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ON UNBOUNDED POSITIVE SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENTS

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1. INTRODUCTION

This paper is concerned with the asymptotic behavior of positive solutions of the nonlinear ordinary differential equation

\[
y^{(n)} + \sum_{i=1}^{N} q_i(t) f_i(y) = 0
\]

subject to the hypotheses

1. \( n \geq 2; \)
2. \( (a) \) each \( q_i: [0, \infty) \to \mathbb{R}, 1 \leq i \leq N, \) is continuous;
   \( (b) \) each \( f_i: [0, \infty) \to (0, \infty), 1 \leq i \leq N, \) is continuous and nondecreasing.

Our attention will be focused on the case where each coefficient \( q_i(t) \) in (1) is oscillating, that is, \( q_i(t) \) changes its sign in any neighbourhood of infinity.

It is known that if, for some integer \( k, \) \( 0 \leq k \leq n - 1, \) there is a constant \( c > 0 \) such that

\[
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1} |q_i(t)| f_i(c t^k) \, dt < \infty ,
\]

then equation (1) has a positive solution \( y(t) \) which is asymptotic to the solution \( t^k \) of the corresponding unperturbed equation \( y^{(n)} = 0 \) in the sense that

\[
\lim_{t \to \infty} \frac{y(t)}{t^k} = \text{const} > 0 ;
\]

see Hale and Onuchic [1], Kitamura [2] and Švec [5].

In this paper we are interested in the situation in which equation (1) possesses a positive solution which is asymptotic to none of the solutions of \( y^{(n)} = 0; \) more precisely, we want to find criteria for the existence of a positive solution \( y(t) \) of (1) with the property

\[
\lim_{t \to \infty} \frac{y(t)}{t^k} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{y(t)}{t^{k-1}} = \infty
\]

for some integer \( k, \) \( 1 \leq k \leq n - 1. \) The desired existence criteria, given in Theorems 1 and 2 below, are formulated in terms of the positive part \( (q_i)_+ (t) \) and the negative
parts \((q_i)^-(t)\) of the coefficients \(q_i(t)\):

\[
(q_i)^+(t) = \max\{q_i(t), 0\}, \quad (q_i)^-(t) = \max\{-q_i(t), 0\}, \quad 1 \leq i \leq N,
\]

and show that, in case \(k\) is such that \(n \not\equiv k \pmod{2}\) [resp. \(n \equiv k \pmod{2}\)], there exists a solution \(y(t)\) satisfying (5) provided the contribution of \((q_i)^+(t)\) is greater than that of \((q_i)^-(t)\) [resp. the contribution of \((q_i)^+(t)\) is greater than that of \((q_i)^-(t)\)] in a suitable sense. Our results include part of the recent results of Kusano and Naito [3, 4] on the same problem for equation (1) in which all \(q_i(t) > 0\) or all \(q_i(t) < 0\) on \([0, \infty)\).

2. MAIN RESULTS

Our first result is the following

**Theorem 1.** (i) Let \(k\) be an integer such that \(1 \leq k \leq n - 1\) and \(n \not\equiv k \pmod{2}\). Then equation (1) has a positive solution \(y(t)\) satisfying (5) if the following conditions are satisfied:

\[
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1}(q_{i})^+(t)f_i(at^k)\,dt < \infty \text{ for some } a > 0, \\
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k}(q_{i})^-(t)f_i(bt^k)\,dt < \infty \text{ for some } b > 0, \\
\int_{0}^{\infty} t^{n-k}(q_{i_{0}})^+(t)f_i(c^k)\,dt = \infty \text{ for some } i_0, 1 \leq i_0 \leq N, \text{ and all } c > 0.
\]

(ii) Let \(k\) be an integer such that \(1 \leq k \leq n - 1\) and \(n \equiv k \pmod{2}\). Then equation (1) has a positive solution \(y(t)\) satisfying (5) if the following conditions are satisfied:

\[
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k}(q_{i})^-(t)f_i(at^k)\,dt < \infty \text{ for some } a > 0, \\
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k}(q_{i})^+(t)f_i(bt^k)\,dt < \infty \text{ for some } b > 0, \\
\int_{0}^{\infty} t^{n-k}(q_{i_{0}})^-(t)f_i(c^k)\,dt = \infty \text{ for some } i_0, 1 \leq i_0 \leq N, \text{ and all } c > 0.
\]

