Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

LOCAL SOLUTION OF PARABOLIC EQUATIONS WITH STRONGLY INCREASING NONLINEARITY BY THE ROTHE METHOD

VOLKER PLUSCHKE, Halle

(Received September 3, 1986)

1. INTRODUCTION

The Rothe method (also called semidiscretization in time) is a frequently used method to prove existence of solutions of time dependent differential equations by constructing approximates (cf. the monographs by Kačur [1] and Rektorys [5] with many references, moreover e.g. [3], [4]). To construct weak solutions to nonlinear equations in general there is necessary a global Lipschitz condition. This restricts the increase of nonlinearity. The aim of the present paper is to replace the global Lipschitz condition by a local one. Thus an arbitrary increase of nonlinearity is possible. The way to this end provide estimates of the approximates u_j in $W_p^1(G)$ with $p > 2$ for equations with linear principle part. By means of an embedding theorem thus we get an estimate for $\sup |u|$ for sufficiently small $t \leqq \hat{T}$.

2. THE PROBLEM AND ASSUMPTIONS

Let $G \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial G, I = [0, T]$, $Q_T = G \times I$, $\Gamma = \partial G \times I$. We consider the problem

$$(1) \qquad Au + \frac{\partial u}{\partial t} = f(x, t, u) \quad \text{in} \quad Q_T\,,$$

$$(2) \qquad u = 0 \quad \text{on} \quad \Gamma\,,$$

$$(3) \qquad u(x, 0) = U_0(x) \quad x \in G$$

where

$$(4) \qquad Au = -\sum_{i,k=1}^{N} \frac{\partial}{\partial x_k}\left(a_{ik}(x, t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i(x, t)\frac{\partial u}{\partial x_i}\,.$$

After subdivision of the time interval I by points $t_j = jh$, $j = 0, \ldots, n$, the problem (1)–(3) is replaced by a sequence of linear elliptic boundary value problems

$$(1_j) \qquad A_j u_j + \frac{1}{h}(u_j - u_{j-1}) = f_j \quad \text{in} \quad G\,,$$

(2_j) $$u_j = 0 \quad \text{on} \quad \partial G\,,$$

(3_0) $$u_0 = U_0\,,$$

$j = 1, \ldots, n$. Here denotes $f_j = f(x, t_j, u_{j-1})$, and A_j stands for a differential operator of the form (4) with coefficients $a_{ik,j} = a_{ik}(x, t_j)$ and $a_{i,j} = a_i(x, t_j)$, respectively. For abbreviation we write $\Delta u_j = u_j - u_{j-1}$. Starting from (3_0) by solution of the problems $(1_j), (2_j)$, $j = 1, \ldots, n$, we obtain the approximates

(5) $$\tilde{u}^n(x, t) = \frac{t_j - t}{h}\, u_{j-1}(x) + \frac{t - t_{j-1}}{h}\, u_j(x)\,, \quad t \in [t_{j-1}, t_j]$$

and

(6) $$\bar{u}^n(x, t) = \begin{cases} u_j(x)\,, & t \in (t_{j-1}, t_j] \\ U_0(x)\,, & t \leqq 0 \end{cases}$$

of the solution of (1)–(3). By the help of investigations of convergence of $\tilde{u}^n$ and $\bar{u}^n$ for n tending to infinity we prove existence of such a solution.

First we formulate the assumptions:

(i) Let $G \subset \mathbb{R}^N$, $N \geqq 2$, be a simply connected, bounded domain; $\partial G \in C^1$.

(ii) $U_0 \in \mathring{W}_p^1(G)$, $A_0 U_0 \in L_p(G)$ with $p > N$.

(iii) Let $a_{ik} \in C(\bar{Q}_T)$, $a_i \in L_\infty(G)$ with ellipticity condition

(7) $$a\xi^2 \leqq \sum_{i,k=1}^{N} a_{ik}\xi_i\xi_k \leqq b\xi^2 \quad \forall \xi \in \mathbb{R}^N\,,$$

and assume for a.a. $x \in G$ and every $t, t' \in I$ the Lipschitz conditions

(8) $$|a_{ik}(x, t) - a_{ik}(x, t')| \leqq l_1 |t - t'|\,,$$

(9) $$|a_i(x, t) - a_i(x, t')| \leqq l_2 |t - t'|\,.$$

(iv) Let $f(\cdot, 0, U_0) \in L_p(G)$, moreover let the condition

(10) $$|f(x, t, u) - f(x, t', u')| \leqq P(x)\,|t - t'| + Q(x)\,|u - u'|\,,$$

$P \in L_p(G)$, $Q \in L_\infty(G)$, on $Q_T \times [-R, R]$ be satisfied for a sufficient large R.

Here, $W_p^1(G)$ denotes the well-known Sobolev space of L_p-integrable functions with first order derivatives. The minimal length of the interval $[-R, R]$ will be fixed later and follows by (23). However, the bounded domain $Q_T \times [-R, R]$ of (10) allows an arbitrary increase of f with respect to u. Of course, this influences the domain of existence of a solution u.

To derive the a priori estimates for some time we will use a suitable continuation of the right-hand side f to $Q_T \times (-\infty, \infty)$, e.g.

$$f^R(x, t, u) = \begin{cases} f(x, t, -R) & \text{for} \quad u < -R\,, \\ f(x, t, u) & \text{for} \quad -R \leqq u \leqq R\,, \\ f(x, t, R) & \text{for} \quad R < u\,, \end{cases} \qquad (x, t) \in Q_T\,.$$

f^R satisfies condition (10) on $Q_T \times (-\infty, \infty)$, which yields

(11) $$\|f^R(\cdot, t, u) - f^R(\cdot, t', u')\|_p \leqq l_3 |t - t'| + l_4 \|u - u'\|_p$$

$\forall t, t' \in I, \forall u, u' \in L_p(G)$. $\|\cdot\|_p$ denotes the norm in $L_p(G)$. Furthermore, we use $\langle \cdot, \cdot \rangle$ for the duality between $L_p(G)$ and $L_q(G)$, $p^{-1} + q^{-1} = 1$, as well as the notation

$$\|u\|_{1,p} = (\int_G \sum_{i=1}^{N} |u_{x_i}|^p \, dx)^{1/p} ,$$

what, by the Friedrichs inequality, represents an equivalent norm in $\mathring{W}_p^1(G)$.

