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# ON REPRESENTATION OF CYCLICALLY ORDERED SETS

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In [5] we have constructed, for any cardinal $m$, an $m$-universal cyclically ordered set. The $m$-universality is meant there in the following sense: For any cyclically ordered set $\boldsymbol{G}$ with cardinality $\leqq m$ there exists a subset $\boldsymbol{G}'$ of the universal set constructed such that $\boldsymbol{G}$ is a strong homomorphic image of $\boldsymbol{G}'$. Here we present a construction of a set with an asymmetric and cyclic ternary relation such that any cyclically ordered set of cardinality $\leqq m$ is isomorphic with its suitable subset.

## 1. POWER OF TERNARY STRUCTURES

Let $G$ be a set and $C$ a ternary relation on $G$. The pair $\boldsymbol{G} = (G, C)$ will be called a *ternary structure*. Sometimes we denote by $\mathscr{C}(\boldsymbol{G})$ the carrier of this structure, i.e. $\mathscr{C}(\boldsymbol{G}) = G$, and by $\mathscr{R}(\boldsymbol{G})$ the relation of this structure, i.e. $\mathscr{R}(\boldsymbol{G}) = C$.

A ternary structure $\boldsymbol{G} = (G, C)$ is called
*reflexive*, iff $x, y, z \in G$, $\operatorname{card}\{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \in C$;
*irreflexive*, iff $x, y, z \in G$, $\operatorname{card}\{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \bar{\in} C$;
*symmetric*, iff $x, y, z \in G$, $(x, y, z) \in C \Rightarrow (z, y, x) \in C$;
*asymmetric*, iff $x, y, z \in G$, $(x, y, z) \in C \Rightarrow (z, y, x) \bar{\in} C$;
*cyclic*, iff $x, y, z \in G$, $(x, y, z) \in C \Rightarrow (y, z, x) \in C$;
*transitive*, iff $x, y, z, u \in G$, $(x, y, z) \in C$, $(x, z, u) \in C \Rightarrow (x, y, u) \in C$.

A *cyclically ordered set* is a ternary structure which is asymmetric, cyclic and transitive. A *cycle* is a cyclically ordered set $\boldsymbol{G} = (G, C)$ which is
*complete*, i.e. $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow (x, y, z) \in C$ or $(z, y, x) \in C$.

Let $\boldsymbol{G} = (G, C)$ be a ternary structure and $H \subseteq G$. We call the subset $H$ *discrete*, iff $H^3 \cap C = \emptyset$. An element $x \in G$ will be called *isolated*, iff $\{x, y, z\}$ is a discrete subset of $G$ for any $y \in G$, $z \in G$.

A direct sum, direct product and a homomorphism of ternary structures are defined in the obvious way. By the symbol $\operatorname{Hom}(\boldsymbol{G}, \boldsymbol{H})$ we denote the set of all homomorphisms of $\boldsymbol{G}$ into $\boldsymbol{H}$. An isomorphism of $\boldsymbol{G}$ onto $\boldsymbol{H}$ is a bijective homomorphism $f$ of $\boldsymbol{G}$ onto $\boldsymbol{H}$ such that $f^{-1}$ is a homomorphism of $\boldsymbol{H}$ onto $\boldsymbol{G}$. An injective homomorphism $f$ of $\boldsymbol{G}$ into $\boldsymbol{H}$ such that $f^{-1}$ is a homomorphism of $f(\boldsymbol{G})$ onto $\boldsymbol{G}$ will be called an embedding.

**1.1. Definition.** Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be ternary structures. A *power* $\boldsymbol{G}^{\boldsymbol{H}}$ is a ternary structure $(K, E)$ where $K = \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$ and for $f, g, h \in K$ we have $(f, g, h) \in E$ iff $(f(x), g(x), h(x)) \in C$ for any $x \in H$.

**1.2. Lemma.** *Let $\boldsymbol{G}, \boldsymbol{H}$ be ternary structures. Let $p$ be any of the properties: reflexivity, irreflexivity, symmetry, asymmetry, cyclicity, transitivity. If the structure $\boldsymbol{G}$ has a property $p$, then the structure $\boldsymbol{G}^{\boldsymbol{H}}$ has the property $p$.*

Proof is straightforward.

