

Said R. Grace; Bikkar S. Lalli

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OSCILLATION THEOREMS FOR CERTAIN NEUTRAL DIFFERENTIAL EQUATIONS

S. R. GRACE, Dhahran and B. S. LALLI, Saskatoon

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1. INTRODUCTION

Consider the neutral differential equation

$$(1) \quad L_n[x(t) + p(t)x[t - \tau]] + q(t)f(x[t - \sigma]) = 0, \quad n \text{ is even,}$$

where

$$L_0 x(t) = x(t), \quad L_k x(t) = \frac{1}{a_k(t)} (L_{k-1} x(t)), \quad k = 1, 2, \dots, n,$$

$a_0 = a_n = 1, a_i \in C[[t_0, \infty), (0, \infty)], i = 1, 2, \dots, n - 1, p, q \in C[[t_0, \infty), R], f \in C[R, R]$  and  $\tau$  and  $\sigma$  are nonnegative constants.

We will assume that

$$(2) \quad \int^\infty a_k(s) ds = \infty, \quad k = 1, 2, \dots, n - 1;$$

$$(3) \quad \frac{f(x)}{x} \geq \gamma > 0 \quad \text{for } x \neq 0.$$

The domain of  $L_n, D(L_n)$  is defined to be the set of all functions  $x: [t_0, \infty) \rightarrow R$  such that  $L_j x(t), 0 \leq j \leq n$  exist and are continuous on  $[t_0, \infty)$ . Our attention is restricted to those solutions  $X \in D(L_n)$  of Eq. (1) which satisfy  $\sup \{|x(t)|: t \geq T\} > 0$  for any  $T \geq t_0$ . A solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. Equation (1) is said to be *oscillatory* if all of its solutions are oscillatory.

Recently, Grammatikopoulos et. al. [4] and Ladas and Spicas [6] obtained some interesting criteria for the oscillation of the neutral differential equations of the form

$$\frac{d^n}{dt^n} [x(t) + p(t)x[t - \tau]] + q(t)x[t - \sigma] = 0.$$

These results extended some of the known results for the nonneutral differential equation

$$\frac{d^n}{dt^n} x(t) + q(t)x[t - \sigma] = 0.$$

The purpose of this paper is to establish some new criteria for the oscillation of Eq. (1). These criteria extend and improve those in [4] and [6]. Our results are new even when  $p(t) \equiv 0$ .

Neutral delay differential equations have applications in electric networks containing lossless transmission lines. Such networks arise for example, in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see [8] and [9]). Second order neutral delay differential equations appear in the study of vibrating masses attached to an elastic bar. We are not aware of any situation where equations of the type (1) arise for  $n > 2$  even when  $a_i(t) \equiv 1$ ,  $i = 1, 2, \dots, n - 1$ .

We assume here that solutions of Eq. (1) are defined for all  $t \geq t_0$ .

## 2. MAIN RESULTS

For any functions  $p_i \in C[[t_0, \infty), R]$ ,  $i = 1, 2, \dots, n$ , we define

$$I_0 = 1, \quad I_i(t, s; p_i, \dots, p_1) = \int_s^t p_i(u) I_{i-1}(u, s; p_{i-1}, \dots, p_1) du.$$

It is easy to verify that for  $1 \leq i \leq n - 1$

$$I_i(t, s; p_1, \dots, p_i) = (-1)^i I_i(s, t; p_i, \dots, p_1)$$

and

$$I_i(t, s; p_1, \dots, p_i) = \int_s^t p_i(u) I_{i-1}(t, u; p_1, \dots, p_{i-1}) du.$$

The following two lemmas will be needed in the proof of our results.

**Lemma 1.** *If  $x \in D(L_n)$ , then for  $t, s \in [t_0, \infty)$  and  $0 \leq i < k \leq n$*

- (i) 
$$L_i x(t) = \sum_{j=i}^{k-1} I_{j-1}(t, s; a_{i+1}, \dots, a_j) L_j x(s) + \int_s^t I_{k-i-1}(t, u; a_{i+1}, \dots, a_{k-1}) a_k(u) L_k x(u) du.$$
- (ii) 
$$L_i x(t) = \sum_{j=i}^{k-1} (-1)^{j-i} I_{j-i}(s, t; a_j, \dots, a_{i+1}) L_j x(s) + (-1)^{k-i} \int_s^t I_{k-i-1}(u, t; a_{k-1}, \dots, a_{i+1}) a_k(u) L_k x(u) du.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is immediate.