Replacing the right-hand sides f_j by f_j^R we look for weak solutions of (1_j), (2_j), (3_0). In the weak formulation we write $A_j(\cdot, \cdot)$ for the bilinear form on $W_p^1(G) \times W_q^1(G)$ generated by the operator A_j (cf. [6]).

Lemma 1. *Let assumptions* (i)–(iv) *be fulfilled. Then for* $h \leqq h_0$ *there exist unique weak solutions* $u_j \in \mathring{W}_p^1(G)$ *of* $(1_j), (2_j), j = 1, \ldots, n$, *satisfying the relations*

$$(12_j) \qquad A_j(u_j, v) + \frac{1}{h} \langle \Delta u_j, v \rangle = \langle f_j^R, v \rangle \quad \forall v \in \mathring{W}_q^1(G) .$$

Proof. Because of (iv) and (11) $u_{j-1} \in L_p(G)$ implies $f_j^R \in L_p(G) \subset (\mathring{W}_q^1(G))^*$. Thus the assertion of the lemma is a consequence of [6], Corollary 7.4. ■

Now we derive the a priori estimates for u_j.

3. A PRIORI ESTIMATES FOR THE APPROXIMATES

The fundamental tool to establish a priori estimates are the relations (12_j) with test functions of the form $v = |u_j|^{p-2} u_j$.

Lemma 2. *Let* $u \in \mathring{W}_p^1(G)$, $p > 2$. *Then* $v = |u|^{p-2} u$ *belongs to* $\mathring{W}_q^1(G)$, $1/p + 1/q = 1$, *and it holds*

$$(13) \qquad \frac{\partial}{\partial x_i} (|u|^{p-2} u) = (p-1) |u|^{p-2} u_{x_i}$$

in the weak sense.

Proof. Obviously, we have $v \in L_q(G)$ because of

$$(14) \qquad \| |u|^{p-2} u \|_q = \|u\|_p^{p-1} .$$

Moreover, $|u|^{p-2} u_{x_i} \in L_q(G)$. We now introduce the "cut-off" functions

$$u^+(x) = \begin{cases} u(x) & \text{for} \quad u > 0 \\ 0 & \text{else} \end{cases}, \qquad u^-(x) = \begin{cases} u(x) & \text{for} \quad u < 0 \\ 0 & \text{else} . \end{cases}$$

According to [2], p. 84, we have $u^+ \in \mathring{W}_p^1(G)$, $u^- \in \mathring{W}_p^1(G)$. Then for every $\varphi \in W_p^1(G)$ we get

$$\int_G [|u|^{p-2} u \varphi_{x_i} + (p-1) |u|^{p-2} u_{x_i} \varphi] \, dx =$$

$$= \int_G \left[(u^+)^{p-1} \varphi_{x_i} + \frac{\partial}{\partial x_i} ((u^+)^{p-1}) \varphi \right] dx -$$

$$-\int_G \left[(-u^-)^{p-1}\varphi_{x_i} + \frac{\partial}{\partial x_i}((-u^-)^{p-1}\varphi\right] dx =$$

$$= \int_{\partial G} (u^+)^{p-1}\varphi \, d\sigma_i - \int_{\partial G} (-u^-)^{p-1}\varphi \, d\sigma_i = 0 \quad \forall\varphi \in W_q^1(G).$$

This yields (13) and the assertion of the lemma. ■

For the following estimations we use the Young's inequality

$$ab \leqq \varepsilon^p \frac{a^p}{p} + \varepsilon^{-q}\frac{b^q}{q}, \quad p^{-1} + q^{-1} = 1, \tag{15}$$

and the generalization

$$\prod_{i=1}^r a_i \leqq \sum_{i=1}^r p_i^{-1} a_i^{p_i}, \quad \sum_{i=1}^r p_i^{-1} = 1. \tag{16}$$

The proof of (16) may be performed by induction with (15).

Lemma 3. *For $j = 0, \ldots, n$ and $u \in W_p^1(G)$ the estimate*

$$A_j(u, |u|^{p-2} u) \geqq k_1 \| |u|^{(p-2)/2} u\|_{1,2}^2 - k_2 \|u\|_p^p \tag{17}$$

holds true with constants k_1, k_2 independent of j and n.

Proof. Owing to (13) and (iii) we have

$$A_j(u, |u|^{p-2} u) = \int_G \sum_{i,k=1}^N a_{ik,j} u_{x_i} (|u|^{p-2} u)_{x_k} dx + \int_G \sum_{i=1}^N a_{i,j} u_{x_i} |u|^{p-2} u \, dx \geqq$$

$$\geqq (p-1)\int_G |u|^{p-2} \sum_{i,k=1}^N a_{ik,j} u_{x_i} u_{x_k} dx - \max_i \sup_{Q_T} \text{ess} |a_i| \sum_{i=1}^N \left|\int_G u_{x_i} |u|^{p-2} u \, dx\right| \geqq$$

$$\geqq (p-1)\, a \int_G \sum_{i=1}^N |u|^{p-2} u_{x_i}^2 dx - c \sum_{i=1}^N \left|\int (|u|^{(p-2)/2} u_{x_i}) (|u|^{(p-2)/2} u) \, dx\right|.$$

