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EMBEDDING m-QUASISTARS INTO n-CUBES

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(Received November 13, 1986)

In the present paper the letters \(i, j, k, m, n\) and \(p\) denote integers. By a graph we mean a graph in the sense of [1]; \(V(G)\) and \(E(G)\) denote the vertex set and the edge set of a graph \(G\), respectively. We shall say that graphs \(G_1\) and \(G_2\) are vertex-disjoint if \(V(G_1) \cap V(G_2) = \emptyset\).

A graph which is homeomorphic to the star \(K(1, m)\), where \(m \geq 3\), will be referred to as an \(m\)-quasistar. We say that an \(m\)-quasistar \(T\) of order \(p\) is balanced if \(p\) is even and there exists a 2-coloring of \(T\) with \(p/2\) blue vertices and \(p/2\) yellow ones. I. Havel [2] conjectured that

if \(3 \leq m \leq n\), then every balanced \(m\)-quasistar of order \(2^n\) can be embedded into the \(n\)-cube.

The conjecture has been proved for \(m = 3\) by Havel [2], for \(m = 4\) and \(5\) by the present author [4], and for \(m = 6\) by N. B. Limaye [3]. In the present paper the conjecture will be proved for every \(m \geq 5\).

Let \(P\) be a nontrivial path. Then \(P\) is a graph homeomorphic to \(K_2\). If \(u\) is a vertex of degree one in \(P\), then we say that \(P\) is a \(u\)-path. If \(P\) is a \(u\)-path, then the only vertex of degree one in \(P\) which is different from \(u\) will be denoted by \(e(P, u)\).

Let \(G\) be an \(n\)-cube, \(n \geq 1\). If \(u_1\) and \(u_2\) are adjacent vertices in \(G\), \(P_1\) and \(P_2\) are vertex-disjoint nontrivial paths in \(G\) such that \(P_1\) is a \(u_1\)-path and \(P_2\) is a \(u_2\)-path, then we denote by \(P_1 + u_1u_2 + P_2\) the path in \(G\) induced by \(E(P_1) \cup \{u_1u_2\} \cup E(P_2)\). Since \(G\) is an \(n\)-cube, where \(n \geq 1\), it is clear that there exist vertex-disjoint \((n - 1)\)-cubes \(G'\) and \(G''\) such that \(V(G') \cup V(G'') = V(G)\) and \(E(G') \cup E(G'') \subseteq E(G)\); the set \(\{G', G''\}\) will be referred to as a canonical partition of \(G\). If \(\{G', G''\}\) is a canonical partition of \(G\) and \(u \in V(G')\), then the only vertex of \(G''\) which is adjacent to \(u\) in \(G\) will be denoted by \(u/G''\).

The proof of Havel’s conjecture (for \(m \geq 5\)) will be divided into two lemmas and two theorems.

**Lemma 1.** Let \(m \geq 1\), let \(G\) be an \(m\)-cube, let \(u \in V(G)\), and let \(W \subseteq V(G)\) such that \(|W| \leq m - 1\). Then there exists a hamiltonian \(u\)-path \(P\) in \(G\) such that \(e(P, u) \notin W\).

**Proof.** Obviously, there exists a 2-coloring of \(G\) with \(2^{m-1}\) blue vertices and \(2^{m-1}\)
yellow ones. Without loss of generality, let $u$ be blue. Havel [2] has shown that for each yellow vertex $v$ of $G$, there exists a hamiltonian path $P$ in $G$ such that $e(P, u) = v$. Since $m - 1 < 2^{m - 1}$, the assertion of the lemma follows.

**Lemma 2.** Let $m \geq 2$, let $G$ be an $m$-cube, let $u, v_1, v_2$ be distinct vertices of $G$ such that $v_1 v_2 \in E(G)$, and let $W \subseteq V(G - v_1 - v_2)$ such that $|W| \leq m - 2$. Then there exists a hamiltonian $u$-path $P$ in $G - v_1 - v_2$ such that $e(P, u) \notin W$.

