Ar. Meenakshi

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ON THE PARTIAL ORDERING OF ALMOST DEFINITE MATRICES

AR. MEENAKSHI, Annamalainagar*)

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INTRODUCTION

Let $C_{m\times n}$ be the linear space of complex matrices of order $m \times n$. Let C_n be the linear space of complex n-tuples. The vectors in C_n are denoted by lower case letters x, y, z etc. Matrices are denoted by capital letters such as A, B, etc. For $A \in C_{m\times n}$, let $A^*, A^+, N(A)$ and $R(A)$ denote conjugate transpose, Moore Penrose inverse, null space and range space respectively. A matrix $A \in C_{n\times n}$ is said to be *almost definite* (a.d [4]) if for $x \in C_n$, $x^*Ax = 0 \Rightarrow Ax = 0$. A is *positive semi definite* [4, 7] if $\operatorname{Re}(x^*Ax) \geqq 0$. A matrix is said to be *almost positive definite* (a.p.d) if it is both a.d and p.s.d. Recently, Mitra and Puri [9] have introduced the notion of *quasi positive definite matrix* (q.p.d). A is said to be q.p.d if A is p.s.d and $\operatorname{Re}(x^*Ax) = 0 \Rightarrow Ax = 0$. A q.p.d matrix is always a.p.d [9]. These special types of matrices are widely used in the study of electrical networks and in linear electromechanical systems [4, 9]. For properties of a.d, a.p.d and q.p.d matrices one may refer [4, 9, 10]. It was pointed out by Duffin and Morley [4] that the unique transfer impedance in a general linear electromechanical system exists for every structure operator if and only if the constitutive operator is a.d. For terminology and representation of a general linear electromechanical system by a pair of equations, one may refer [4].

We are concerned with the hermitian positive semi definite partial ordering on complex matrices. For $A, B \in C_{n\times n}$, $A \geqq B \Leftrightarrow A - B \geqq 0 \Leftrightarrow (A - B)$ is *hermitian positive semi definite* (h.p.s.d). It is well known [2, p. 59] that for non-singular matrices A, B if $A \geqq B \geqq 0$ then $B^{-1} \geqq A^{-1} \geqq 0$. This was extended to generalized inverses of certain types of pairs of singular matrices $A \geqq B \geqq 0$ by Hans J. Werner [5] and independently by Hartwig [6]. Here we have extended their results for a wider class of a.p.d matrices. Further, some well known [3] matrix inequalities ona pair of h.p.s.d matrices have been extended to a pair of a.p.d matrices. As an application, it is shown that under certain conditions the monotonicity of the constitutive operators in linear electromechanical systems having the same structure operator is preserved for the corresponding transfer impedances.

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RESULTS

First we shall prove certain lemmas which will simplify the proof of the main result.

Lemma 1. *Let $A, B \in C_{n\times n}$ such that $A \geqq B$. If B is a.p.d (q.p.d) then A is a.p.d (q.p.d).*

Proof. Since $A \geqq B$, $A - B = P$ is h.p.s.d. $\Rightarrow x^*Px$ is real and $x^*Px \geqq 0$ for all $x \in C_n \Rightarrow \operatorname{Re}(x^*Ax) \geqq \operatorname{Re}(x^*Bx)$ and $I_m(x^*Ax) = \operatorname{Im}(x^*Bx)$. For $x \in C_n$, $x^*Ax = 0 \Rightarrow \operatorname{Im}(x^*Ax) = 0$ and $\operatorname{Re}(x^*Ax) = 0 \Rightarrow \operatorname{Im}(x^*Bx) = 0$ and $\operatorname{Re}(x^*Bx) + x^*Px = 0$.

Since B and P are p.s.d, $\operatorname{Re}(x^*Bx) \geqq 0$ and $x^*Px \geqq 0$. Hence $x^*Ax = 0 \Rightarrow \operatorname{Im}(x^*Bx) = 0$, $\operatorname{Re}(x^*Bx) = 0$ and $x^*Px = 0 \Rightarrow x^*Bx = 0$ and $x^*Px = 0$.

On account of B a.d and P h.p.s.d, $x^*Bx = 0 \Rightarrow Bx = 0$ and $x^*Px = 0 \Rightarrow Px = 0$. Therefore, $Ax = Bx + Px = 0$. Thus for $x \in C_n$ if $x^*Ax = 0$ then $Ax = 0$, and hence A is a.d. Since B and P are p.s.d. A is also p.s.d. Thus A is a.p.d. Similarly we can prove that A is q.p.d whenever B is q.p.d. Hence the Lemma.