**Lemma 2.** *Suppose condition (2) holds. If  $x \in D(L_n)$  is of constant sign and not identically zero for all large  $t$ , then there exist a  $t_x \geq t_0$  and an integer  $l$ ,  $0 \leq l \leq n$  with  $n + l$  even for  $x(t) L_n x(t)$  nonnegative or  $n + l$  odd for  $x(t) L_n x(t)$  nonpositive and such that for every  $t \geq t_x$*

$$l > 0 \text{ implies } x(t) L_k x(t) > 0, \quad (k = 0, 1, \dots, l)$$

and

$$l \leq n - 1 \text{ implies } (-1)^{l-k} x(t) L_k x(t) > 0, \quad (k = l, l + 1, \dots, n).$$

This lemma generalizes a well known lemma of Kiguradze and can be proved similarly.

It will be convenient to make use of the following notation in the remainder of this paper. For any  $T \geq t_0 \geq 0$  and all  $t \geq T$  we let

$$\omega_l[t, T] = \int_T^t I_{l-1}(t, s; a_1, \dots, a_{l-1}) a_l(s) I_{n-l-1}(t, s; a_{n-1}, \dots, a_{l+1}) ds$$

for

$$1 \leq l \leq n - 1.$$

**Theorem 1.** *Let conditions (2) and (3) hold, and*

$$(4) \quad q(t) \geq 0 \quad \text{and} \quad 0 \leq p(t) \leq 1 \quad \text{for} \quad t \geq t_0.$$

If, for each  $l \in \{1, 3, \dots, n - 1\}$

$$(5) \quad \liminf_{t \rightarrow \infty} \int_{t-\sigma}^t \omega_l[s - \sigma, T] q(s) [1 - p(s - \sigma)] ds > \frac{1}{\gamma e}$$

for all large  $T, t > T + \sigma$ , then Eq. (1) is oscillatory.

*Proof.* Let  $x(t)$  be a nonoscillatory solution of Eq. (1) say  $x(t) > 0$  for  $t \geq t_0$ . There exists  $t_1 \geq t_0$  such that  $x[t - \tau] > 0$  and  $x[t - \sigma] > 0$  for all  $t \geq t_1$ . Set

$$z(t) = x(t) + p(t) x[t - \tau].$$

Then

$$z(t) > 0 \quad \text{for all} \quad t \geq t_1,$$

and

$$L_n z(t) \leq 0 \quad \text{for} \quad t \geq t_1.$$

The hypotheses of Lemma 2 are satisfied for  $t \geq t_1$  and hence there exist  $t_2 \geq t_1$  and  $l \in \{1, 3, \dots, n - 1\}$  such that

$$(6) \quad L_k z(t) > 0 \quad (k = 0, 1, \dots, l), \quad t \geq t_2$$

and

$$(-1)^{l-k} L_k z(t) > 0 \quad (k = l, l + 1, \dots, n), \quad t \geq t_2.$$

Clearly

$$\dot{z}(t) > 0 \quad \text{and} \quad L_{n-1} z(t) > 0 \quad \text{for} \quad t \geq t_2.$$

In view of condition (3), Eq. (1) becomes

$$L_n z(t) + \gamma q(t) x[t - \sigma] \leq 0; \quad t \geq t_0,$$

or

$$(7) \quad L_n z(t) + \gamma q(t) \{z[t - \sigma] - p(t - \sigma) x[t - \tau - \sigma]\} \leq 0.$$

Since  $z(t) > x(t)$ , we obtain

$$L_n z(t) + \gamma q(t) \{z[t - \sigma] - p(t - \sigma) z[t - \tau - \sigma]\} \leq 0.$$