**Proof.** We proceed by induction on $m$. The case when $m = 2, 3$ is obvious. Let $m \geq 4$. Assume that the lemma is proved for $m - 1$. It is clear that there exists a canonical partition $\{G', G''\}$ of $G$ such that

$$|W \cap V(G'')| \leq m - 3 \text{ and } v_1, v_2 \in V(G'').$$

We distinguish two cases.

1. Let $u \in V(G')$. Recall that $m - 1 \geq 3$. According to Lemma 1 there exists a hamiltonian $u$-path $P'$ in $G'$ such that $e(P', u) \notin \{v_1|G', v_2|G'\}$. Denote $u' = e(P', u)$ and $u'' = u'|G''$. According to the induction hypothesis, there exists a hamiltonian $u''$-path $P''$ in $G'' - v_1 - v_2$ such that $e(P'', u'') \notin W \cap V(G'')$. Clearly,

(1) $P' + u'u'' + P''$ is a hamiltonian $u$-path in $G - v_1 - v_2$ such that $e(P' + u'u'' + P'', u) \notin W$.

2. Let $u \in V(G'')$. According to the induction hypothesis, there exists a hamiltonian $u$-path $P''$ in $G'' - v_1 - v_2$. Denote $u'' = e(P'', u)$ and $u' = u''|G'$. According to Lemma 1, there exists a hamiltonian $u'$-path $P'$ in $G'$ such that $e(P', u') \notin W \cap V(G')$. Clearly, (1). Thus the proof is complete.

The following theorem is the main step in our proof of Havel's conjecture.

**Theorem 1.** Let $k$ and $m$ be integers such that

$$1 \leq k \leq m \text{ if } 1 \leq m \leq 3 \text{ and } 1 \leq k < m \text{ if } m \geq 4.$$ 

Then $Q(k, m)$, where $Q(k, m)$ is the statement as follows:

for any $G, u_1, \ldots, u_k, a_1, \ldots, a_k, W_1, \ldots, W_k$ such that

(2) $G$ is an $m$-cube,

(3) $u_1, \ldots, u_k$ are distinct vertices of $G$,

(4) $a_1, \ldots, a_k$ are positive even integers with $a_1 + \ldots + a_k = 2^m$,

(5) $W_1, \ldots, W_k$ are subsets of $V(G)$ fulfilling

$$|W_i| \leq m - k, \ldots, |W_k| \leq m - k,$$

there exist vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ in $G$ such that

(6) $P_{(i)}$ is a $u_i$-path of order $a_i$ such that $e(P_{(i)}, u_i) \notin W_i$, for each $i, 1 \leq i \leq k$.

**Proof.** It is easy to prove $Q(1, 1)$, $Q(2, 2)$ and $Q(3, 3)$ by an immediate inspection. Thus, we shall prove that if $m \geq 2$ then $Q(k, m)$, for each $k, 1 \leq k \leq m - 1$. We

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proceed by induction on \( m \). The case \( m = 2 \) is obvious. Let \( m \geq 3 \). Assume that we have proved \( Q(k^*, m - 1) \) for each \( k^*, 1 \leq k^* \leq m - 2 \).

Let \( 1 \leq k \leq m - 1 \). Consider \( G, u_1, \ldots, u_k, a_1, \ldots, a_k, W_1, \ldots, W_k \) such that (2)–(5). For any canonical partition \( \{G_1, G_2\} \) of \( G \) and any \( f \in \{1, 2\} \), we define

\[
I(G_f) = \{i; 1 \leq i \leq k \text{ and } u_i \in V(G_f)\},
\]

\[
k(G_f) = |I(G_f)|,
\]

\[
U(G_f) = \{u_i; i \in I(G_f)\}, \quad \text{and}
\]

\[
A(G_f) = \sum_{i \in I(G_f)} a_i.
\]

We distinguish several cases and subcases.

1. Assume that there exists a canonical partition \( \{G_1, G_2\} \) of \( G \) such that \( A(G_1) = A(G_2) \).

Consider \( f \in \{1, 2\} \). Obviously, \( A(G_f) = 2^{m-1} \) and \( 1 \leq k(G_f) \leq k - 1 < m - 1 \).