Lemma 2. *Let $A, B \in C_{n\times n}$ such that $A \geqq B$. If B is a.p.d then $R(B) \subseteq R(A)$ or equivalently $N(A) \subseteq N(B)$.*

Proof. Since B is a.p.d, by Lemma 1, A is a.p.d. On account of Lemma (2.1) of [9] both A and B are EP matrices. Therefore $R(B) \subseteq R(A)$ is equivalent to $N(A) \subseteq N(B)$. Now we shall prove that $N(A) \subseteq N(B)$ as follows:

$x \in N(A) \Rightarrow Ax = 0 \Rightarrow x^*Ax = 0 \Rightarrow x^*Bx + x^*Px = 0$. Since B is p.s.d and P is h.p.s.d. as in the proof of Lemma 1, we get, $x^*Bx = 0$ and $x^*Px = 0$. Since B is a.d, $x^*Bx = 0 \Rightarrow Bx = 0. \Rightarrow x \in N(B)$. Thus $N(A) \subseteq N(B)$. Hence the Lemma.

Remark 1. We note that the condition on B cannot be relaxed. For example, let

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

Both A and B are neither a.d nor p.s.d. Hence A and B are not a.p.d.

$$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geqq 0$$

but $R(B) \not\subseteq R(A)$.

Thus Lemma 2 fails.

Lemma 3. *Let $A, B \in C_{n\times n}$ such that $A \geqq B$. If B is a.p.d then $R(A) = R(B) \Leftrightarrow$ rank A = rank B.*

Proof. $(\Rightarrow)$ is trivial and $(\Leftarrow)$ is a direct consequence of Lemma 2.

Theorem 1. *Let $A, B \in C_{n\times n}$ such that $A \geqq B$ and B be a.p.d. Then $B^+ \geqq A^+ \Rightarrow R(A) = R(B)$.*

Proof. Since $A \geqq B$ and B is a.p.d, by Lemma 1, A is a.p.d and by Lemma 2, $R(B) \subseteq R(A)$. Since A and B are a.p.d, on account of Lemma 2.5 of [9], A^+ and B^+

are a.p.d with $B^+ \geqq A^+$; another application of Lemma 2, yields $R(A^+) \subseteq R(B^+) \Rightarrow \Rightarrow R(A^*) \subseteq R(B^*) \Rightarrow R(A) \subseteq R(B)$. Thus $R(A) = R(B)$. Hence the Theorem.

For a complex matrix V, let $\frac{1}{2}(V + V^*)$ be denoted by Sym V. It is known that V is q.p.d $\Leftrightarrow$ Sym V is h.p.s.d and rank (Sym V) = rank (V). [refer Lemma (2.8) in 9]

Theorem 2. *Let $A, B \in C_{n \times n}$ such that $A \geqq B$ and B be q.p.d. Then the following are equivalent.*

i) $R(A) = R(B)$;
ii) rank (A) = rank (B);
iii) $(\text{Sym } B)^+ \geqq (\text{Sym } A)^+$.

Proof. (i) $\Leftrightarrow$ (ii). Since B is q.p.d, B is also a.p.d. Hence by Lemma 3, this equivalence is clear.

(ii) $\Rightarrow$ (iii): Since A and B are q.p.d, by Lemma (2.8) of [9], Sym $A \geqq 0$; Sym $B \geqq 0$ and rank (Sym A) = rank (A); rank (Sym B) = rank (B). Further $A \geqq B \Rightarrow A^* \geqq \geqq B^* \Rightarrow A + A^* \geqq B + B^* \Rightarrow$ Sym $A \geqq$ Sym B. Thus (Sym A) $\geqq$ (Sym B) $\geqq 0$ and by (ii) rank (Sym A) = rank (Sym B). Now by Theorem 1 of [5], $(\text{Sym } B)^+ \geqq \geqq (\text{Sym } A)^+$. Thus (iii) holds.

Conversely, A and B are q.p.d with $A \geqq B \Rightarrow$ Sym $A \geqq$ Sym $B \geqq 0$ and by (iii) $(\text{Sym } B)^+ \geqq (\text{Sym } A)^+$. Now by Theorem 1 of [5], rank (Sym A) = rank (Sym B) which implies rank (A) = rank (B). Thus (ii) holds. Hence the Theorem.