Now, using the fact that  $z(t)$  is increasing on  $[t_2, \infty)$ , we have

$$(8) \quad L_n z(t) + \gamma q(t) \{1 - p(t - \sigma)\} z[t - \sigma] \leq 0 \quad \text{for} \quad t \geq t_2.$$

By Lemma 1 (ii) we get

$$L_l z(s) = \sum_{j=1}^{n-2} (-1)^{j-l} I_{j-l}(t, s; a_j, \dots, a_{l+1}) L_j z(t) + \\ + (-1)^{n-l-1} \int_s^t I_{n-l-2}(u, s; a_{n-2}, \dots, a_{l+1}) a_{n-1}(u) L_{n-1} z(u) du \\ \text{for } t_2 \leq s \leq t.$$

Using (6) and the fact that  $L_{n-1} z(t)$  is decreasing on  $[t_2, \infty)$  we obtain

$$(9) \quad L_l z(s) \geq I_{n-l-1}(t, s; a_{n-1}, \dots, a_{l+1}) L_{n-1} z(t) \quad \text{for } t_2 \leq s \leq t.$$

Next, we use Lemma 1 (i), and obtain

$$z(t) = \sum_{j=0}^{l-1} I_j(t, t_2; a_1, \dots, a_j) L_j z(t_2) + \\ + \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1}) a_l(s) L_l z(s) ds, \quad t \geq t_2.$$

By (6) we get

$$(10) \quad z(t) \geq \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1}) a_l(s) L_l z(s) ds, \quad \text{for } t \geq t_2.$$

By combining (9) and (10) we obtain

$$z(t) \geq \int_{t_2}^t I_{l-1}(t, s; a_1, \dots, a_{l-1}) a_l(s) I_{n-l-1}(t, s; a_{n-1}, \dots, a_{l+1}) ds L_{n-1} z(t) \\ \text{or}$$

$$z(t) \geq \omega_l[t, t_2] L_{n-1} z(t) \quad \text{for } t \geq t_2.$$

There exists a  $t_3 \geq t_2$  so that  $t > t_2 + \sigma$  for all  $t \geq t_3$ . Then

$$z[t - \sigma] \geq \omega_l[t - \sigma, t_2] L_{n-1} z[t - \sigma] \quad \text{for } t > t_3,$$

and (8) becomes

$$(11) \quad L_n z(t) + \gamma q(t) \omega_l[t - \sigma, t_2] [1 - p(t - \sigma)] L_{n-1} z[t - \sigma] \leq 0 \\ \text{for } t \geq t_3.$$

Now, set  $y(t) = L_{n-1} z(t)$ . Thus (11) takes the form

$$(12) \quad \dot{y}(t) + \gamma q(t) \omega_l[t - \sigma, t_2] [1 - p(t - \sigma)] y[t - \sigma] \leq 0, \quad t \geq t_3.$$

In view of (5) and Theorem 1 in [7], Inq. (12) has no eventually positive solutions. This contradicts the fact that  $z(t) > 0$  for  $t \geq t_1$ , and the proof is complete.

The following examples are illustrative.

Example 1. The neutral differential equation

$$(E_1) \quad \left( t \left( t \left( \frac{1}{t} \left( x(t) + \frac{1}{2} x[t - 2\pi] \right) \right) \right) \right)' + \frac{3e^{-\sin t}}{2(t - \pi)t^2} x[t - \pi] e^{\sin x[t - \pi]} = 0 \quad \text{for } t > \pi$$

has a nonoscillatory solution  $x(t) = t$ . All conditions of Theorem 1 are satisfied except condition (5).

**Example 2.** Consider the neutral differential equation

$$(E_2) \quad (e^{-t}(x(t) + \frac{1}{2}x[t - 2\pi]))' + (2(1 + \frac{1}{2}e^{-2\pi})e^{\pi/4})e^{-t}x[t - \pi/4] = 0$$

for  $t \geq 2\pi$ .

All conditions of Theorem 1 are satisfied and hence every solution of Eq. (E<sub>2</sub>) is oscillatory. One such solution is  $x(t) = e^t \sin t$ .