**Proof.** In either of the cases (i) and (ii) the desired solution of equation (1) will be obtained, via the Schauder-Tychonoff fixed point theorem, as a solution of the integral equation

\[
y(t) = \frac{at^{k-1}}{(k-1)!} + (-1)^{n-k-1} \int_{T}^{\infty} \int_{T}^{\infty} \frac{(t-s)^{k-1}}{(k-1)!} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r)f_i(y(r))\,dr\,ds, \\
t \geq T,
\]

for suitably chosen \(\alpha > 0\) and \(T > 0\).

(i) Let \(k, 1 \leq k \leq n - 1\), be such that \(n \not\equiv k \pmod{2}\). Let \(\alpha, 0 < \alpha < \min\{a, b\}\), be fixed, where \(a\) and \(b\) are positive constants appearing in (7) and (8). Because of (7)
and (8) there is a constant $T > 0$ such that 
\[
\sum_{i=1}^{N} \int_{T}^{t} t^{n-k-1}(q_i)(t)f_i(\alpha(t+1)^k)\,dt \leq \alpha
\]
and 
\[
\sum_{i=1}^{N} \int_{T}^{t} t^{n-k}(q_i)(t)f_i(\alpha(t+1)^k)\,dt \leq \frac{1}{2}\alpha.
\]

Let $C[T, \infty)$ be the locally convex space of all continuous functions on $[T, \infty)$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$, and consider the closed convex subset of $C[T, \infty)$ defined by 
\[
Y = \left\{ y \in C[T, \infty) : \frac{\alpha t^{k-1}}{2(k-1)!} \leq y(t) \leq \frac{\alpha t^{k-1}}{(k-1)!} + \frac{\alpha t^k}{k!}, \quad t \geq T \right\}.
\]

Define the mapping $F : Y \to C[T, \infty)$ by 
\[
F(y)(t) = \frac{\alpha t^{k-1}}{(k-1)!} + \int_{T}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r)f_i(y(r))\,dr\,ds, \quad t \geq T.
\]

It can be shown that $F$ is a continuous mapping which sends $Y$ into a compact subset of $Y$.

(a) $F(Y) \subseteq Y$. Noting that $q_i(t) = (q_i)^+(t) - (q_i)^-(t)$ and using (14), we have for $y \in Y$
\[
F(y)(t) \leq \frac{\alpha t^{k-1}}{(k-1)!} + \int_{T}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r)f_i(y(r))\,dr\,ds \leq \frac{\alpha t^{k-1}}{(k-1)!} + \frac{\alpha t^k}{k!} \leq \alpha(t+1)^k, \quad t \geq T.
\]

On the other hand, $y \in Y$ implies 
\[
F(y)(t) \geq \frac{\alpha t^{k-1}}{(k-1)!} - \int_{T}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r)f_i(y(r))\,dr\,ds = \frac{\alpha t^{k-1}}{(k-1)!} - \int_{T}^{t} \frac{(t-s)^{k-2}}{(k-2)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r)f_i(y(r))\,dr\,ds \geq \frac{\alpha t^{k-1}}{(k-1)!} - \int_{T}^{t} \frac{(t-s)^{k-2}}{(k-2)!} ds \sum_{i=1}^{N} \int_{T}^{t} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)(r)f_i(\alpha(r+1)^k)\,dr\,ds
\]
for $t \geq T$. Since, by (15), 
\[
\sum_{i=1}^{N} \int_{T}^{t} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)(r)f_i(\alpha(r+1)^k)\,dr\,ds =
\]
135
\[
\frac{N}{p} \sum_{i=1}^{N} \int_{T}^{r} \left( \int_{T}^{r-s} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \, ds \right) (q_i)^{-} (r) f_i(\alpha(r + 1)^k) \, dr + \\
+ \sum_{i=1}^{N} \int_{T}^{\infty} \left( \int_{T}^{r-s} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \, ds \right) (q_i)^{-} (r) f_i(\alpha(r + 1)^k) \, dr 
\]
\[
\leq \sum_{i=1}^{N} \int_{T}^{\infty} \frac{(r-T)^{n-k}}{(n-k)!} (q_i)^{-} (r) f_i(\alpha(r + 1)^k) \, dr \leq \frac{\alpha}{2}, \quad t \geq T, 
\]

we see from (19) that for \( y \in Y \)

\[
F_y(t) \geq \frac{\alpha t^{k-1}}{(k-1)!} - \frac{\alpha t^{k-1}}{2(k-1)!} = \frac{\alpha t^{k-1}}{2(k-1)!}, \quad t \geq T. 
\]