**1.3. Corollary.** *Let $\boldsymbol{G}$ be a cyclically ordered set and $\boldsymbol{H}$ a ternary structure. Then $\boldsymbol{G}^{\boldsymbol{H}}$ is a cyclically ordered set.*

For further purposes we now define a new operation of a power of ternary structures $\boldsymbol{G}, \boldsymbol{H}$. Its carrier is the same as for $\boldsymbol{G}^{\boldsymbol{H}}$; its relation is, however, an extension of $\mathscr{R}(\boldsymbol{G}^{\boldsymbol{H}})$.

**1.4. Definition.** Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be ternary structures. A *strong power* ${}^{\boldsymbol{H}}\boldsymbol{G}$ is a ternary structure $(K, E)$ where $K = \operatorname{Hom}(\boldsymbol{H}, \boldsymbol{G})$, and for $f, g, h \in K$ we have $(f, g, h) \in E$ iff

(1) there exists $x \in H$ such that $\{f(x), g(x), h(x)\}$ is a nondiscrete subset of $\boldsymbol{G}$;

(2) for any $x \in H$ with the property (1) we have $(f(x), g(x), h(x)) \in C$.

**1.5. Lemma.** *Let $\boldsymbol{G} = (G, C)$, $\boldsymbol{H} = (H, D)$ be ternary structures. Let $p$ be any of the properties: reflexivity, irreflexivity, symmetry, asymmetry, cyclicity. If the structure $\boldsymbol{G}$ has a property $p$, then the structure ${}^{\boldsymbol{H}}\boldsymbol{G}$ has the property $p$.*

Proof is easy in all cases. Let us show, for instance, that cyclicity of $\boldsymbol{G}$ implies cyclicity of ${}^{\boldsymbol{H}}\boldsymbol{G}$. Thus, let ${}^{\boldsymbol{H}}\boldsymbol{G} = (K, E)$ and $f, g, h \in K$, $(f, g, h) \in E$. Then there exists $x \in H$ such that $\{f(x), g(x), h(x)\}$ is nondiscrete in $\boldsymbol{G}$ and $(f(x), g(x), h(x)) \in C$ for any such $x$. Then $(g(x), h(x), f(x)) \in C$ which shows $(g, h, f) \in E$.

**1.6. Corollary.** *Let $\boldsymbol{G}$ be a cyclically ordered set and $\boldsymbol{H}$ a ternary structure. Then the ternary structure ${}^{\boldsymbol{H}}\boldsymbol{G}$ is asymmetric and cyclic.*

## 2. EMBEDDING OF A CYCLICALLY ORDERED SET INTO A STRONG POWER

Let us denote by the symbol $\mathbf{3}$ a 3-element cycle, i.e. $\mathbf{3} = (\{0, 1, 2\}, \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\})$. Further, let $\mathbf{3} + \mathbf{1}$ be the direct sum of a 3-element cycle and a one-element set $\{\omega\}$, i.e. $\mathbf{3} + \mathbf{1} = (\{0, 1, 2, \omega\}, \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\})$.

If $M$ is any (abstract) set, then $M$ will be considered as a discrete ternary structure, i.e. $M = (M, \emptyset)$.

**2.1. Theorem.** *Let $\boldsymbol{G} = (G, C)$ be a cyclically ordered set. Then there exists a set $M$ and an isomorphic embedding of $\boldsymbol{G}$ into ${}^{M}(\mathbf{3} + \mathbf{1})$.*

Proof. First note that by 1.5, ${}^{M}(\mathbf{3} + \mathbf{1})$ is an asymmetric and cyclic ternary structure. The carrier of this structure consists of all mappings $f: M \to \mathbf{3} + \mathbf{1}$. Denote $E = \mathscr{R}({}^M(\mathbf{3} + \mathbf{1}))$.