Denote

\[
I_f = I(G_f),
\]

\[
u_{if} = u_i, \quad a_{if} = a_i \quad \text{and} \quad W_{if} = W_i \cap V(G_f) \quad \text{for each} \quad i \in I_f.
\]

It is clear that

\[
(7)_f \quad u_{if} \ (i \in I_f) \text{ are distinct vertices of } G_f,
\]

and

\[
(8)_f \quad a_{if} \ (i \in I_f) \text{ are even positive integers such that } \sum_{i \in I_f} a_{if} = 2^{m-1}.
\]

Obviously, \( |W_{if}| \leq |W_i| \leq m - k \) for \( i \in I_f \). Since \( m - k \leq (m - 1) - |I_f| \),

\[
(9)_f \quad |W_{if}| \leq (m - 1) - |I_f|, \quad \text{for each} \quad i \in I_f.
\]

According to \( Q(k(G_f), m - 1) \), there exists a set of \( |I_f| \) vertex-disjoint paths \( P_{if} \ (i \in I_f) \) in \( G_f \) such that

\[
(10)_f \quad P_{if} \text{ is a } u_{if}-\text{path of order } a_{if} \text{ with the property that } d(P_{if}, u_{if}) \notin W_{if} \text{ for each } i \in I_f.
\]

Denote

\[
P_{(i)} = P_{i1} \text{ if } i \in I_1, \quad \text{and} \quad P_{(i)} = P_{i2} \text{ if } i \in I_2.
\]

Clearly, \( P_{(1)}, \ldots, P_{(k)} \) are vertex-disjoint paths in \( G \) such that (6).

2. Assume that \( A(G^*) \neq A(G^{**}) \) for any canonical partition \( \{G^*, G^{**}\} \) of \( G \).

2.1. Let \( k = 1 \). Then \( a_1 = 2^m \). Lemma 1 implies that there exists a path \( P_{(1)} \) in \( G \) such that (6).

2.2. Let \( k = 2 \). Clearly, \( a_1 \neq a_2 \). Without loss of generality we assume that \( a_1 > a_2 \).

2.2.1. Let \( a_2 = 2 \). Since \( |W_2| \leq m - 2 \), there exists \( u_2^* \in V(G) - (\{u_1\} \cup W_2) \) such that \( u_2u_2^* \in E(G) \). We denote by \( P_{(2)} \) the path in \( G \) induced by \( \{u_2u_2^*\} \). Since \( |W_1| \leq m - 2 \), it follows from Lemma 2 that there exists a hamiltonian \( u_1 \)-path \( P_{(1)} \)
in $G - u_2 - u_2^*$ such that $e(P_{(1)}, u_1) \notin W_1$. Hence, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in $G$ such that (6).

2.2.2. Let $a_2 \geq 4$. Since $a_1 > a_2$, $m \geq 4$. Clearly, there exists a canonical partition $\{G_1, G_2\}$ of $G$ such that

\[
|W_1 \cap V(G_f)| \leq m - 3 \quad \text{for} \quad f = 1 \quad \text{and} \quad 2.
\]

Without loss of generality we assume that $u_1 \in V(G_1)$.

2.2.2.1. Let $u_2 \in V(G_1)$ and $W_2 \cap V(G_1) = \emptyset$. Denote

\[
I_1 = \{1, 2\}, \quad u_{11} = u_1, \quad u_{21} = u_2, \quad a_{11} = 2^{m-1} - a_2, \\
\quad a_{21} = a_2, \quad W_{11} = \emptyset = W_{21}.
\]

It is clear that $(7)_1 -(9)_1$. According to Q(2, $m - 1$), there exist vertex-disjoint paths $P_{11}$ and $P_{21}$ in $G_1$ such that (10)$_1$. Denote $v = e(P_{11}, u_{11})$ and $u_{12} = v \mid G_2$. As follows from (11) and Lemma 1, there exists a hamiltonian $u_{12}$-path $P_{12}$ in $G_2$ such that $e(P_{12}, u_{12}) \notin W_1 \cap V(G_2)$. Define $P_{(1)} = P_{11} + v_{12} + P_{12}$ and $P_{(2)} = P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in $G$ such that (6).