**Remark.** Theorem 1 is new even if  $p \equiv 0$ .

**Theorem 2.** Let conditions (2)–(4) hold. If

$$(13) \quad \int_{t_0}^{\infty} q(s) [1 - p(s - \sigma)] ds = \infty,$$

then Eq. (1) is oscillatory.

**Proof.** Let  $x(t)$  be a nonoscillatory solution of Eq. (1). Assume  $x(t) > 0$  for  $t \geq t_0$ . As in the proof of Theorem 1, we get (8). Integrating (8) from  $t_2$  to  $t$  we have

$$L_{n-1} z(t) \leq L_{n-1} z(t_2) - \gamma z[t_2 - \sigma] \int_{t_2}^t q(s) [1 - p(s - \sigma)] ds$$

which, as  $t \rightarrow \infty$ , leads to a contradiction. The proof is complete.

**Example 3.** The neutral differential equation

$$(E_3) \quad (e^{3t/2}(x(t) + \frac{1}{2}x[t - \pi]))' + \frac{1}{2}(1 + \frac{1}{2}e^{\pi})e^{\pi}e^{3t/2}x[t - \pi] = 0, \quad t \geq 0$$

has a bounded nonoscillatory solution  $x(t) = e^{-t}$ . Only condition (2) of Theorem 2 is violated.

**Remark.** If  $a_i \equiv 1, i = 1, \dots, n - 1$  and  $f(x) = x$ , then Theorem 10 in [4] and Theorem 2 are the same.

For convenience of notation we define

$$\beta^n(t) = -\frac{\gamma q(t)}{p[t - \sigma + \tau]} > 0 \quad \text{and} \quad \delta = \frac{\sigma - \tau}{n} > 0.$$

**Theorem 3.** Let conditions (2) and (3) hold and let there exist constants  $p_1, p_2$  and  $q$  such that

$$(14) \quad -1 \leq p_1 \leq p(t) \leq p_2 < 0 \quad \text{for} \quad t \geq t_0$$

and

$$(15) \quad q(t) \geq q > 0 \quad \text{for} \quad t \geq t_0.$$

If

$$(16) \quad a_{n-i}[t - i\delta] - \left(\frac{\beta[t - \delta]}{\beta(t)}\right)^{i-1} \geq (i-1) \frac{\beta(t)}{\beta^2(t)} \geq 0, \quad i = 1, 2, \dots, n,$$

$t \geq t_0$

and

$$(17) \quad \liminf_{t \rightarrow \infty} \int_{t-\delta}^t \beta(s) ds > 1/e,$$

then every solution of Eq. (1) is oscillatory.

Proof. Let  $x(t)$  be a nonoscillatory solution of Eq. (1). Assume  $x(t) > 0$  for  $t \geq t_0$ . Then there exists a  $t_1 \geq t_0$  so that  $x[t - \tau] > 0$  and  $x[t - \sigma] > 0$  for  $t \geq t_1$ . Set

$$z(t) = x(t) + p(t)x[t - \tau].$$

Then

$$L_n z(t) \leq 0 \quad \text{for } t \geq t_1.$$

We claim that  $z(t)$  is eventually a negative solution of Eq. (1). From Eq. (1) we have

$$(18) \quad L_n z(t) = -q(t) \frac{f(x[t - \sigma])}{x[t - \sigma]} x[t - \sigma] \leq -\gamma q(t)x[t - \sigma] \leq \\ \leq -\gamma q x[t - \sigma] \quad \text{for } t \geq t_1,$$

which implies that  $L_{n-1} z(t)$  is strictly decreasing for  $t \geq t_1$ , while  $L_i z(t)$ ,  $n = 0, 1, \dots, n-2$  are strictly monotone functions of  $t$ . Therefore, either

$$(19) \quad \lim_{t \rightarrow \infty} L_{n-1} z(t) = -\infty$$

or

$$(20) \quad \lim_{t \rightarrow \infty} L_{n-1} z(t) = m \quad \text{is finite.}$$

Assume that (19) holds. Then we can easily obtain

$$(21) \quad \lim_{t \rightarrow \infty} L_i z(t) = -\infty, \quad i = 0, 1, \dots, n-1.$$

