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ASYMPTOTIC STABILITY FOR A CLASS OF INTEGRODIFFERENTIAL EQUATIONS

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1. Introduction. We shall be concerned with the asymptotic stability of a class of abstract semi-linear Volterra equations which involve infinite delay and are of the form

$$\dot{x}(\phi)(t) + A\,x(\phi)(t) = \int_{-\infty}^{t} g(t-s, x(\phi)(s))\,\mathrm{d}s \tag{1.1a}$$

$$x(\phi)(0) = \phi(\theta)\,, \quad \theta \in (-\infty, 0]\,, \quad \phi \in C_A\,. \tag{1.1b}$$

Here $-A$ is the infinitesimal generator of a strongly continuous semigroup on a Banach space X and $g(\ ,\ )$ is in general an unbounded nonlinear mapping of $R \times X$ to X. We let X_A denote the Banach space obtained by imposing the graph norm on $D(A)$ and specify C_A to be the space of bounded uniformly continuous functions from the interval $(-\infty, 0]$ to X_A.

In this setting we can consider partial integrodifferential equations. Such equations can arise in a variety of applications including problems treating heat flow with memory, [1], [10], [11], [12]. Equation (1.1a–b) and equations related to it have attracted considerable attention in recent years and the interested reader is referred to [2], [3], [4], [5], [9], [15], [16], [17], [18].

One of the distinguishing features of this study is that we are able to obtain asymptotic stability results when the initial history space is the space of bounded uniformly continuous functions.

2. The results. In what follows X is a general Banach space and A is a one to one closed linear operator such that $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear transformations, $\{T(t) \mid t \geqq 0\}$. We further require that there exist positive constant ω such that

$$\|T(t)\| \leqq \mathrm{e}^{-\omega t} \quad \text{for} \quad t > 0\,. \tag{2.1}$$

We make the domain of A, $D(A)$, into a Banach space by imposing the graph norm $\|\ \|_A$, i.e.

$$\|x\|_A = \|Ax\| \quad \text{for} \quad x \in D(A)\,. \tag{2.2}$$

We place the following assumptions or $g(,)$

(2.3) $g(,): R^+ \times X_A \to X$ is continuous, continuously differentiable with respect to the first place and there exist positive constants K_1, K_2, α, β such that

$$\|g(s, x_1) - g(s, x_2)\| \leqq e^{-\alpha s} K_1 \|x_1 - x_2\|_A ,$$
$$\|g_1(s, x_1) - g_1(s, x_2)\| \leqq e^{-\beta s} K_2 \|x_1 - x_2\|_A , \quad g_1 = \partial g / \partial t .$$

In [2] we establish the following global existence theorem.

Theorem 1. *Let $-A$ be the infinitesimal generator of a strongly continuous semigroup of linear operators $\{T(t) \mid t \geqq 0\}$ and assume that $g(,): R^+ \times X_A \to X$ satisfies* (2.3). *If $T > 0$ and $\phi \in C_A$, then there exists a unique function $x(\phi)$: $(-\infty, T] \to X$ such that*

$$\dot{x}(\phi)(t) + A\, x(\phi)(t) = \int_{-\infty}^{t} g(t - s, x(\phi)(s))\, ds , \quad t > 0 ,$$
$$x(\phi)(\theta) = \phi(\theta) , \quad \theta \in (-\infty, 0] .$$

We shall utilize two lemmas. The first lemma appears in [10, p. 485] and was extensively used in [16] in the context of abstract Volterra integrodifferential equations.

Lemma 2.4. *Let $K( ): [0, T] \to X$ be such that $K( )$ is continuously differentiable if $q( ): [0, T] \to X$ is defined via*

$$q(t) = \int_0^t T(t - s) K(s)\, ds$$

then $q(t) \in D(A)$, q is continuously differentiable and

$$q'(t) = A\, q(t) + K(t) = \int_0^t T(t - s) K'(s)\, ds + T(t) K(0) .$$

We introduce a scalar integral operator as follows

(2.5) Let $h_1( )$, $h_2( )$ and $h_3( )$ be nonnegative scalar functions such that

$$\int_0^c h_1(s)\, ds < \infty , \quad \int_{-\infty}^c h_2(s)\, ds \leqq \infty \quad \text{and} \quad \int_{-\infty}^c h_3(s)\, ds < \infty$$

for all c; let $p( )$ be a continuous scalar function on $[0, \infty)$. If $y( )$ is a continuous nonnegative on $(-\infty, T]$ function then the integral operator S is given by,

$$(Sy)(t) = p(t) + \int_0^t h_1(t - s)\, y(s)\, ds +$$
$$+ \int_0^t h_1(t - s) \int_{-\infty}^s h_2(s - r)\, y(r)\, dr\, ds + \int_{-\infty}^t h_3(t - s)\, y(s)\, ds .$$

Our next lemma provides a comparison principle and it is adapted from a result of R. Redlinger [14].