Let $G_1$ be the set of all nonisolated elements in $G$, $G_2$ the set of all isolated elements in $G$. Then $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \emptyset$. Choose any linear ordering $<$ on the set $G_1$ and call a triple $(x, y, z) \in C$ notable, if $x < y, x < z$. Note that if $(x, y, z) \in C$, then exactly one of the triples $(x, y, z)$, $(y, z, x)$, $(z, x, y)$ is notable. Let $M_1$ be the set of all notable triples in $C$ and put $M = M_1 \cup G_2$. Finally, for any $x \in G$ let us define a mapping $f_x: M \to \mathbf{3} + \mathbf{1}$ in the following manner:

(1) Let $x \in G_1$ and $m \in M$. If $m \in M_1$, $m = (x_0, x_1, x_2)$, we put

$$f_x(m) = \begin{cases} 0\,, & \text{if} \quad x = x_0 \\ 1\,, & \text{if} \quad x = x_1 \\ 2\,, & \text{if} \quad x = x_2 \\ \omega\,, & \text{if} \quad x \neq x_0,\ x \neq x_1,\ x \neq x_2\,. \end{cases}$$

If $m \in G_2$, we put $f_x(m) = \omega$.

(2) Let $x \in G_2$. Then we put

$$f_x(x) = 0\,, \quad f_x(m) = \omega \quad \text{for any} \quad m \in M - \{x\}\,.$$

Clearly, $f_x \in \mathscr{C}({}^M(\mathbf{3} + \mathbf{1}))$ for any $x \in G$. We show that the mapping $x \mapsto f_x$ is injective. Let $x, y \in G$, $x \neq y$. If $x \in G_1$, $y \in G_2$, then there exists $m \in M_1$, such that $x \in m$; then $f_x(m) \in \{0, 1, 2\}$, $f_y(m) = \omega$ and thus $f_x \neq f_y$. If $x, y \in G_2$, then $f_x(x) = 0$, $f_y(x) = \omega$ and $f_x \neq f_y$. Suppose finally that $x, y \in G_1$ and choose any $m \in M_1$ with $x \in m$. If $y \bar{\in} m$, then $f_x(m) \in \{0, 1, 2\}$, $f_y(m) = \omega$, thus $f_x \neq f_y$. If $y \in m = (x_0, x_1, x_2)$, then $x = x_i$, $y = x_j$ where $i, j \in \{0, 1, 2\}$, $i \neq j$. By definition of the mapping $f_x$ we then have $f_x(m) \neq f_y(m)$ so that $f_x \neq f_y$.

Further we show that the mapping $x \to f_x$ is a homomorphism of $\boldsymbol{G}$ into ${}^M(\mathbf{3} + \mathbf{1})$. Let $x, y, z \in G$, $(x, y, z) \in C$. Then $x, y, z \in G_1$ and there exists $m \in M_1$ such that $m$ is a cyclic permutation of $(x, y, z)$, say, $m = (y, z, x)$. Then $f_y(m) = 0, f_z(m) = 1$, $f_x(m) = 2$ and the subset $\{f_x(m), f_y(m), f_z(m)\}$ is nondiscrete in $\mathbf{3} + \mathbf{1}$. If $m \in M$ is any element such that $\{f_x(m), f_y(m), f_z(m)\}$ is a nondiscrete subset of $\mathbf{3} + \mathbf{1}$, then necessarily $m \in M_1$ and $x, y, z \in m$; otherwise some of the elements $f_x(m), f_y(m)$, $f_z(m)$ would be $\omega$. As $(x, y, z) \in C$, $m$ is a cyclic permutation of $(x, y, z)$; say, $m = (z, x, y)$. Then $f_x(m) = 1$, $f_y(m) = 2$, $f_z(m) = 0$ and $(f_x(m), f_y(m), f_z(m)) \in \mathscr{R}(\mathbf{3} + \mathbf{1})$. Thus $(f_x, f_y, f_z) \in E$.