Since $(|u|^{(p-2)/2} u)_{x_i} = \frac{1}{2}p|u|^{(p-2)/2} u_{x_i}$ and $\| |u|^{(p-2)/2} u\|_2 = \|u\|_p^{p/2}$ we can continue

$$A_j(u, |u|^{p-2} u) \geqq \frac{4p-4}{p^2} a \| |u|^{(p-2)/2} u\|_{1,2}^2 -$$

$$- \frac{2}{p} c \sum_{i=1}^N |\langle(|u|^{(p-2)/2} u)_{x_i}, |u|^{(p-2)/2} u\rangle| \geqq$$

$$\geqq \frac{4p-4}{p^2} a \| |u|^{(p-2)/2} u\|_{1,2}^2 - \frac{2c}{p} \sum_{i=1}^N \|(|u|^{(p-2)/2} u)_{x_i}\|_2 \|u\|_p^{p/2} \geqq$$

$$\geqq \frac{(4p-4)\, a - \varepsilon}{p^2} \| |u|^{(p-2)/2} u\|_{1,2}^2 - \frac{c^2 N}{\varepsilon} \|u\|_p^p.$$

Here Young's inequality (15) for $p = q = 2$ has been used. For fixed $\varepsilon < (4p-4)\, a$ this provides us the assertion (17). ■

For the proof of Lemma 5 we still need an auxiliary estimate. We derive

$$|A_v(w, |u|^{p-2} u)| = |\int_G \sum_{i,k=1}^{N} a_{ik,v} w_{x_i}(p-1) |u|^{p-2} u_{x_k} \,dx| +$$
$$+ |\int_G \sum_{i=1}^{N} a_{i,v} w_{x_i} |u|^{p-2} u \,dx| \leqq$$
$$\leqq \frac{2p-2}{p} \max_{i,k} \sup_G |a_{ik,v}| \int_G \sum_{i,k=1}^{N} |w_{x_i}| |(|u|^{(p-2)/2} u)_{x_k}| |u|^{(p-2)/2} \,dx +$$
$$+ \max_i \operatorname{sup\,ess}_G |a_{i,v}| \int_G \sum_{i=1}^{N} |w_{x_i}| |u|^{p-1} \,dx .$$

Applying (16) to the first item with $p_1 = p$, $p_2 = 2$, $p_3 = 2p/(p-2)$ $(p_1^{-1} + p_2^{-1} + p_3^{-1} = 1)$ and to the second item with $p_1 = p$, $p_2 = q = p/(p-1)$ we obtain

$$(18) \qquad |A_v(w, |u|^{p-2} u)| \leqq (\max_{i,k} \sup_G |a_{ik,v}|^p + \max_i \operatorname{sup\,ess}_G |a_{i,v}|^p) \|w\|_{1,p}^p +$$
$$+ \varepsilon \| |u|^{(p-2)/2} u\|_{1,2}^2 + k_3(\varepsilon) \|u\|_p^p .$$

$k_3(\varepsilon)$ depends besides on ε only on p and N.

For estimation of the solutions u_j of the elliptic problems there will be used the following

Lemma 4. *Suppose $u \in \mathring{W}_p^1(G)$, $p > N$, is a solution of*

$$A_j(u, v) = \langle F, v \rangle \quad \forall v \in \mathring{W}_q^1(G) .$$

Then the estimates

$$(19) \qquad \|u\|_{1,p} \leqq K_1 \|F\|_p ,$$
$$(20) \qquad \|u\|_{C(\bar{G})} \leqq K_2 \|F\|_p$$

hold true.

Proof. Inequality (19) is proved in [6], Theorem 6.3. Moreover, Sobolev's embedding theorem (cf. e.g. [2], p. 77, [6], p. 225) implies $\mathring{W}_p^1(G) \subset C(\bar{G})$ and the estimate $\|u\|_{C(\bar{G})} \leqq K \|u\|_{1,p}$. Due to (19) this yields (20). ■

The following estimate for $\Delta u_j = u_j - u_{j-1}$ is an essential base for the proof of convergence of the approximation scheme.

Lemma 5. *There exists a constant $M_1(t)$ independent of h and j, such that for $h \leqq h_0 \leqq t \leqq T$*

$$(21) \qquad \|\Delta u_j\|_p \leqq M_1(t)\, h \quad \forall j : jh \leqq t .$$

Proof. First we will estimate Δu_1. For this purpose we have to choose $v = |\Delta u_1|^{p-2} \Delta u_1$ in the relation (12_1) and obtain after subtraction of $A_1(U_0, v)$ and owing to (14)

$$A_1(\Delta u_1, |\Delta u_1|^{p-2} \Delta u_1) + \frac{1}{h} \langle \Delta u_1, |\Delta u_1|^{p-2} \Delta u_1 \rangle =$$
$$= \langle f_1^R, |\Delta u_1|^{p-2} \Delta u_1 \rangle - A_1(U_0, |\Delta u_1|^{p-2} \Delta u_1) \leqq$$

$$\leqq \|f_1^R\|_p \, \|\Delta u_1\|_p^{p-1} + \left|A_0(U_0, |\Delta u_1|^{p-2} \Delta u_1)\right| +$$
$$+ \left|A_1(U_0, |\Delta u_1|^{p-2} \Delta u_1) - A_0(U_0, |\Delta u_1|^{p-2} \Delta u_1)\right| .$$

In virtue of (17), the Lipschitz conditions (8), (9), and (11), together with (18) we obtain

$$k_1 \| \, |\Delta u_1|^{(p-2)/2} \Delta u_1\|_{1,2}^2 - k_2\|\Delta u_1\|_p^p + \frac{1}{h} \|\Delta u_1\|_p^p \leqq$$
$$\leqq (\|f^R(\cdot, 0, U_0)\|_p + l_3 h)\| \, \Delta u_1\|_p^{p-1} +$$
$$+ \|A_0 U_0\|_p \, \|\Delta u_1\|_p^{p-1} + (l_1^p + l_2^p) \, h^p \|U_0\|_{1,p}^p +$$
$$+ \varepsilon \| \, |\Delta u_1|^{(p-2)/2} \Delta u_1\|_{1,2}^2 + k_3(\varepsilon) \, \|\Delta u\|_p^p .$$