In deriving (21) from (19) we have assumed that \( k \geq 2 \). It is simpler to verify that (21) also holds for \( k = 1 \). It follows that \( y \in Y \) implies \( F_y \in Y \), that is, \( F(Y) \subset Y \).

(b) \( F \) is continuous. Let \( \{y_v\} \) be a sequence of elements of \( Y \) converging to \( y \in Y \) in the \( C[T, \infty) \) topology. We then have

\[
|F_{y_v}(t) - F_y(t)| \leq \frac{\alpha t^{k-1}}{k!} \int_{T}^{\infty} r^{n-k-1} \sum_{i=1}^{N} |q_i(r)| |f_i(y_v(r)) - f_i(y(r))| \, dr 
\]

for \( t \geq T \). The integrand on the right-hand side of the above tends to zero pointwise on \( [T, \infty) \) as \( v \to \infty \) and is bounded by

\[
2r^{n-k-1} \sum_{i=1}^{N} [(q_i)^+(r) + (q_i)^-(r)] f_i(\alpha(r + 1)^k) 
\]

which is integrable on \( [T, \infty) \) by (7) and (8), and so the Lebesgue dominated convergence theorem shows that \( F_{y_v}(t) \to F_y(t) \) as \( v \to \infty \) uniformly on each compact subinterval of \( [T, \infty) \). Therefore, \( F_{y_v} \to F_y \) as \( v \to \infty \) in \( C[T, \infty) \), which implies the continuity of \( F \).

(c) \( F(Y) \) is relatively compact. This follows from the observation that, for \( y \in Y \), \( (F_y)' \) is given by

\[
(F_y)'(t) = \int_{T}^{\infty} \frac{(r-t)^{n-2}}{(n-2)!} \sum_{i=1}^{N} q_i(r) f_i(\alpha(r + 1)^k) \, dr, \quad k = 1, \\
(F_y)'(t) = \frac{\alpha t^{k-2}}{(k-2)!} + \int_{T}^{t} \frac{(t-s)^{k-2}}{(k-2)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(r) f_i(\alpha(r + 1)^k) \, dr \, ds, \quad k \geq 2, 
\]

and satisfies

\[
|(F_y)'(t)| \leq \int_{T}^{\infty} r^{n-2} \sum_{i=1}^{N} |q_i(r)| f_i(\alpha(r + 1)^k) \, dr, \quad k = 1, \\
|(F_y)'(t)| \leq \frac{\alpha t^{k-2}}{(k-2)!} + \frac{t^{k-1}}{(k-1)!} \int_{T}^{\infty} r^{n-k-1} \sum_{i=1}^{N} |q_i(r)| f_i(\alpha(r + 1)^k) \, dr, \quad k \geq 2. 
\]

Therefore the Schauder-Tychonoff fixed point theorem is applicable to \( F \) and there
exists an element \( y \in Y \) such that \( y = Fy \). This function \( y = y(t) \) satisfies (13), so that it is a positive solution of equation (1) on \([T, \infty)\).