Finally we show that the inverse mapping $f_x \mapsto x$ is a homomorphism from ${}^M(\mathbf{3} + \mathbf{1})$ onto $\boldsymbol{G}$. Let $x, y, z \in G$, $(f_x, f_y, f_z) \in E$. Then there exists $m \in M$ such that $\{f_x(m), f_y(m), f_z(m)\}$ is a nondiscrete subset of $\mathbf{3} + \mathbf{1}$, i.e. $(f_x(m), f_y(m), f_z(m)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Suppose, for instance, that $f_x(m) = 1$, $f_y(m) = 2$, $f_z(m) = 0$. This implies, by definition of the functions $f_x, f_y, f_z$, that $m \in M_1$ and $m = (z, x, y)$. Thus $(z, x, y) \in C$, i.e. $(x, y, z) \in C$ and we have shown that $(f_x, f_y, f_z) \in E$ implies $(x, y, z) \in C$.

**2.2.** Let $\boldsymbol{G} = (G, C)$ be a cyclically ordered set. By 2.1 there exists a set $M$ and a subset $\boldsymbol{G}(M)$ of a strong power ${}^{M}(\mathbf{3} + \mathbf{1})$ isomorphic with $\boldsymbol{G}$. Let us call this set $\boldsymbol{G}(M)$ a *representation of* $\boldsymbol{G}$ *in the set* $M$ and denote

$$\text{rep}\ \boldsymbol{G} = \min\{\text{card}\ M;\ \text{there exists a representation of } \boldsymbol{G} \text{ in } M\}$$

From the proof of 2.1 we immediately see

**2.3. Theorem.** *Let* $\boldsymbol{G} = (G, C)$ *be a cyclically ordered set and let* $G_2$ *be the set of all isolated elements in* $G$. *Then*

$$\text{rep}\ \boldsymbol{G} \leqq \tfrac{1}{3}\,\text{card}\ C + \text{card}\ G_2\ .$$

**2.4.** Let $m > 0$ be a cardinal. We call a ternary structure $\boldsymbol{H}$ *m-universal for cyclically ordered sets* iff for any cyclically ordered set $\boldsymbol{G} = (G, C)$ with card $G \leqq m$ there exists an isomorphic embedding of $\boldsymbol{G}$ into $\boldsymbol{H}$.

From 2.1 and its proof we obtain

**2.5. Theorem.** *Let* $m > 0$ *be a cardinal and* $n = \binom{m}{3} + m$. *Then a ternary structure of type* ${}^{n}(\mathbf{3} + \mathbf{1})$ *is m-universal for cyclically ordered sets; this structure is asymmetric and cyclic.*

## 3. CHARACTERIZATION OF NUMBER REP $G$

In the preceding section we have proved that any cyclically ordered set can be embedded into a strong power with base $\mathbf{3} + \mathbf{1}$ and discrete exponent. Here we show that to any cyclically ordered set $\boldsymbol{G}$ it is possible to assign a certain ternary structure – we call it a dominant of $\boldsymbol{G}$ – with the properties:

(1) knowing a dominant of $\boldsymbol{G}$ we know also $\boldsymbol{G}$,

(2) dominant of $\boldsymbol{G}$ can be embedded into a power of structures in the usual sense.

**3.1. Definition.** Let $\boldsymbol{G} = (G, C)$ be a cyclically ordered set, let $\boldsymbol{G}' = (G, D)$ be a ternary structure with $\mathscr{C}(\boldsymbol{G}') = \mathscr{C}(\boldsymbol{G})$. We call $\boldsymbol{G}'$ a *dominant of* $\boldsymbol{G}$ iff for any elements $x, y, z \in G$ the following equivalence holds:

$$(x, y, z) \in C \Leftrightarrow (x, y, z) \in D\ , \quad (z, y, x) \mathrel{\bar{\in}} D$$

Let us denote by $\mathbf{3} \oplus \mathbf{1}$ the following ternary structure:
$\mathscr{C}(\mathbf{3} \oplus \mathbf{1}) = \{0, 1, 2, \omega\}$,
$\mathscr{R}(\mathbf{3} \oplus \mathbf{1}) = \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \cup \{(x, y, z);\ x, y, z \in \{0, 1, 2, \omega\}$ and either $x = \omega$ or $y = \omega$ or $z = \omega$ or card $\{x, y, z\} \leqq 2\}$.