Hence, for $\varepsilon = k_1$ we have

$$\|\Delta u_1\|_p^p \leqq (\|f^R(\cdot, 0, U_0)\|_p + l_3 t + \|A_0 U_0\|_p) \, h \|\Delta u_1\|_p^{p-1} +$$
$$+ c_1 h^{p+1} + c_2 h \|\Delta u_1\|_p^p ,$$

which yields by means of (16) with $p_1 = p$, $p_2 = q = p/(p-1)$

$$\|\Delta u_1\|_p^p \leqq \frac{1}{p} (\|f^R(\cdot, 0, U_0)\|_p + l_3 t + \|A_0 U_0\|_p)^p \, h^p + \frac{p-1}{p} \|\Delta u_1\|_p^p +$$
$$+ c_1 t h^p + c_2 h_0 \|\Delta u_1\|_p^p .$$

For fixed h_0 where $pc_2 h_0 < 1$ we have thus proved

$$\|\Delta u_1\|_p \leqq (1 - pc_2 h_0)^{-1/p} (\|f^R(\cdot, 0, U_0)\|_p + l_3 t + \|A_0 U_0\|_p) \, h \leqq K_3(t) \, h , \tag{22}$$

that is the assertion for $j = 1$.

To estimate now Δu_j for $j = 2, \ldots, n$ take the difference $(12_j)-(12_{j-1})$ and set $v = |\Delta u_j|^{p-2} \Delta u_j$,

$$A_j(u_j, |\Delta u_j|^{p-2} \Delta u_j) - A_{j-1}(u_j, |\Delta u_j|^{p-2} \Delta u_j) +$$
$$+ A_{j-1}(u_j, |\Delta u_j|^{p-2} \Delta u_j) - A_{j-1}(u_{j-1}, |\Delta u_j|^{p-2} \Delta u_j) +$$
$$+ \frac{1}{h} \langle \Delta u_j - \Delta u_{j-1}, |\Delta u_j|^{p-2} \Delta u_j \rangle = \langle \Delta f_j^R, |\Delta u_j|^{p-2} \Delta u_j \rangle .$$

Similarly as above from (17), (18), (8), (9), and (14) we conclude

$$k_1 \| \, |\Delta u_j|^{(p-2)/2} \Delta u_j\|_{1,2}^2 - k_2 \|\Delta u_j\|_p^p + \frac{1}{h} \|\Delta u_j\|_p^p - \frac{1}{h} \|\Delta u_{j-1}\|_p \, \|\Delta u_j\|_p^{p-1} \leqq$$
$$\leqq \|\Delta f_j^R\|_p \, \|\Delta u_j\|_p^{p-1} + (l_1^p + l_2^p) \, h^p \|u_j\|_{1,p}^p +$$
$$+ \varepsilon \| \, |\Delta u_j|^{(p-2)/2} \Delta u_j\|_{1,2}^2 + k_3(\varepsilon) \, \|\Delta u_j\|_p^p ,$$

where for $\varepsilon = k_1$ due to (11) and (16) follows

$$\frac{1}{h} \|\Delta u_j\|_p^p - \frac{1}{hp} \|\Delta u_{j-1}\|_p^p - \frac{p-1}{hp} \|\Delta u_j\|_p^p \leqq$$

$$\leqq \frac{1}{p}(l_3 h + l_4 \|\Delta u_{j-1}\|_p)^p + \left(\frac{p-1}{p} + k_2 + k_3\right)\|\Delta u_j\|_p^p +$$
$$+ (l_1^p + l_2^p)\, h^p \|u_j\|_{1,p}^p .$$

In virtue of Lemma 4 and (11) $\|u_j\|_{1,p}$ can be estimated by

$$\|u_j\|_{1,p} \leqq \frac{1}{h}\|\Delta u_j\|_p + \|f_j^R\|_p \leqq$$
$$\leqq \frac{1}{h}\|\Delta u_j\|_p + \|f^R(\cdot, 0, U_0)\|_p + l_3 h_0 + l_4 \sum_{s=1}^{j-1} \|\Delta u_s\|_p .$$

Taking into account the inequality $(a_1 + \ldots + a_m)^p \leqq m^{p-1}(a_1^p + \ldots + a_m^p)$ derived from the Hölder inequality in spaces l_p, l_q we continue in the estimation with

$$\frac{1}{h}\|\Delta u_j\|_p^p - \frac{1}{h}\|\Delta u_{j-1}\|_p^p \leqq 2^{p-1} l_3^p h^p + 2^{p-1} l_4^p \|\Delta u_{j-1}\|_p^p +$$
$$+ (p - 1 + k_2 p + k_3 p)\|\Delta u_j\|_p^p + 4^{p-1} p (l_1^p + l_2^p)\,[\|\Delta u_j\|_p^p +$$
$$+ h^p \|f^R(\cdot, 0, U_0)\|_p^p + l_3^p h_0^p h^p + (j-1)^{p-1} h^p l_4^p \sum_{s=1}^{j-1} \|\Delta u_s\|_p^p] \leqq$$
$$\leqq c_3 h^p + c_4 \|\Delta u_{j-1}\|_p^p + c_5 \|\Delta u_j\|_p^p + c_6 t_{j-1}^{p-1} h \sum_{s=1}^{j-1} \|\Delta u_s\|_p^p .$$