2.2.2.2. Let $u_2 \in V(G_1)$ and $W_2 \cap V(G_1) = \emptyset$. Hence,

\[
|W_2 \cap V(G_2)| \leq m - 3.
\]

Denote

\[
I_1 = \{1, 2\}, \quad u_{11} = u_1, \quad u_{21} = u_2, \quad a_{11} = 2^{m-1} - 2, \quad a_{21} = 2, \\
\quad W_{11} = \emptyset = W_{21}.
\]

It is clear that $(7)_1 -(9)_1$. According to Q(2, $m - 1$), there exist vertex-disjoint paths $P_{11}$ and $P_{21}$ in $G_1$ such that (10)$_1$. Denote

\[
I_2 = \{1, 2\}, \quad v_1 = e(P_{11}, u_{11}), \quad v_2 = e(P_{21}, u_{21}), \quad u_{12} = v_1 \mid G_2, \\
\quad u_{22} = v_2 \mid G_2, \quad a_{12} = a_1 + 2 - 2^{m-1}, \quad a_{22} = a_2 - 2, \\
\quad W_{12} = W_1 \cap V(G_2), \quad W_{22} = W_2 \cap V(G_2).
\]

It is clear that $(7)_2$ and $(8)_2$. It follows from (11) and (12) that $(9)_2$. According to Q(2, $m - 1$), there exist vertex-disjoint paths $P_{12}$ and $P_{22}$ in $G_2$ such that (10)$_2$. Define $P_{(1)} = P_{11} + v_{12}u_{12} + P_{12}$ and $P_{(2)} = P_{21} + v_{22}u_{22} + P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in $G$ such that (6).

2.2.2.3. Let $u_2 \in V(G_2)$ and $W_2 \cap V(G_2) = \emptyset$. According to Lemma 1 there exists a hamiltonian $u_1$-path $P_{11}$ in $G_1$ such that $e(P_{11}, u_1) + u_2 \mid G_1$. Denote

\[
v_1 = e(P_{11}, u_1), \quad I_2 = \{1, 2\}, \quad u_{12} = v_1 \mid G_2, \quad u_{22} = u_2, \\
\quad a_{12} = a_1 - 2^{m-1}, \quad a_{22} = a_2, \quad W_{12} = W_1 \cap V(G_2), \quad W_{22} = W_2 \cap V(G_2).
\]

It is clear that $(7)_2 -(9)_2$. According to Q(2, $m - 1$), there exist vertex-disjoint paths $P_{12}$ and $P_{22}$ in $G_2$ such that (10)$_2$. Define $P_{(1)} = P_{11} + v_{12}u_{12} + P_{12}$ and $P_{(2)} = P_{22}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in $G$ such that (6).

2.2.2.4. Let $u_2 \in V(G_2)$ and $V(G_2) \cap W_2 = \emptyset$. Hence,

\[
|W_2 \cap V(G_1)| \leq m - 3.
\]
There exists $v_2 \in V(G_2 - u_2)$ such that $v_2$ is adjacent to $u_2$ in $G_2$ and $v_2 \neq u_1/G_2$. We denote by $P_{22}$ the path in $G_2$ induced by $\{u_2v_2\}$. Denote
\begin{align*}
I_1 &= \{1, 2\}, & u_{11} = u_1, & u_{21} = v_2/G_1, & a_{11} = 2^{n-1} + 2 - a_2, \\
ap_{21} = a_2 - 2, & W_{11} = \{u_2/G_1\} & \text{and} & W_{21} = W_2 \cap V(G_1).
\end{align*}

It is clear that $(7)_1$ and $(8)_1$. Since $m - 1 \geq 3$, $(13)$ implies that $(9)_1$. As follows from $Q(2, m - 1)$, there exist vertex-disjoint paths $P_{11}$ and $P_{21}$ such that $(10)_1$. Denote $v_1 = e(P_{11}, u_{11})$ and $u_{12} = v_1/G_2$. It is easy to see that $u_{12} \notin \{u_2, v_2\}$. It follows from Lemma 2 and $(11)$ that there exists a Hamiltonian $u_{12}$-path $P_{12}$ in $G_2 - u_2 - v_2$ such that $e(P_{12}, u_{12}) \notin W_1 \cap V(G_2)$. Define $P_{(1)} = P_{11} + v_1u_{12} + P_{12}$ and $P_{(2)} = P_{22} + v_2u_{21} + P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in $G$ such that $(6)$.