To study the asymptotic behavior of \( y(t) \) we note from (13) that

\[
y^{(k-1)}(t) = \alpha + \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r)) \, dr \, ds, \quad t \geq T
\]

and

\[
y^{(k)}(t) = \int_T^t \frac{(r-t)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r)) \, dr, \quad t \geq T.
\]

It is obvious that \( y^{(k)}(t) \to 0 \) as \( t \to \infty \). Use of (20) shows that

\[
y^{(k-1)}(t) = \alpha - \sum_{i=1}^N \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_- (r) f_i(y(r)) \, dr \, ds
\]

\[
+ \sum_{i=1}^N \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} (q_i)_+ (r) f_i(y(r)) \, dr \, ds
\]

\[
\geq \frac{\alpha}{2} + \sum_{i=1}^N \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \frac{a^{r-1}}{2(k-1)!} \, dr \, ds, \quad t \geq T,
\]

which, combined with (9), implies that \( y^{(k-1)}(t) \to \infty \) as \( t \to \infty \). Thus, the solution \( y(t) \) satisfies \( \lim_{t \to \infty} y^{(k)}(t) = 0 \) and \( \lim_{t \to \infty} y^{(k-1)}(t) = \infty \) which is equivalent to (5) by L'Hospital's rule. This completes the proof of (i).

(ii) Let \( k, 1 \leq k \leq n-1 \), be an integer such that \( n \equiv k \pmod{2} \). Let \( \alpha, 0 < \alpha < \min \{a, b\} \) be fixed, take \( T > 0 \) so large that

\[
\sum_{i=1}^N \int_T^\infty t^{n-k-1}(q_i)_-(t) f_i(\alpha(t + 1)^k) \, dt \leq \alpha,
\]

\[
\sum_{i=1}^N \int_T^\infty t^{n-k}(q_i)_+(t) f_i(\alpha(t + 1)^k) \, dt \leq \frac{1}{2}\alpha
\]

and define the operator \( F \) by

\[
Fy(t) = \frac{\alpha t^{k-1}}{(k-1)!} - \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^N q_i(r) f_i(y(r)) \, dr \, ds, \quad t \geq T.
\]

Then, applying the same argument as in the preceding case, we obtain a fixed point \( y \) of \( F \) in the set \( Y \) defined by (16), which gives rise to a positive solution of equation (1) existing on \([T, \infty)\) and having the asymptotic behavior (5). This finishes the proof.

Example 1. Consider the mixed sublinear-superlinear equation

\[
y^{(n)} + \varphi(t) y^\gamma + \psi(t) y^\delta = 0,
\]

where \( n \geq 2, \; 0 < \gamma < 1, \; \delta > 1 \) and \( \varphi, \psi: [0, \infty) \to \mathbb{R} \) are continuous.
Conditions (7) and (8) reduce, respectively, to
\[
\int_0^\infty t^{n-k-1+k\gamma} \varphi_+(t) \, dt < \infty, \quad \int_0^\infty t^{n-k-1+k\delta} \psi_+(t) \, dt < \infty
\]
and
\[
\int_0^\infty t^{n-k+k\gamma} \varphi_-(t) \, dt < \infty, \quad \int_0^\infty t^{n-k+k\delta} \psi_-(t) \, dt < \infty,
\]
and condition (9) is equivalent to requiring that either
\[
\int_0^\infty t^{n-k+(k-1)\gamma} \varphi_+(t) \, dt = \infty
\]
or
\[
\int_0^\infty t^{n-k+(k-1)\delta} \psi_+(t) \, dt = \infty.
\]
It is easily seen that (26) and the second condition of (23) are not consistent because \((n - k - 1 + k\delta) - (n - k + (k - 1) \delta) = \delta - 1 > 0\). Therefore, by (i) of Theorem 1 we conclude that if \(k\) is an integer such that \(1 \leq k \leq n - 1\) and \(n \not\equiv k \pmod{2}\), then conditions (23), (24) and (25) are sufficient for equation (22) to have a positive solution \(y(t)\) satisfying (5). Similarly, the conditions
\[
\int_0^\infty t^{n-k+k\gamma} \varphi_+(t) \, dt < \infty, \quad \int_0^\infty t^{n-k+k\delta} \psi_+(t) \, dt < \infty,
\]
and
\[
\int_0^\infty t^{n-k+(k-1)\gamma} \varphi_+(t) \, dt = \infty
\]
guarantee the existence of a positive solution \(y(t)\) satisfying (5). It should be noticed that Theorem 1 cannot be applied to the purely superlinear case of (22).

Theorem 1 is a local existence theorem in that the solution is guaranteed to exist on an interval \([T, \infty)\), \(T > 0\) being sufficiently large, that is, in a “small” neighborhood of infinity. Under stronger sublinearity hypotheses on \(f_i(y)\) the global existence of a solution satisfying (5) can be established as the following theorem shows.