Hence, by summation for $j = 2, \ldots, i$ follows

$$\|\Delta u_i\|_p^p \leqq \|\Delta u_1\|_p^p + c_3(i-1)\, h h^p + (c_4 + c_5)\, h \sum_{j=1}^{i} \|\Delta u_j\|_p^p +$$
$$+ c_6 t_i^{p-1} h^2 \sum_{j=2}^{i} \sum_{s=1}^{j-1} \|\Delta u_s\|_p^p \leqq$$
$$\leqq \|\Delta u_1\|_p^p + c_3 t h^p + (c_4 + c_5 + c_6 t^p) \sum_{j=1}^{i} \|\Delta u_j\|_p^p\, h .$$

Recall the estimate (22) for Δu_1 thus we have established the inequalities

$$\|\Delta u_i\|_p^p \leqq (K_3^p + c_3 t)\, h^p + L \sum_{j=1}^{i} \|\Delta u_j\|_p^p\, h , \quad i = 1, \ldots, n .$$

Applying the discret form of Gronwall's lemma (see [1], p. 29) we obtain for $h \leqq$ $\leqq h_0 < 1/L$ the desired estimate

$$\|\Delta u_i\|_p^p \leqq \frac{K_3 + c_3 t}{1 - L h_0} \exp\left(\frac{L t}{1 - L h_0}\right) h^p = (M_1(t))^p\, h^p . \quad \blacksquare$$

Owing to the embedding theorem u_j belongs to $C(\bar{G})$. Now Lemma 5 gives us the possibility to estimate the maximum of $|u_j|$ since from Lemma 4 and (12_j) we obtain

$$\|u_j\|_{C(\bar{G})} \leqq K_2 \left(\frac{1}{h}\|\Delta u_j\|_p + \|f_j^R\|_p\right) \leqq$$

$$\leqq K_2(M_1(t) + \|f^R(\cdot, 0, U_0)\|_p + l_3 t + l_4 \|U_0 - u_{j-1}\|_p) \leqq$$
$$\leqq K_2(M_1(t) + \|f^R(\cdot, 0, U_0)\|_p + l_3 t + l_4 M_1(t)\, t) =: M(t) \quad \forall t_j \leqq t\,.$$

Assume now

$$R > R_0 = \max\{\|U_0\|_{C(\bar{G})};\ K_2(\|A_0 U_0\|_p + 2\|f(\cdot, 0, U_0)\|_p)\}\,. \tag{23}$$

Because of $\lim_{t \to 0} M(t) = K_2(\|A_0 U_0\|_p + 2\|f^R(\cdot, 0, U_0)\|_p)$ and $f^R(x, 0, U_0) = f(x, 0, U_0)$ we can find a $\hat{T} > 0$ with $M(\hat{T}) = R$. Consequently, we have proved

Theorem 1. *Let assumptions* (i)–(iv) *be fulfilled with* $R > R_0$, R_0 *given by* (23). *Then there exists a* $\hat{T} > 0$ *independent of the subdivision such that for* $h \leqq h_0$

$$\|u_j\|_{C(\bar{G})} \leqq R \quad \forall j: jh \leqq \hat{T}\,.$$

Remark. It is obvious from (23) that R_0 does not depend on any Lipschitz constants. However, the time $\hat{T}$ as well as the maximal step length h_0 especially depend on l_4 and thus in general on the choosen R.

In the following we always suppose $R > R_0$ and $jh \leqq \hat{T}$. For the approximates (5) and (6) which are piecewise linear and piecewise constant interpolations with respect to time, respectively, making use of the notation $\tau_h u(x, t) = u(x, t - h)$ Lemma 5 yields the relations

$$\|\tilde{u}^n(\cdot, t) - \tilde{u}^n(\cdot, t')\|_p \leqq M_1 |t - t'| \tag{24}$$

$$\left\| \frac{\partial^-}{\partial t} \tilde{u}^n(\cdot, t) \right\|_p \leqq M_1 \tag{25}$$

$$\|\tilde{u}^n(\cdot, t) - \bar{u}^n(\cdot, t)\|_p \leqq M_1 h_n \tag{26}$$

$$\|\tilde{u}^n(\cdot, t) - \tau_h \bar{u}^n(\cdot, t)\|_p \leqq M_1 h_n \tag{27}$$

for $t, t' \in \hat{I}$, $\hat{I} = [0, \hat{T}]$, with $M_1 = M_1(\hat{T})$. Moreover, by the above considerations the uniform boundedness

$$|\tilde{u}^n(x, t)| \leqq R\,, \quad |\bar{u}^n(x, t)| \leqq R \quad \forall t \in \hat{I},\ \forall x \in \bar{G} \tag{28}$$

and

$$\|\tilde{u}^n(\cdot, t)\|_{1,p} \leqq M_2\,, \quad \|\bar{u}^n(\cdot, t)\|_{1,p} \leqq M_2 \quad \forall t \in \hat{I} \tag{29}$$

has been proved. In this new notation from (12_j) we obtain an integral relation defined for all $t \in \hat{I}$

$$\bar{A}^n(\bar{u}^n(\cdot, t), v) + \left\langle \frac{\partial^-}{\partial t} \tilde{u}^n(\cdot, t), v \right\rangle = \langle \bar{f}^n, v \rangle \quad \forall v \in \mathring{W}_q^1(G)\,, \tag{30^n}$$

where $\bar{f}^n = f(x, \bar{t}^n, \tau_h \bar{u}^n)$ and $\bar{A}^n$ is the operator with coefficients being piecewise constant with respect to t, i.e. t is replaced by $\bar{t}^n$.

The aim of this section is to investigate the behaviour of $\tilde{u}^n$, $\bar{u}^n$ for $n \to \infty$, that means if the step length h tends to zero.