Next, assume that (20) is satisfied. Then integrating (18) from  $t_1$  to  $t$  and letting  $t \rightarrow \infty$ , we have

$$\int_{t_1}^{\infty} q x[s - \sigma] ds \leq L_{n-1} z(t_1) - m$$

which implies that  $x \in L_1[t_1, \infty)$ . Thus, by (14),  $z \in L_1[t_1, \infty)$ . Since  $z(t)$  is monotonic, it follows that

$$(22) \quad \lim_{t \rightarrow \infty} z(t) = 0$$

and so also  $m = 0$ . From (22) we conclude that for each  $i = 0, 1, \dots, n-1$

$$(23) \quad L_i z(t) L_{i+1} z(t) < 0 \quad \text{for } t \geq t_1.$$

Clearly, either (21) or (22) and (23) implies that

$$z(t) < 0 \quad \text{for } t \geq t_1.$$

Now,

$$x(t) < -p(t)z[t - \tau] \leq -p_1 x[t - \tau] \leq x[t - \tau]$$

which implies that  $x(t)$  is a bounded function.

Since

$$z[t - \sigma + \tau] = p[t - \sigma + \tau]x[t - \sigma] + x[t - \sigma + \tau] \geq \\ \geq p[t - \sigma + \tau]x[t - \sigma] \quad \text{for } t \geq t_1.$$

Thus

$$\frac{q(t)}{p[t - \sigma - \tau]} z[t - \sigma + \tau] \leq q(t) x[t - \sigma], \text{ for } t \geq t_1.$$

Using the above inequality in (18) we get

$$L_n z(t) - \left( \frac{-\gamma q(t)}{p[t - \sigma + \tau]} \right) z[t - (\sigma - \tau)] \leq 0 \text{ for } t \geq t_1$$

or

$$(14) \quad L_n z(t) - \beta^n(t) z[t - n\delta] \leq 0 \text{ for } t \geq t_1.$$

By Theorem 2 in [1] it follows that the above inequality has no eventually negative bounded solution. This contradicts the fact that  $z(t) < 0$  for  $t \geq t_1$  and the proof of theorem is complete.

**Theorem 4.** Let condition (14) in Theorem 3 be replaced by

$$(14) \quad p_1 \leq p(t) \leq p_2 < 0 \text{ for } t \geq t_0.$$

Then every bounded solution of Eq. (1) is oscillatory.

Proof. Let  $x(t)$  be a bounded nonoscillatory solution of Eq. (1) say  $x(t) > 0$  for  $t \geq t_0$ . As in the proof of Theorem 3 we have  $z(t) < 0$  for  $t \geq t_1$ . The rest of the proof is similar to that of Theorem 3 and hence is omitted.

**Theorem 5.** Let condition (16) in Theorem 3 (resp. Theorem 4) be replaced by

$$(25) \quad a_{n-i}[t - i\delta] \geq \beta(t) \text{ for } i = 1, 2, \dots, n.$$

Then the conclusion of Theorem 3 (resp. Theorem 4) holds.

Proof. The proof of Theorem 5 is similar to that of Theorem 3 (resp. Theorem 4) except that we use Theorem 3 in [1] instead of Theorem 2 in [1]. The details are omitted.

Remark. Theorems 3–5 include as special cases Theorems 8 and 11 in [4]. They also extend our Theorems 2 and 3 in [1] to more general equations of the form of Eq. (1). We also mention that conditions (16) and (25) are discarded if  $a_i \equiv 1$ ,  $i = 1, \dots, n - 1$ .

The following examples are illustrative.

Example 4. The neutral differential equation

$$(E_4) \quad \left( \frac{1}{t} [x(t) - x[t - 1]] \right)' + \frac{e - 1}{e^2} \left[ \frac{1}{t} + \frac{1}{t^2} \right] x[t - 2] = 0, \quad t > 0,$$

has a nonoscillatory solution  $x(t) = e^{-t}$ . All conditions of Theorem 5 are satisfied except condition (17).