**3.2. Theorem.** *Let* $\boldsymbol{G} = (G, C)$ *be a cyclically ordered set, let* $\boldsymbol{G}(M) = (H, E)$ *be ite representation in a set* $M$. *Then* $\boldsymbol{H} = (H, H^3 \cap \mathscr{R}((\mathbf{3} \oplus \mathbf{1})^M)$ *is a dominant of this representation.*

Proof. Denote $H^3 \cap \mathscr{R}((\mathbf{3} \oplus \mathbf{1})^M) = D$. Let $f, g, h \in H$, $(f, g, h) \in E$. Then there exists $m_0 \in M$ such that $\{f(m_0), g(m_0), h(m_0)\}$ is a nondiscrete subset of $\mathbf{3} + \mathbf{1}$

and for any $m \in M$ with this property we have $(f(m), g(m), h(m)) \in \mathscr{R}(\mathbf{3} + \mathbf{1}) = \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Now, let $m \in M$ be any element. If either $f(m) = \omega$ or $g(m) = \omega$ or $h(m) = \omega$ or card $\{f(m), g(m), h(m)\} \leqq 2$, then $(f(m), g(m), h(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$. In all the other cases $(f(m), g(m), h(m)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \subseteq \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$. Hence we have $(f, g, h) \in D$. Suppose $(h, g, f) \in D$. Then $(h(m), g(m), f(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ for any $m \in M$, in particular $(h(m_0), g(m_0), f(m_0)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ and this is a contradiction. Thus, $(f, g, h) \in E$ implies $(f, g, h) \in D$, $(h, g, f) \bar{\in} D$. On the other hand, let $f, g, h \in H$, $(f, g, h) \in D$, $(h, g, f) \bar{\in} D$. Then $(f(m), g(m), h(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ for any $m \in M$. If $f(m) = \omega$ or $g(m) = \omega$ or $h(m) = \omega$ or card $\{f(m), g(m), h(m)\} \leqq 2$ for any $m \in M$, then $(h(m), g(m), f(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ for any $m \in M$, i.e. $(h, g, f) \in D$, a contradiction. Thus there exists $m_0 \in M$ with $\{f(m_0), g(m_0), h(m_0)\} = \{0, 1, 2\}$ so that $\{f(m_0), g(m_0), h(m_0)\}$ is a nondiscrete subset of $\mathbf{3} + \mathbf{1}$; further. for any $m \in M$ with this property we have $(f(m), g(m), h(m)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} = \mathscr{R}(\mathbf{3} + \mathbf{1})$. Thus $(f, g, h) \in E$.

**3.3. Theorem.** *Let $\mathbf{G} = (G, C)$ be a cyclically ordered set. Then* rep $\mathbf{G} = \min \{c \in \text{Card};$ *structure of type $(\mathbf{3} \oplus \mathbf{1})^c$ contains a subset isomorphic with a suitable dominant of $\mathbf{G}\}$.*