2.3. Let $k \geq 3$. Then $m \geq 4$. Recall that $A(G^*) = A(G^{**})$ for any canonical partition $\{G^*, G^{**}\}$ of $G$. We first prove that

\begin{equation}
(14) \quad \text{there exists a canonical partition $\{G_1, G_2\}$ of $G$ such that $A(G_1) > A(G_2)$ and $1 \leq k(G_2) \leq k - 2$.}
\end{equation}

To the contrary, let us assume that

\begin{equation}
(14) \quad \text{for any canonical partition $\{G^*, G^{**}\}$ of $G$, if $A(G^*) > A(G^{**})$ and $1 \leq k(G^{**})$, then $k(G^{**}) = k - 1$.}
\end{equation}

Since $k \geq 3$, there exists a canonical partition $\{G_{11}, G_{12}\}$ of $G$ such that $A(G_{11}) > A(G_{12})$ and $k(G_{12}) \geq 1$. According to $(14)$, $k(G_{12}) = k - 1$, and therefore $k(G_{11}) = 1$. Obviously, there exists $i$, $1 \leq i \leq k$, such that $U(G_{11}) = \{u_i\}$. Since $A(G_{11}) > A(G_{12})$, $u_i > 2^{n-1}$.

Since $k(G_{12}) = k - 1 \geq 2$, there exists a canonical partition $\{G_{21}, G_{22}\}$ of $G$ such that

\[ U(G_{12}) \cap V(G_{21}) \neq 0 \neq U(G_{12}) \cap V(G_{22}). \]

Without loss of generality we assume that $A(G_{21}) > A(G_{22})$. Since $U(G_{12}) \cap V(G_{22}) \neq 0$, $k(G_{22}) \geq 1$. According to $(14)$, $k(G_{22}) = k - 1$, and therefore $k(G_{21}) = 1$. There exists $j$, $1 \leq j \leq k$, such that $U(G_{21}) = \{u_j\}$. Since $A(G_{21}) > A(G_{22})$, $a_j > 2^{n-1}$. Since $U(G_{12}) \cap V(G_{21}) \neq 0$ and $U(G_{21}) = \{u_j\}$, we can see that $u_j \in V(G_{12})$. Hence $i \neq j$. As follows from $(4)$, $a_i + a_j < 2^n$, which is a contradiction. Thus, we have proved $(14)$.

Denote
\[ a = \min_{i \in I(G_1)} a_i. \]

We shall prove that

\begin{equation}
(15) \quad a \leq 2^{n-1} - 2(k(G_1) - 1).
\end{equation}

To the contrary, let

\[ a > 2^{n-1} - 2(k(G_1) - 1). \]
Since $a$ is even, we have that
\[ a \geq 2^{m-1} - 2(k(G_1) - 2). \]
Since $k(G_2) \geq 1$, $A(G_2) \geq 2$. Hence,
\[ a \leq \frac{2^m - 2}{k(G_1)}. \]
If $k(G_1) = 2$, then combining (15) and (16) we get that $2^{m-1} - 1 \geq 2^{m-1}$, which is a contradiction. Let $k(G_1) \geq 3$. Obviously, $m - 2 \geq k(G_1)$. Thus according to (15) and (16) we get that
\[ \frac{2^m - 2}{3} \geq \frac{2^m - 2}{k(G_1)} \geq 2^{m-1} - 2(k(G_1) - 2) \geq 2^{m-1} - 2(m - 4). \]
Hence, $6m - 26 \geq 2^{m-1}$, which is a contradiction. Thus, we have proved (15).