**Theorem 2.** Suppose that, for each \(i\), \(1 \leq i \leq N\), \(y^{-1} f_i(y)\) is nonincreasing and satisfies
\[
\lim_{y \to \infty} y^{-1} f_i(y) = 0.
\]

(i) Let \(k\) be an integer such that \(1 \leq k \leq n - 1\) and \(n \not\equiv k \pmod{2}\). Then, condition (7), (8) and (9) are sufficient for equation (1) to have infinitely many positive solutions \(y(t)\) which exist on \([0, \infty)\) and satisfy (5).

(ii) Let \(k\) be an integer such that \(1 \leq k \leq n - 1\) and \(n \equiv k \pmod{2}\). Then conditions (10), (11) and (12) are sufficient for equation (1) to have infinitely many positive solutions \(y(t)\) which exist on \([0, \infty)\) and satisfy (5).

**Proof.** We only prove the statement (i), since the statement (ii) is proved similarly. Suppose that \(k\) satisfies \(1 \leq k \leq n - 1\) and \(n \not\equiv k \pmod{2}\). Let \(\alpha, 0 < \alpha < \min\{a, b\}\), be fixed. Then, by (7), the function \(\sum_{i=1}^N r_i^{n-k-1}(q_i)_+(t)f_i(\alpha(t+1)^k)\)
is integrable on \([0, \infty)\), and, in view of the nonincreasing nature of \(y^{-1}f_i(y)\), \(\beta > \alpha\) implies
\[
\beta^{-1} \sum_{i=1}^{N} t^{n-k-1}(q_i)_{+} (t) f_i(\beta(t + 1)^k) \leq \alpha^{-1} \sum_{i=1}^{N} t^{n-k-1}(q_i)_{+} (t) f_i(\alpha(t + 1)^k),
\]
\(t \geq 0\).

Moreover, by (30), the left hand side of the above tends to zero as \(\beta \to \infty\) pointwise on \([0, \infty)\). So, the Lebesgue dominated convergence theorem shows that
\[
\lim_{\beta \to \infty} \beta^{-1} \sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1}(q_i)_{+} (t) f_i(\beta(t + 1)^k) \, dt = 0,
\]
and similarly
\[
\lim_{\beta \to \infty} \beta^{-1} \sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1}(q_i)_{-} (t) f_i(\beta(t + 1)^k) \, dt = 0.
\]

Because of (31) and (32), a constant \(\beta_0 > 0\) can be chosen so that
\[
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1}(q_i)_{+} (t) f_i(\beta(t + 1)^k) \, dt \leq \beta
\]
and
\[
\sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1}(q_i)_{-} (t) f_i(\beta(t + 1)^k) \, dt \leq \frac{1}{2}\beta
\]
for all \(\beta \geq \beta_0\). If we define, for a fixed \(\beta \geq \beta_0\),
\[
Y = \left\{ y \in C[0, \infty) : \frac{\beta t^{k-1}}{2(k-1)!} \leq y(t) \leq \frac{\beta t^{k-1}}{(k-1)!} + \frac{\beta t^k}{k!}, \quad t \geq 0 \right\}
\]
\[
Fy(t) = \frac{\beta t^{k-1}}{(k-1)!} + \int_{0}^{t} \frac{(t-s)^{k-2}}{(k-1)!} \int_{s}^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \sum_{i=1}^{N} q_i(s)f_i(y(r)) \, dr \, ds, \quad t \geq 0,
\]
and proceed as in the proof of (i) of Theorem 1, then we can show that \(F\) has a fixed element \(y \in Y\) and that this element gives a solution of equation (1) existing on \([0, \infty)\) and satisfying (5). Since any positive number \(\beta\) greater than \(\beta_0\) can be taken in defining \(Y\) and \(F\), there exist infinitely many such solutions of (1). This completes the proof of the first statement of the theorem.