Example 5. Consider the neutral differential equation

$$(E_5) \quad L_n[x(t) - \frac{1}{2}x[t - 1]] + \frac{1}{2}x[t - (n + 1)] = 0, \quad n \text{ even}, \quad t > n,$$

where  $L_0 x(t) = x(t)$ ,  $L_k x(t) = (1/t)(L_{k-1} x(t))'$ ,  $k = 1, 2, \dots, n$ ,  $a_0 = a_n = 1$ .



Here  $a_i(t) = t, i = 1, 2, \dots, n - 1, \beta(t) = 1$  and  $\delta = 1$ . All conditions of Theorem 3 (or 5) are satisfied and hence all solutions of Eq. (E<sub>5</sub>) are oscillatory.

We note that none of the results in [1]–[6] can be used to investigate the oscillatory character of Eq. (E<sub>5</sub>).

The following theorem includes Theorem 12 in [4] as a special case.

**Theorem 6.** *Let conditions (2) and (3) hold, and*

$$(26) \quad -1 \leq p(t) \leq 0 \quad \text{and} \quad q(t) \geq 0 \quad \text{for} \quad t \geq t_0.$$

If

$$(27) \quad \int_{t_0}^{\infty} q(s) \, ds = \infty,$$

then every unbounded solution of Eq. (1) is oscillatory.

*Proof.* Let  $x(t)$  be an unbounded nonoscillatory solution of Eq. (1) say  $x(t) > 0$  for  $t \geq t_0$ . There exists a  $t_1 \geq t_0$  so that  $x[t - \sigma] > 0$  and  $x[t - \tau] > 0$  for  $t \geq t_1$ . Set

$$z(t) = x(t) + p(t)x[t - \tau].$$

We have

$$(28) \quad L_n z(t) \leq -\gamma q(t)x[t - \sigma] \leq 0 \quad \text{for} \quad t \geq t_1,$$

and so  $L_i z(t)$  for  $i = 0, 1, \dots, n - 1$ , are monotone functions. We claim that  $z(t) \geq 0$  for  $t \geq t_1$ . Otherwise  $z(t) < 0$  for  $t \geq t_1$  and hence

$$x(t) < -p(t)x[t - \tau] \leq x[t - \tau] \quad \text{for} \quad t \geq t_1$$

which is impossible since  $x(t)$  is unbounded. Thus  $x(t) \geq 0$  for  $t \geq t_1$  and by Lemma 2, there exists a  $t_2 \geq t_1$  so that  $L_{n-1} z(t) \geq 0$  and  $\dot{z}(t) \geq 0$  for  $t \geq t_2$ .

Integrating (28) from  $t_2$  to  $t$  we have

$$\begin{aligned} L_{n-1} z(t) &\leq L_{n-1} z(t_2) - \gamma \int_{t_2}^t q(s)x[s - \sigma] \, ds \leq \\ &\leq L_{n-1} z(t_2) - \gamma z[t_2 - \sigma] \int_{t_2}^t q(s) \, ds \rightarrow -\infty \quad \text{as} \quad t \rightarrow \infty, \end{aligned}$$

a contradiction. This completes the proof.

For illustration we consider the following example.

**Example 6.** The neutral differential equation

$$(E_6) \quad \left( t \left( t \left( \frac{1}{t} \left( x(t) - \frac{1}{2}x[t - 2] \right) \right) \right) \right)' + \frac{e^{-\sin t}}{2(t - \pi)t^2} x[t - \pi] e^{\sin x[t - \pi]} = 0 \quad \text{for} \quad t > \pi,$$

has an unbounded nonoscillatory solution  $x(t) = t$ . Only condition (27) of Theorem 5 is violated. However, all unbounded solutions of the equation

$$(E_7) \quad \left( t \left( t \left( \frac{1}{t} \left( x(t) - \frac{1}{2}x[t - 2\pi] \right) \right) \right) \right)' + \frac{e^{-\sin t}}{t} x[t - \pi] e^{\sin x[t - \pi]} = 0$$

are oscillatory by Theorem 6. It is easy to check that Theorems 3–5 fail to apply to Eq. (E<sub>7</sub>).

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*Authors' addresses:* S. R. Grace, Department of Mathematical Sciences, University of Petroleum and Minerals, Dhahran, Saudi Arabia; B. S. Lalli, Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan, S7N 0W0 Canada.