Denote $I_1 = I(G_1)$. It follows from (15) that there exist disjoint nonempty subsets $I^\sharp$ and $I^\flat$ of $I_1$ and even positive integers $a_{11}$ (for each $i \in I_1$) satisfying
\[
I_2 = I^\sharp \cup I^\flat, \\
a_{11} = a_i, \text{ if } i \in I^\sharp, \\
a_{11} \leq a_i - 2, \text{ if } i \in I^\flat, \text{ and} \\
\sum_{i \in I_1} a_{11} = 2^{m-1}.
\]
Denote
\[
u_{11} = u_i \text{ if } i \in I_1, \\
W_{j1} = W_j \cap V(G_1) \text{ if } j \in I^\sharp, \text{ and} \\
W_{j1} = \{ v; v \in G_2 \in I(G_2) \} \text{ if } j \in I^\flat.
\]
Since $k(G_2) = k - k(G_1) \leq (m - 1) - k(G_1)$, we can see that (9)_1. According to $Q(k(G_1), m - 1)$, there exists a set of $|I_1|$ vertex-disjoint paths $P_{i1}$ ($i \in I_1$) in $G_1$ such that (10)_1. Denote
\[ v_j = e(P_j, u_{j1}) \text{ for each } j \in I^\flat.
\]
Moreover, denote
\[
I_2 = I^\flat \cup I(G_2), \\
u_{12} = u_i \text{ if } i \in I(G_2), \text{ } u_{i1} = v_i | G_2 \text{ if } i \in I^\flat, \\
a_{12} = a_i \text{ if } i \in I(G_2), \\
a_{12} = a_i - a_{11} \text{ if } i \in I^\flat, \text{ and} \\
W_{j2} = W_j \cap V(G_2) \text{ if } j \in I_2.
\]
It is clear that $(7)_2 - (9)_2$. As follows from $Q(I_2, m - 1)$, there exists a set of $|I_2|$ vertex-disjoint paths $P_{i2}$ ($i \in I_2$) such that (10)_2.

Define
\[
P_{(i)} = P_{i1} \text{ if } i \in I^\sharp, \\
P_{(i)} = P_{i1} + v_i u_{i2} + P_{i2} \text{ if } i \in I^\flat, \text{ and} \\
P_{(i)} = P_{i2} \text{ if } i \in I(G_2).
\]
It is obvious that $P_{(1)}, \ldots, P_{(k)}$ are vertex disjoint paths in $G$ such that (6).

Thus, the proof of the theorem is complete.

Remark 1. Let $k \geq m \geq 4$. Consider $G, u_1, \ldots, u_k, a_1, \ldots, a_k, W_1, \ldots, W_k$ such that (2)–(5), $a_1 \geq 4, \ldots, a_k \geq 4$, and $u_1 u, \ldots, u_k u \in E(G)$, where $u$ is a vertex of $G$. Then (6) holds for no set of $k$ vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ of $G$. This means that for $k \geq m \geq 4$, $Q(k, m)$ does not hold. (It is also clear that $Q(k, m)$ does not hold for $m \leq 3$ and $k > m$.)

Remark 2. Let $2 \leq k < m$. Consider $G, u_1, \ldots, u_k, a_1, \ldots, a_k, W_1, \ldots, W_k$ such that (2)–(4), $a_1 = 2$, and

$$|W_1| \geq m - k + 1.$$

Let $u_1, \ldots, u_k$ be chosen so that there exist $m - k + 1$ vertices of $W_1$, say vertices $w_1, \ldots, w_{m-k+1}$, such that $u_1 w_1, \ldots, u_1 w_{m-k+1} \in E(G), u_1 u_2, \ldots, u_1 u_k \in E(G)$, and

$$\{u_2, \ldots, u_k\} \cap \{w_1, \ldots, w_{m-k+1}\} = \emptyset.$$

Hence, no set of $k$ vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ in $G$ satisfies (6). Let $j \geq 1$. We can see that in Theorem 1 the inequalities

$$|W_1| \leq m - k, \ldots, |W_k| \leq m - k$$

cannot be replaced by the inequalities

$$|W_1| \leq m - k + j, \ldots, |W_k| \leq m - k + j.$$

We are now prepared to show that Havel’s conjecture is true.