Lemma 6. *There exists a function $u \in C(\hat{I}, L_p(G))$ such that*

$$\tilde{u}^n \to u \quad \text{in} \quad C(\hat{I}, L_p(G)) \quad \text{holds for} \quad n \to \infty . \tag{31}$$

Proof. By construction we have $\tilde{u}^n \in L_\infty(\hat{I}, \mathring{W}_p^1(G))$ with weak derivatives $\tilde{u}_t^n \in L_\infty(\hat{I}, L_p(G))$, consequently formula (13) can be applied also on differentiation with respect to t. We regard two different subdivisions of I into m and n subintervals, respectively. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{u}^m - \tilde{u}^n\|_p^p = \frac{\mathrm{d}}{\mathrm{d}t} \langle \tilde{u}^m - \tilde{u}^n, |\tilde{u}^m - \tilde{u}^n|^{p-2} (\tilde{u}^m - \tilde{u}^n)\rangle =$$
$$= \left\langle \frac{\partial}{\partial t} (\tilde{u}^m - \tilde{u}^n), |\tilde{u}^m - \tilde{u}^n|^{p-2} (\tilde{u}^m - \tilde{u}^n) \right\rangle +$$
$$+ \left\langle \tilde{u}^m - \tilde{u}^n, (p-1) |\tilde{u}^m - \tilde{u}^n|^{p-2} \frac{\partial}{\partial t} (\tilde{u}^m - \tilde{u}^n) \right\rangle =$$
$$= p \left\langle \frac{\partial}{\partial t} (\tilde{u}^m - \tilde{u}^n), |\tilde{u}^m - \tilde{u}^n|^{p-2} (\tilde{u}^m - \tilde{u}^n) \right\rangle .$$

Integration over $t \leqq \hat{T}$ and estimation with (25) yields

$$\|\tilde{u}^m(\cdot, t_0) - \tilde{u}^n(\cdot, t_0)\|_p^p \leqq \tag{32}$$
$$\leqq \int_0^{t_0} \left[p \left\langle \frac{\partial}{\partial t} (\tilde{u}^m - \tilde{u}^n), |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n) \right\rangle + \right.$$
$$\left. + 2pM_1 \| |\tilde{u}^m - \tilde{u}^n|^{p-2} (\tilde{u}^n - \tilde{u}^n) - |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n)\|_q \right] \mathrm{d}t .$$

To estimate the last term in the integral we take up an auxiliary consideration. Due to (13) it holds for $u, v \in L_p(G)$

$$\| |u|^{p-2} u - |v|^{p-2} v\|_q =$$
$$= \left\| \int_0^1 \frac{\partial}{\partial s} [|su + (1-s) v|^{p-2} (su + (1-s) v) \, \mathrm{d}s \right\|_q =$$
$$= \|\textstyle\int_0^1 (p-1) |su + (1-s) v|^{p-2} (u - v) \, \mathrm{d}s\|_q \leqq$$
$$\leqq (p-1) \|(|u| + |v|)^{p-2} |u - v| \|_{q=p/(p-1)} \leqq$$
$$\leqq (p-1) [\textstyle\int_G (|u| + |v|)^{p(p-2)/(p-1)} |u - v|^{p/(p-1)} \, \mathrm{d}x]^{(p-1)/p} .$$

Then Hölder's inequality with $p' = (p-1)/(p-2)$, $q' = p - 1$ implies

$$\| |u|^{p-2} u - |v|^{p-2} v\|_q \leqq (p-1) (\|u\|_p + \|v\|_p)^{p-2} \|u - v\|_p .$$

Using this relation we continue in estimation of (32) by

$$\|\tilde{u}^n(\cdot, t_0) - \tilde{u}^n(\cdot, t_0)\|_p^p \leqq$$

$$\leqq p \int_0^{t_0} \Bigg[\left\langle \frac{\partial}{\partial t} (\tilde{u}^m - \tilde{u}^n), |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n) \right\rangle +$$

$$+ 2(p-1) M_1 (\|\tilde{u}^m - \tilde{u}^n\|_p + \|\bar{u}^m - \bar{u}^n\|_p)^{p-2} \|(\tilde{u}^m - \bar{u}^m) + (\bar{u}^n - \tilde{u}^n)\|_p \Bigg] \mathrm{d}t .$$

We form now the difference of integral relations $(31^m)-(31^n)$ inserting the test function $v(\cdot, t) = |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n) \in \mathring{W}_q^1(G)$, replace thus the first item in our estimate and obtain with (26)

$$(33) \quad \|\tilde{u}^m(\cdot, t_0) - \tilde{u}^n(\cdot, t_0)\|_p^p \leqq$$
$$\leqq p \textstyle\int_0^{t_0} [\langle \bar{f}^m - \bar{f}^n, |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n)\rangle -$$
$$- \bar{A}^n(\bar{u}^m - \bar{u}^n, |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n))] \, \mathrm{d}t +$$
$$+ p \textstyle\int_0^{t_0} |(\bar{A}^n - \bar{A}^m)(\bar{u}^m, |\bar{u}^m - \bar{u}^n|^{p-2} (\bar{u}^m - \bar{u}^n))| \, \mathrm{d}t +$$
$$+ 2p(p-1) M_1^2 \textstyle\int_0^{t_0} (2\|\tilde{u}^m - \tilde{u}^n\|_p + M_1(h_m + h_n))^{p-2} (h_m + h_n) \, \mathrm{d}t .$$