Remark 2. We note that the condition on B cannot be relaxed. This is illustrated in the following.

$$A = \begin{bmatrix} 2+i & 1+i \\ 1+i & 1+i \end{bmatrix} \text{ is q.p.d by Lemma (2.8) of [9]},$$

$$B = \begin{bmatrix} 1+i & i \\ i & i \end{bmatrix} \text{ is a.p.d but not q.p.d}.$$

A and B are nonsingular, hence $R(A) = R(B)$ and rank A = rank B.

$$A - B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \geqq 0,$$

$$\text{Sym } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (\text{Sym } B)^+,$$

$$\text{Sym } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad (\text{Sym } A)^+ = (\text{Sym } A)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

$$(\text{Sym } B)^+ - (\text{Sym } A)^+ = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \not\geqq 0.$$

Thus the Theorem fails.

Remark 3. In particular if $A \geqq B \geqq 0$, then Sym $A = A$ and Sym $B = B$; and the above Theorem reduces to the following known results.

Corollary 1 (Theorem 1 in [6]). *Let $A, B \in C_{n\times n}$ such that $A \geqq B \geqq 0$ then $B^+ \geqq A^+ \Leftrightarrow R(A) = R(B)$.*

Corollary 2. (Theorem 1 in [5]): *For $A, B \in C_{n\times n}$, any two of the following three conditions imply the other one.*

1) $A \geqq B \geqq 0$;
2) $\operatorname{rank}(A) = \operatorname{rank}(B)$;
3) $B^+ \geqq A^+ \geqq 0$.

Proof. (1) and (2) $\Rightarrow$ (3), (1) and (3) $\Rightarrow$ (2) follow from Theorem 2, using $\operatorname{Sym} A = A$; and $\operatorname{Sym} B = B$.

The proof for (2) and (3) $\Rightarrow$ (1) runs as follows: Since $\operatorname{rank} A = \operatorname{rank} A^+$ and $\operatorname{rank} B = \operatorname{rank} B^+$; $B^+ \geqq A^+ \geqq 0$ and $\operatorname{rank} A^+ = \operatorname{rank} B^+ \Rightarrow (A^+)^+ \geqq (B^+)^+ \geqq 0 \Rightarrow A \geqq B \geqq 0$. Thus (1) holds. Hence the corollary.

Corollary 3. *Let $A, B \in C_{n\times n}$ such that $A \geqq B$ and B be q.p.d. Then $B^+ \geqq A^+$ imply the following equivalent statements.*

i) $R(A) = R(B)$;
ii) $\operatorname{rank}(A) = \operatorname{rank}(B)$;
iii) $(\operatorname{Sym} B)^+ \geqq (\operatorname{Sym} A)^+$.

Proof. The equivalence of these statements follows from Theorem 2. Now by Theorem 1, $A \geqq B$ and $B^+ \geqq A^+ \Rightarrow R(A) = R(B)$ holds for any a.p.d matrix B. Since B is q.p.d, B is also a.p.d. Thus, $A \geqq B$ and $B^+ \geqq A^+ \Rightarrow R(A) = R(B)$. Hence the corollary.

Remark 4. In particular for $A \geqq B \geqq 0$, the converse holds. However in general the converse need not be true can be seen from the following example.

Let

$$A = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

both A and B are q.p.d.

$$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geqq 0\,.$$

$$\operatorname{Sym} A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \operatorname{Sym} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(\operatorname{Sym} A)^+ = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad (\operatorname{Sym} B)^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(\operatorname{Sym} B)^+ \geqq (\operatorname{Sym} A)^+\,.$$

Since A and B are nonsingular, $R(A) = R(B)$. $\operatorname{rank} A = \operatorname{rank} B$. Thus statement (i), (ii) and (iii) hold.

$$B^+ = B^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad \text{and} \quad A^+ = A^{-1} = \tfrac{1}{5}\begin{bmatrix} 2 & -i \\ -i & 2 \end{bmatrix},$$

$$10(B^+ - A^+) = \begin{bmatrix} 1 & -3i \\ -3i & 1 \end{bmatrix} \ngeqq 0 \Rightarrow B^+ \ngeqq A^+ .$$

Thus the converse is not true.

For a partitioned matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

the matrix $D - CA^+B$ is called *generalized Schur complement of A in M* and is denoted by M/A. Now we shall generalize the results found in [3] in the following way.