Example 2. Consider the differential equation
\[
y^{(n)} + q(t) y'[\log (1 + y)]^{\delta} = 0,
\]
where \(n \geq 2, 0 < \gamma < 1, 0 < \gamma + \delta < 1\), and \(q: [0, \infty) \to \mathbb{R}\) is continuous. This is a special case of (1) in which \(N = 1, q_1(t) = q(t), f_1(y) = y'[\log (1 + y)]^{\delta}\). Clearly, \(y^{-1}f_1(y)\) is nonincreasing for \(y > 0\) and satisfies \(\lim_{y \to \infty} y^{-1}f_1(y) = 0\). Conditions (7), (8) and (9) for equation (33) reduce, respectively, to
\[
\int_{0}^{\infty} t^{n-k-1+\gamma}(\log t)^{\delta} q_+(t) \, dt < \infty,
\]
and
\[
\int_{0}^{\infty} t^{n-k+\gamma}(\log t)^{\delta} q_-(t) \, dt < \infty.
\]
and
\[ (36) \quad \int_{-\infty}^{\infty} t^{n-k+\gamma(k-1)}((k-1)\log t)^{q(t)} \, dt = \infty ; \]

conditions (10), (11) and (12) can be formulated by interchanging the role of \( q_+(t) \) and \( q_-(t) \) in (34), (35) and (36).

If in particular, \( n = 2, \ k = 1, \) and \( q(t) \) is given by
\[ (37) \quad q(t) = \begin{cases} \frac{\sin(\log t)}{t^2} & \text{for } \sin (\log t) \geq 0, \ i.e., \ for \ t \in \bigcup_{i=0}^{\infty} \left[ e^{2i\pi}, e^{(2i+1)\pi} \right] \\ \frac{\sin(\log t)}{t^3} & \text{for } \sin (\log t) \geq 0, \ i.e., \ for \ t \in \bigcup_{i=1}^{\infty} \left[ e^{(2i-1)\pi}, e^{2i\pi} \right], \end{cases} \]

then conditions (34) – (36) are satisfied, because
\[ \int_{e}^{\infty} t^\gamma (\log t)^{q} q_+(t) \, dt < \int_{e}^{\infty} \frac{(\log t)^{\delta}}{t^{2-\gamma}} \, dt < \infty, \]
\[ \int_{e}^{\infty} t^{\gamma+1} (\log t)^{q} q_-(t) \, dt < \int_{e}^{\infty} \frac{(\log t)^{\delta}}{t^{2-\gamma}} \, dt < \infty, \]

and
\[ \int_{1}^{\infty} t q_+(t) \, dt = \sum_{i=0}^{\infty} \int_{e^{2i\pi}}^{e^{(2i+1)\pi}} \frac{\sin(\log t)}{t} \, ds = \sum_{i=0}^{\infty} \int_{2i\pi}^{(2i+1)\pi} \sin s \, ds = \infty. \]

Consequently, by (i) of Theorem 2, equation (33) with \( n = 2 \) and \( q(t) \) defined by (37) possesses infinitely many positive solution \( y(t) \) which exist on \([e, \infty)\) and have the asymptotic behavior
\[ \lim_{t \to \infty} \frac{y(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} y(t) = \infty. \]

Remark 1. Condition (7) and (8) \[ \text{resp. (10) and (11)} \] clearly imply
\[ \sum_{i=1}^{N} \int_{0}^{\infty} t^{n-k-1} |q_i(t)| f_i(x^k) \, dt < \infty \]
for \( k, \ 1 \leq k' \leq n - 1, \) such that \( n \equiv k \mod 2 \) \[ \text{resp. } n \equiv k \mod 2 \], where \( x = \min \{ a, b \} \). It follows that, under the hypotheses of Theorems 1 and 2, equation (1) also has a positive solution \( y(t) \) satisfying (4):
\[ \lim_{t \to \infty} \frac{y(t)}{t^k} = \text{const} > 0. \]

Remark 2. If assumption (2 – c) is replaced by
(2) \[ (c') \quad \text{each } f_i: \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing and } y f_i(y) > 0 \text{ for } y \neq 0, \]

then one can easily formulate criteria for the existence of a negative solution \( y(t) \) of (1) with the property
\[ \lim_{t \to \infty} \frac{y(t)}{t^k} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{y(t)}{t^{k-1}} = -\infty. \]
References


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