For simplicity let us denote the integrands in the right-hand side of (33) by S_1, S_2, and S_3, respectively. Due to (26) we have

$$(34) \quad \|\bar{u}^m - \bar{u}^n\|_p \leqq M_1(h_m + h_n) + \|\tilde{u}^m - \tilde{u}^n\|_p$$

and by construction (see (6))

$$(35) \quad |\bar{t}^m - \bar{t}^n| \leqq h_m + h_n .$$

Therefore, S_1 can be estimated with the aid of (11), (14), (15), (17), and (27) by

$$(36) \quad S_1 \leqq (l_3 |\bar{t}^m - \bar{t}^n| + l_4 \|\tau_h \bar{u}^m - \tau_h \bar{u}^n\|_p) \|\bar{u}^m - \bar{u}^n\|_p^{p-1} -$$
$$- k_1 \| |\bar{u}^m - \bar{u}^n|^{(p-2)/2} (\bar{u}^m - \bar{u}^n)\|_{1,2}^2 + k_2 \|\bar{u}^m - \bar{u}^n\|_p^p \leqq$$
$$\leqq c_1(h_m + h_n)^p - k_1 \| |\bar{u}^m - \bar{u}^n|^{(p-2)/2} (\bar{u}^m - \bar{u}^n)\|_{1,2}^2 + c_2 \|\tilde{u}^m - \tilde{u}^n\|_p^p .$$

The integrand S_2 is estimated by application of (18) under consideration of (iii), (29), (34), and (35):

$$(37) \quad S_2 \leqq (l_1^p + l_2^p) |\bar{t}^m - \bar{t}^n|^p \|\bar{u}^m\|_{1,p}^p +$$
$$+ \varepsilon \| |\bar{u}^m - \bar{u}^n|^{(p-2)/2} (\bar{u}^m - \bar{u}^n)\|_{1,2}^2 + k_3(\varepsilon) \|\bar{u}^m - \bar{u}^n\|_p^p \leqq$$
$$\leqq [(l_1^p + l_2^p) M_2^p + k_3(\varepsilon) 2^{p-1} M_1^p] (h_m + h_n)^p +$$
$$+ \varepsilon \| |\bar{u}^m - \bar{u}^n|^{(p-2)/2} (\bar{u}^m - \bar{u}^n)\|_{1,2}^2 + k_3(\varepsilon) 2^{p-1} \|\tilde{u}^m - \tilde{u}^n\|_p^p .$$

Here, as well as in the following estimate, we have used the inequality $(a + b)^p \leqq \leqq 2^{p-1}(a^p + b^p)$. In a similar way we estimate S_3 and obtain by the aid of (16) with $p_1 = p/(p-2)$, $p_2 = p/2$

$$(38) \quad S_3 \leqq \frac{(p-2)}{p} 2^{2p-1} \|\tilde{u}^m - \tilde{u}^n\|_p^p + \frac{(p-2)}{p} 2^{p-1} M_1^p (h_m + h_n)^p + \frac{2}{p} (h_m + h_n)^{p/2} .$$

Inserting (36)–(38) into (33) this yields an integral inequality of the form

$$\|\tilde{u}^m(\cdot, t_0) - \tilde{u}^n(\cdot, t_0)\|_p^p \leqq K_4 t_0 (h_m + h_n)^p + $$
$$+ K_5 t_0 (h_m + h_n)^{p/2} + K_6 \int_0^{t_0} \|\tilde{u}^m(\cdot, t) - \tilde{u}^n(\cdot, t)\|_p^p \, dt$$

$\forall t_0 \in \hat{I}$. Therefore, from Gronwall's lemma (cf. [1], p. 28) we conclude

(39) $$\|\tilde{u}^m(\cdot, t) - \tilde{u}^n(\cdot, t)\|_p^p \leqq t e^{K_6 t} [K_4 (h_m + h_n)^p + K_5 (h_m + h_n)^{p/2}]$$

$\forall t \in \hat{I}$. Since the space $C(\hat{I}, L_p(G))$ is complete the relation (39) immediately implies the assertion of the lemma. ■

Remark. Passing to the limit $m \to \infty$ in relation (39) we obtain the error estimate

(40) $$\|u(\cdot, t) - \tilde{u}^n(\cdot, t)\|_p \leqq K t^{1/p} e^{\bar{K}_6 t} h_n^{1/2} .$$

An immediate consequence of Lemma 6 is, owing to (26) and (27),

(41) $$\bar{u}^n \to u \, , \quad \tau_h \bar{u}^n \to u \quad \text{in} \quad L_\infty(\hat{I}, L_p(G)) \quad \text{as} \quad n \to \infty \, .$$

Finally we have to show that the limit function u from Lemma 6 is the solution of the stated problem.

Theorem 2. *Let assumptions* (i)–(iv) *be fulfilled. Then the following statements are true*:

a) *There exists a* $\hat{T} > 0$ *such that the problem* (1)–(3) *has a unique weak solution* $u \in L_\infty(\hat{I}, \mathring{W}_p^1(G)) \cap C^{0,1}(\hat{I}, L_p(G)) \subset L_\infty(\hat{I}, C(\bar{G})) \cap C^{0,1}(\hat{I}, L_p(G))$, $u_t \in L_\infty(\hat{I}, L_p(G))$, *defined in* $Q_{\hat{T}} = G \times [0, \hat{T}]$.

b) *The Rothe approximates* $\tilde{u}^n, \bar{u}^n$ *established by solution of* (1_j), (2_j), (3_0), $j = 1, \ldots, \hat{n}$, *have the convergence properties*

(42) $$\tilde{u}^n \to u \quad in \quad C(\hat{I}, L_p(G)) \, , \quad \bar{u}^n \to u \quad in \quad L_\infty(\hat{I}, L_p(G)) \, ,$$

(43) $$\tilde{u}^n, \bar{u}^n \rightharpoonup^* u \quad in \quad L_\infty(\hat{I}, \mathring{W}_p^1(G)) \, ,$$

(44) $$\partial \tilde{u}^n / \partial t \rightharpoonup^* \partial u / \partial t \quad in \quad L_\infty(\hat{I}, L_p(G))$$

for n tending to infinity.

c) *The error can be estimated by* (40).

