} |x|^\alpha & \text{if } |x| \geqq 1 \\ 1 & \text{if } |x| < 1 \end{cases}, \quad v(x) = \begin{cases} |x|^\beta & \text{if } |x| \geqq 1 \\ 1 & \text{if } |x| < 1 \end{cases}.$$

By standard calculation we obtain that condition (2.3) is fulfilled if and only if $\alpha > 0$. Then $(w, v) \in A_r(\Omega \setminus Q_m(0))$ (for each m such that $\tilde{\Omega} \subset Q_m(0)$) if and only if

$$\beta < N(r-1) \quad \text{and} \quad \alpha \leqq \beta$$

or

$$\beta \geqq N(r-1) \quad \text{and} \quad \alpha \leqq N(r-1) .$$

The functions w, v are both continuous and positive on Ω and $1/N > 1/p - 1/q$, so the imbedding (2.4) is an easy consequence of the Sobolev imbedding theorem. Thus, using Theorem 2.1, we can conclude that the imbedding (2.5) takes place if

$$0 < \alpha \leqq \beta < N(r-1)$$

or

$$0 < \alpha \leqq N(r-1) \leqq \beta .$$

Using the fact that $|x|^{\eta_1} \leqq |x|^{\eta_2}$ for $|x| \geqq 1$ and $\eta_1 \leqq \eta_2$, we can omit the condition $\alpha > 0$, so we have imbedding (2.5) under the conditions

$$0 < \beta < N(r-1) \quad \text{and} \quad \alpha \leqq \beta$$

or

$$\beta \geqq N(r-1) \geqq \alpha .$$

Example 2.2. Let α, β be real numbers and $\Omega \in C^{0,1}$, $0 \in \Omega$. For $x \in \Omega$ we put

$$w(x) = |x|^{\alpha}, \quad v_0(x) = |x|^{\beta-p}, \quad v_1(x) = |x|^{\beta} .$$

It is easy to prove that condition (2.11) is fulfilled if and only if

$$-N < \alpha , \quad \beta \leqq \alpha \quad \text{and} \quad \beta < N(r-1) ,$$

while condition (2.12) is fulfilled if and only if

$$\alpha < -N + \frac{p}{q-p} = -N + \frac{Nr-p}{Nr} .$$

The imbedding (2.13) takes place by the same argument as in the previous example. So we can conclude that the imbedding (2.14) holds if

$$-N < \alpha < -N + \frac{Nr-p}{Nr} \quad \text{and} \quad \beta \leqq \alpha .$$

Using the fact that $|x|^{\eta_1} \geqq |x|^{\eta_2}$ for $|x| \leqq 1$ and $\eta_1 \leqq \eta_2$ we can omit the condition $\alpha < -N + (Nr - p)/Nr$, so we have the imbedding (2.14) under the conditions

$$-N < \alpha , \quad \beta \leqq \alpha \quad \text{and} \quad \beta < -N + \frac{Nr-p}{Nr} .$$

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