**Theorem 2.** If $3 \leq m \leq n$, then every balanced $m$-quasistar of order $2^n$ can be embedded into the $n$-cube.

**Proof.** We proceed by induction on $m$. In our proof we make use of the fact that the case $m = 3$ has been proved in [2] and the case $m = 4$ has been proved in [4]. Let $m \geq 5$. Assume that we have proved that for any $j, m - 1 \leq j$, every balanced $(m - 1)$-quasistar of order $2^j$ can be embedded into the $j$-cube.

Let $T$ be a balanced $m$-quasistar of order $2^n$. Then $T$ contains exactly one vertex of degree $m$, say a vertex $s$, and exactly $m$ vertices of degree one, say vertices $t_1, \ldots, t_m$. We denote by $b_i$ the distance between $s$ and $t_i$ in $T$ for each $i, 1 \leq i \leq m$. Without loss of generality we assume that $b_1 \geq \ldots \geq b_m$. Clearly, $b_1 + \ldots + b_m = 2^n - 1$. Since $T$ is balanced, it is easy to see that there exists exactly one $h, 1 \leq h \leq m$, such that $b_h$ is odd.

We shall first prove that

$$b_1 + \ldots + b_{m-2} \geq 2^{n-1} + 2(m - 4) + 1. \quad (17)$$

To the contrary, let

$$b_1 + \ldots + b_{m-2} \leq 2^{n-1} + 2(m - 4). \quad (17)$$

Since $b_1 + \ldots + b_m = 2^n - 1$ and $b_1 \geq \ldots \geq b_m$, it follows from (17) that

$$2 \cdot 2^{n-1} - 1 = 2^n - 1 \leq m(2^{n-1} + 2m - 8)/(m - 2),$$
and thus
\[ 2(m - 2) \cdot 2^n - 1 - (m - 2) \leq m 2^n - 1 + 2m^2 - 8m. \]
Since \( m \leq n \), we get that
\[ (m - 4) 2^n - 1 \leq 2m^2 - 7m - 2. \]
Hence \( m \leq 4 \), which is a contradiction. Thus, we have proved (17).
This means that there exist \( I \subseteq \{1, \ldots, m - 2\} \), even positive integers \( a_i \) for each \( i \in I \), and exactly one \( f \in I \) such that
\[
\begin{align*}
a_f &= b_f, \\
a_i &< b_i \quad \text{for each} \quad i \in I - \{f\}, \quad \text{and} \\
\sum_{i \in I} a_i &= 2^n - 1.
\end{align*}
\]
For each \( i \in I \) we denote by \( v_i \) and \( w_i \) the vertices which belong to the path connecting \( s \) and \( t_i \) in \( T \) and such that the distance between \( s \) and \( v_i \) equals \( b_i - a_i \), and the distance between \( s \) and \( w_i \) equals \( b_i - a_i + 1 \). Obviously, the vertices \( v_i \) (\( i \in I \)) are mutually distinct, and \( v_f = s \). Denote
\[ C = \{v_iw_i; i \in I\}. \]
Moreover, we denote by \( T' \) the component of \( T - C \) which contains the vertex \( s \). It is clear that \( T' \) is a balanced \((m - 1)\)-quasistarr of order \( 2^{n-1} \).
Let \( G \) be an \( n \)-cube, and let \( \{G', G''\} \) be a canonical partition of \( G \). According to the induction hypothesis, \( T' \) can be embedded into \( G' \). Thus, we can assume that \( T' \) is a subgraph of \( G' \). Denote
\[ u_i = v_i|G'' \quad \text{for} \quad i \in I. \]
It follows from Theorem 1 that there exists a set of \( |I| \) vertex-disjoint paths \( P_{(i)} \) (\( i \in I \)) in \( G'' \) such that \( P_{(i)} \) is a \( u_i \)-path of order \( a_i \), for each \( i \in I \). The subgraph of \( G \) induced by
\[ E(T') \cup \{v_iu_i; i \in I\} \cup \bigcup_{i \in I} E(P_{(i)}) \]
is isomorphic to \( T \), which completes the proof of the theorem.

**Acknowledgement.** The author wishes to thank I. Havel and P. Liebl for their stimulating interest.

**References**


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