Proof. 1. First we prove the convergence properties b). The relation (42) is the assertion of Lemma 6 and (41). Because of (29) $\tilde{u}^n$ and $\bar{u}^n$ belong to $L_\infty(\hat{I}, \mathring{W}_p^1(G))$ with uniformly bounded norm. Hence, there exist subsequences $\{n_k\}$ and $\{n_l\}$ with

$$\tilde{u}^{n_k} \rightharpoonup^* q_1 \, , \quad \bar{u}^{n_l} \rightharpoonup^* q_2 \quad \text{in} \quad L_\infty(\hat{I}, \mathring{W}_p^1(G)) \, ,$$

where this is valid also in the weaker topology of $L_\infty(\hat{I}, L_p(G))$. Particularly, due to (42) we have

$$\tilde{u}^n \rightharpoonup^* u \, , \quad \bar{u}^n \rightharpoonup^* u \quad \text{in} \quad L_\infty(\hat{I}, L_p(G)) \, ,$$

hence $q_1 = q_2 = u$. Moreover, uniqueness of the limit for every subsequence implies the correctness of (43) for the whole sequence.

In similar way, by (25) there exists a subsequence $\{n_k\}$ with

$$\frac{\partial \tilde{u}^{n_k}}{\partial t} \rightharpoonup^* q \quad \text{in} \quad L_\infty(\hat{I}, L_p(G)) .$$

Passing to the limit $n = n_k \to \infty$ in the relation

$$\int_0^{\hat{T}} \left[\left\langle \tilde{u}^n, \frac{\partial v}{\partial t} \right\rangle + \left\langle \frac{\partial \tilde{u}^n}{\partial t}, v \right\rangle \right] \mathrm{d}t = 0 \quad \forall v \in C_0^\infty(Q_{\hat{T}})$$

we obtain $q = \partial u / \partial t$ and thus (44) for the hole sequence.

2. From properties (42)–(44) and completness of the used spaces follows $u \in L_\infty(\hat{I}, \mathring{W}_p^1(G)) \cap C(\hat{I}, L_p(G))$, $u_t \in L_\infty(\hat{I}, L_p(G))$. Since $p > N$ is assumed we have the continuous embedding $L_\infty(\hat{I}, \mathring{W}_p^1(G)) \subset L_\infty(\hat{I}, C(\bar{G}))$. If we pass to the limit $n \to \infty$ in (24) furthermore we get the Lipschitz condition

$$\|u(\cdot, t) - u(\cdot, t')\|_p \leqq M_1 |t - t'| \quad \forall t, t' \in \hat{I} ,$$

that means $u \in C^{0,1}(\hat{I}, L_p(G))$.

3. To prove that u solves (1)–(3) we choose an arbitrary test function $v \in L_1(\hat{I}, \mathring{W}_q^1(G))$, insert $v(\cdot, t)$ into (30″) and integrate over $t \in \hat{I}$. Taking into account the Lipschitz conditions from (iii) and (iv) and the properties (41)–(44) the limiting process $n \to \infty$ yields the integral relation

$$\int_0^{\hat{T}} \left[A(u, v) + \left\langle \frac{\partial u}{\partial t}, v \right\rangle \right] \mathrm{d}t = \int_0^{\hat{T}} \langle f(\cdot, t, u), v \rangle \, \mathrm{d}t \quad \forall v \in L_1(\hat{I}, \mathring{W}_q^1(G)) ,$$

that means u is a weak solution of (1). By construction of $\tilde{u}^n$ due to (42), (43) the initial boundary condition is satisfied.

4. Uniqueness: In [1], Chapter 2.2, uniqueness of a solution w in $L_\infty(\hat{I}, \mathring{W}_2^1(G)) \cap C(\hat{I}, L_2(G))$ is proved for the problem corresponding to (1)–(3) with the global Lipschitz continuous right-hand side $f^R(x, t, w)$ instead of $f(x, t, w)$ (cf. Theorem 2.2.4 and Example 2.2.17). Because of (28) and (42) for $t \leqq \hat{T}$ there is valid $|u| \leqq R$, thus we have $u = w$ as a consequence of $f(x, t, u) = f^R(x, t, u)$ for $t \leqq \hat{T}$ and the above stated uniqueness. Since $|U_0| < R$ we get uniqueness of a solution of (1)–(3) even in $L_\infty(\hat{I}, \mathring{W}_2^1(G)) \cap C(\hat{I}, L_2(G))$. ∎

Corollary. 1. *Due to* (28) *and* (42) *we have* $\sup_{Q_{\hat{T}}} |u(x, t)| \leqq R$.

2. *Particularly, u belongs to $W_p^1(Q_{\hat{T}})$.*

3. *If $p > N + 1$ then the embedding $W_p^1(\bar{Q}_{\hat{T}}) \subset C(\bar{Q}_{\hat{T}})$ implies $u \in C(\bar{Q}_{\hat{T}})$.*

References

[1] *J. Kačur:* Method of Rothe in Evolution Equations. BSB B. G. Teubner Verlagsges., Leipzig 1985.

[2] *О. А. Ладыженская, Н. Н. Уральцева*: Линейные и квазилинейные уравнения эллиптического типа. Изд. Наука, Мссква 1973.

[3] *V. Pluschke:* Solution of Nonlinear Pseudoparabolic Equations by Semidiscretization in Time. Complex Variables *7* (1987), 321—336.

[4] *V. Pluschke:* An Existence Theorem for a Quasilinear Hyperbolic Boundary Value Problem Solved by Semidiscretization in Time. Z. Anal. Anwend. *5* (1986), 157—168.

[5] *K. Rektorys:* The Method of Discretization in Time and Partial Differential Equations. Reidel Publ. Comp., Dortrecht—Boston—London 1982.

[6] *Ch. G. Simader:* On Dirichlet's Boundary Value Problem. Lecture Notes in Math. 268, Springer-Verl., Berlin—Heidelberg—New York 1972.

Author's address: Martin-Luther-University Halle-Wittenberg, Dept. of Mathematics, Universitätsplatz 6, Halle (S.), 4020, G.D.R.