Lemma 2.6. *Let S be defined via* (2.5) *and act on continuous nonnegative functions $y( )$ and $z( )$ for $t \in (-\infty, T)$ $(0 < T \leqq \infty)$. If*

$$y(t) - (Sy)(t) < z(t) - (Sz)(t) \quad \textit{for} \quad 0 \leqq t < T$$

and

$$y(t) < z(t) \quad for \quad -\infty < t < 0$$

then

$$y(t) < z(t) \quad for \quad -\infty < t < T.$$

Proof. If we set $t_0 = \inf\{t: y(t) = z(t)\}$, we may observe that $z(t_0) = y(t_0) \leqq \leqq y(t_0) + (Sz)(t_0) - (Sy)(t_0) < z(t_0)$ and reach a contradiction.

We are now in a position to prove our main result.

Theorem 2. *Let A and $g(\ ,\ )$ satisfy the conditions of Theorem 1 and assume that $K_1/\alpha + K_2/\beta\omega + K_1/\omega < 1$. If $\phi, \psi \in C_A$ then there exists $\delta < \min\{\omega, \alpha, \beta\}$: and $D \geqq \|\phi - \psi\|_{C_A}$ such that*

$$\|x(\phi)(t) - x(\psi)(t)\|_A \leqq De^{-\delta t}.$$

Proof. The theory of abstract semilinear equations implies that solutions to (1.1) have variation of parameters representation,

$$x(\phi)(t) = T(t)\,\phi(0) + \int_0^t T(t-s) \int_{-\infty}^s g(s-r, x(\phi)(r))\,dr\,ds.$$

Thus we may apply Lemma 2.4 to observe that

$$A\,x(\phi)(t) = A\,T(t)\,\phi(0) + T(t)\left(\int_{-\infty}^0 g(-s, x(\phi)(s))\,ds + \right.$$
$$+ \int_0^t T(t-s)\left\{g(0, x(\phi)(s)) + \int_{-\infty}^s g_1(s-r, x(\phi)(r))\,dr\right\} ds -$$
$$- \int_{-\infty}^t g(t-s, x(\phi)(s))\,ds.$$

Consequently, we may estimate,

$$\|x(\phi)(t) - x(\psi)(t)\|_A \leqq \|A\,T(t)\,\phi(0) - A\,T(t)\,\psi(0)\| + \tag{2.7}$$
$$+ \left\|T(t)\left(\int_{-\infty}^0 g(-s, x(\phi)(s))\,ds - g(-s, x(\psi)(s))\,ds\right\| + \right.$$
$$+ \int_0^t \|T(t-s)\{g(0, x(\phi)(s)) - g(0, x(\psi)(s)) +$$
$$+ \int_{-\infty}^s (g_1(s-r, x(\phi)(r)) - g_1(s-r, x(\psi)(r))\,dr\}\|\,ds +$$
$$+ \int_{-\infty}^t \|g(t-s, x(\phi)(s)) - g(t-s, x(\psi)(s))\|\,ds \leqq$$
$$\leqq \|\phi(0) - \psi(0)\|_A\,e^{-\omega t} + K_1/\alpha\|\phi - \psi\|_{C_A}\,e^{-\omega t} +$$
$$+ \int_0^t e^{-\omega(t-s)} K_1 \|x(\phi)(s) - x(\psi)(s)\|_A\,ds +$$
$$+ \int_0^t e^{-\omega(t-s)} \int_{-\infty}^s K_2 e^{-\beta(s-r)} \|x(\phi)(r) - x(\psi)(r)\|_A\,dr\,ds +$$
$$+ \int_{-\infty}^t e^{-\alpha(t-s)} K_1 \|x(\phi)(s) - x(\psi)(s)\|_A\,ds.$$

We let $\delta > 0$ and set $z(t) = De^{-\delta t}$. We now observe that

$$(1 + K_1/\alpha)\,\|\phi - \psi\|_{C_A}\,e^{-\omega t} + \int_0^t e^{-\omega(t-s)} K_1\,z(s)\,ds + \tag{2.8}$$
$$+ \int_0^t e^{-\omega(t-s)} \int_{-\infty}^s K_2^{-\beta(s-r)}\,z(r)\,dr\,ds + \int_{-\infty}^t e^{-\alpha(t-s)} K_1\,z(s)\,ds \leqq$$
$$\leqq (1 + K_1/\alpha)\,\|\phi - \psi\|_{C_A}\,e^{-\omega t} + \{K_1/(\omega - \delta) +$$
$$+ K_2/(\omega - \delta)(\beta - \delta) + K_1/(\alpha - \delta)\}\,De^{-\delta t}.$$