Proof. Denote rep $\mathbf{G} = r$, $\min \{c \in \text{Card}$; structure of type $(\mathbf{3} \oplus \mathbf{1})^c$ contains a subset isomorphic with a suitable dominant of $\mathbf{G}\} = s$. By definition of the number $r$, there exists a representation $(H, E)$ of $\mathbf{G}$ in a set $M$ with card $M = r$. By 3.2, $(H, H^3 \cap \mathscr{R}(\mathbf{3} \oplus \mathbf{1})^M)$ is a dominant of this representation, which is a substructure of the structure $(\mathbf{3} \oplus \mathbf{1})^M$ of type $(\mathbf{3} \oplus \mathbf{1})^r$. This dominant is isomorphic with a certain dominant of $\mathbf{G}$ and this implies $s \leqq r$. Conversely, let $M$ be a set with card $M = s$; by definition there exists a dominant $(G, D)$ of the structure $\mathbf{G}$ and an embedding of $(G, D)$ into $(\mathbf{3} \oplus \mathbf{1})^M$. Suppose that this embedding assigns to an element $x \in G$ an element $f_x \in \mathscr{C}((\mathbf{3} \oplus \mathbf{1})^M)$. Put $H = \{f_x; x \in G\}$, $E = H^3 \cap \mathscr{R}(^M(\mathbf{3} + \mathbf{1}))$ and $\mathbf{G}(M) = (H, E)$. We show that $\mathbf{G}(M)$ is a representation of $\mathbf{G}$ in the set $M$ where the corresponding isomorphism is the mapping $x \mapsto f_x$. The definition implies that this mapping is a bijection of $G$ onto $H$. Let $x, y, z \in G$, $(x, y, z) \in C$. Then $(x, y, z) \in D$, $(z, y, x) \bar{\in} D$. Hence $(f_x(m), f_y(m), f_z(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ for any $m \in M$ but there exists $m_0 \in M$ with $(f_z(m_0), f_y(m_0), f_x(m_0)) \bar{\in} \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$. Thus neither $f_x(m_0) = \omega$ nor $f_y(m_0) = \omega$ nor $f_z(m_0) = \omega$ nor card $\{f_x(m_0), f_y(m_0), f_z(m_0)\} \leqq 2$, i.e. $\{f_x(m_0), f_y(m_0), f_z(m_0)\} = \{0, 1, 2\}$ and for any $m \in M$ with this property we have, of course, $(f_x(m), f_y(m), f_z(m)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. This means $(f_x, f_y, f_z) \in \mathscr{R}(^M(\mathbf{3} + \mathbf{1}))$. Thus $(x, y, z) \in C$ implies $(f_x, f_y, f_z) \in E$.

Let $x, y, z \in G$, $(f_x, f_y, f_z) \in E$. Then there exists $m_0 \in M$ with $\{f_x(m_0), f_y(m_0), f_z(m_0)\} = \{0, 1, 2\}$ and for any $m \in M$ with this property we have $(f_x(m), f_y(m), f_z(m)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Let $m \in M$ be any element. If either $f_x(m) = \omega$ or $f_y(m) = \omega$ or $f_z(m) = \omega$ or card $\{f_x(m), f_y(m), f_z(m)\} \leqq 2$, then $(f_x(m), f_y(m), f_z(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$. In all the other cases we have, by the above, also $(f_x(m), f_y(m), f_z(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$. Thus $(f_x, f_y, f_z) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})^M)$ and, as a mapping $x \mapsto f_x$ is an isomorphism of $(G, D)$ into $(\mathbf{3} \oplus \mathbf{1})^M$, we have $(x, y, z) \in D$. Suppose $(z, y, x) \in D$. Then $(f_z, f_y, f_x) \in \mathscr{R}((\mathbf{3} \oplus \mathbf{1})^M)$, i.e. $(f_z(m), f_y(m), f_x(m)) \in \mathscr{R}(\mathbf{3} \oplus \mathbf{1})$ for any $m \in M$. But this contradicts the fact that $(f_x(m_0), f_y(m_0), f_z(m_0)) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$. Thus $(x, y, z) \in D$, $(z, y, x) \,\bar{\in}\, D$ and, as $(G, D)$ is a dominant of $\boldsymbol{G}$, we have $(x, y, z) \in$ $\in C$. We have proved that $\boldsymbol{G}(M)$ is a representation of $\boldsymbol{G}$ in the set $M$ which implies $r = \operatorname{rep} \boldsymbol{G} \leqq \operatorname{card} M = s$. Altogether we have $r = s$.

*References*


[1] *Alles, P., Nešetřil, J., Poljak, S.:* Extendability, Dimensions and Diagrams of Cyclic Orders. Preprint Nr. 944, TH Darmstadt, 1986.

[2] *Alles, P.:* Erweiterungen, Diagramme und Dimension zyklischer Ordnungen. Dissertation, TH Darmstadt, 1986.

[3] *Birkhoff, G.:* Generalized Arithmetic. Duke Math. Journ. *9* (1942), 283–302.

[4] *Birkhoff, G.:* Lattice Theory. New York, 1967.

[5] *Novák, V., Novotný, M.:* Universal Cyclically Ordered Sets. Czech. Math. Journ. *35* (*110*) (1985), 158–161.

[6] *Novák, V.:* On a Power of Relational Structures. Czech. Math. Journ. *35* (*110*) (1985), 167–172.



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