Thus if $\delta > 0$ and $D > 0$ are chosen so that

(2.9)
$$D \geqq (1 + K_1/\alpha) \|\phi - \psi\|_{C_A} + \\ + \{K_1/(\omega - \delta) + K_2/(\omega - \delta)(\beta - \delta) + K_1/(\alpha - \delta\} De^{(\omega - \delta)t}.$$

We see the right hand side of (2.8) can be bounded by $z(t) = De^{-\delta t}$. Combining (2.7) and (2.9) we have,

$$\|x(\phi)(t) - x(\psi)(t)\|_A - \int_0^t e^{-\omega(t-s)} K_1 \|x(\phi)(s) - x(\psi)(s)\|_A \, ds - \\ - \int_0^t e^{-\omega(t-s)} \int_{-\infty}^s K_2 e^{-\beta(s-r)} \|x(\phi)(r) - x(\psi)(r)\|_A \, dr \, ds - \\ - \int_{-\infty}^t K_1 e^{-\alpha(t-s)} \|x(\phi)(s) - x(\psi)(s)\|_A \, ds \leqq \\ \leqq (1 + K_1/\alpha) \|\phi - \psi\|_{C_A} e^{-\omega t} \leqq z(t) - \int_0^t e^{-\omega(t-s)} K_1 \, z(s) \, ds - \\ - \int_0^t e^{-\omega(t-s)} \int_{-\infty}^s e^{-\beta(s-r)} K_2 \, z(r) \, dr \, ds - \int_{-\infty}^t e^{-\alpha(t-s)} K_1 \, z(s) \, ds.$$

We now apply Lemma 2.6 to deduce

$$y(t) = \|x(\phi)(t) - x(\psi)(t)\|_A < z(t) = De^{-\delta t}$$

and reach our conclusion.

3. An example. We consider the following parabolic integrodifferential equation:

(3.1a)
$$W_t(x, t) - W_{xx}(s, t) = \int_{-\infty}^t F(t - s, W_{xx}(x, s)) \, ds$$

(3.1b)
$$W(x, \theta) = \phi(x, \theta) \quad x \in (0, \pi), \ \theta \in (-\infty, 0]$$

(3.1c)
$$0 = W(0, t) = W(\pi, t) \quad t > 0.$$

The function $F: R^+ \times R \to R$ is continuous and is continuously differentiable in the first variable. We further stipulate that $F(,)$ and $F_1 (,)$ be Lipschitz continuous in the second place and decay exponentially, i.e., there exist positive constants K_1, K_2, α, β, such that

$$|F(s, x) - F(s, y)| \leqq e^{-\alpha s} K_1 |x - y|,$$
$$|F_1(s, x) - F_1(s, y)| \leqq e^{-\beta s} K_2 |x - y|.$$

We work in the Banach space $X = L^2(0, \pi)$ and define $A: X \to X$ pointwise as

$$(Au)(x) = u''(x)$$

when domain

$$D(A) = H_0^1(0, \pi) \cap H^2(0, \pi).$$

It is well known [8], that $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t) \mid t > 0\}$ which satisfies (2.1) with any $\omega < 1$. The initial function $\phi(x, \cdot)$ is required to belong to C_A. If the nonlinear function $g(,): R^+ \times X_A \to R$ is defined by

$$g(s, u) = F(s, -Au)$$

is not difficult to verify that (2.3) is satisfied.

In the abstract setting (3.1a–c) assumes the form:

(3.2a)
$$\dot{x}(\phi)(t) + A\, x(\phi)(t) = \int_{-\infty}^t g(t - s, x(\phi)(s)) \, ds,$$

(3.2b) $$x(\phi)(\theta) = \phi(\theta) \quad \theta \in (-\infty, 0].$$

Because $-A$ is the infinitesimal generator of an analytic semigroup the regularity theory of inhomogeneous linear equations [13] guarantees that (3.2a–b) provides classical solutions to (3.1a–c). Theorem 2 provides criteria which guarantees the exponential convergence of $\|x(\phi)(t) - x(\psi)(t)\|_A$. The interpolation theory for generators of analytic semigroups can be used to show the exponential convergence of $\sup_{u\in(0,\pi)} |x(\phi)(t)(u) - x(\psi)(t)(u)|$, cf [13].

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