Lemma 4. *Let H and K be a.p.d matrices of order n such that $H \geqq -K$. Let X and Y be $n \times m$ matrices satisfying*

$$N(H) \subseteq N(X^*) ; \quad N(K) \subseteq N(Y^*) \tag{1.1}$$

and

$$X^*H^+ = (H^+X)^* ; \quad Y^*K^+ = (K^+Y)^* . \tag{1.2}$$

Then the $m \times m$ Hermitian matrix

$$Q = X^*H^+X + Y^*K^+Y - (X + Y)^* (H + K)^+ (X + Y) \geqq 0 .$$

Proof. Let us consider

$$L = \begin{bmatrix} H & X \\ X^* & X^*H^+X \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} K & Y \\ Y^* & Y^*K^+Y \end{bmatrix}$$

Since H is a.p.d, by Lemma (2.1) of [9], H is EP. Hence $N(H^*) = N(H) \subseteq N(X^*)$. Further the generalized Schur complement of H in L, $L/H = 0$. Hence by corollary under Theorem 1 of [3], rank $L =$ rank H. By applying Theorem 3 of [8] and using $X^*H^+ = (H^+X)^*$; H is a.p.d. implies L is a.p.d. Hence L is p.s.d. Similarly we can see that M is p.s.d. Hence $L + M$ is p.s.d. Since $H \geqq -K$, $H + K \geqq 0$, which implies $H + K$ is hermitian. By (1.2) $X^*H^+X + Y^*K^+Y$ is hermitian. Hence,

$$L + M = \begin{bmatrix} H + K & X + Y \\ (X + Y)^* & X^*H^+X + Y^*K^+Y \end{bmatrix} \text{ is hermitian} .$$

Thus $L + M \geqq 0$. Now, by a result of Albert [1],

$$Q = X^*H^+X + Y^*K^+Y - (X + Y)^* (H + K)^+ (X + Y) = L + M/H + K \geqq 0 .$$

Hence the Lemma.

Remark 5. In particular if H and K are h.p.s.d. then the conditions $H \geqq -K$ and (1.2) hold automatically and the above Lemma 4, reduces to Theorem 5 of [3].

Theorem 3. *Let H and K be a.p.d. matrices of order n such that $H \geqq -K$; partitioned in the form.*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{1.3}$$

with

(1.4) $$H_{21}H_{11}^{+} = (H_{11}^{+}H_{21}^{*})^{*} = (H_{11}^{+}H_{12})^{*}$$

and

(1.5) $$K_{21}K_{11}^{+} = (K_{11}^{+}K_{21}^{*})^{*} = (K_{11}^{+}K_{12})^{*} .$$

Then, $H + K/H_{11} + K_{11} \geqq H/H_{11} + K/K_{11} \geqq 0$.

Proof. Since H and K are a.p.d. by Lemmas (2.3) and (2.4) of [9], H_{11} and K_{11} are a.p.d. and (1.1) hold for the matrices H and K. Now, by definition of generalized Schur complement [3], we have

$$H + K/H_{11} + K_{11} = H_{22} + K_{22} - (H_{21} + K_{21})(H_{11} + K_{11})^{+}(H_{12} + K_{12}) .$$

By using (1.4), (1.5) and applying Lemma 4, we get

$$H + K/H_{11} + K_{11} \geqq H_{22} + K_{22} - (H_{21}H_{11}^{+}H_{21}^{*} + K_{21}K_{11}^{+}K_{21}^{*}) =$$

$$= H_{22} + K_{22} - H_{21}H_{11}^{+}H_{12} - K_{21}K_{11}^{+}K_{12} =$$

$$= (H_{22} - H_{21}H_{11}^{+}H_{12}) + (K_{22} - K_{21}K_{11}^{+}K_{12}) = H/H_{11} + K/K_{11} .$$

Thus, $H + K/H_{11} + K_{11} \geqq H/H_{11} + K/K_{11}$.

Since $H_{21}H_{11}^{+} = (H_{11}^{+}H_{12})^{*}$ and $K_{21}K_{11}^{+} = (K_{11}^{+}K_{12})^{*}$ by applying Theorem 1 of [8] for the a.p.d. matrices H and K, we see that H/H_{11} and K/K_{11} are both a.p.d., hence p.s.d. Since $H \geqq -K$, $H + K$ is hermitian $\Rightarrow H_{22} + K_{22}$ is hermitian. By using (1.4), $H_{21}H_{11}^{+}H_{12}$ is hermitian and using (1.5), $K_{21}K_{11}^{+}K_{12}$ is hermitian. Hence $H/H_{11} + K/K_{11} = H_{22} + K_{22} - H_{21}H_{11}^{+}H_{12} - K_{21}K_{11}^{+}K_{22}$ is hermitian. Since H/H_{11} and K/K_{11} are p.s.d. $H/H_{11} + K/K_{11}$ is also p.s.d. Thus $H/H_{11} + K/K_{11} \geqq 0$. Hence the theorem.

Remark 6. In a special case if H and K are h.p.s.d. matrices of the form (1.3) then $H \geqq -K$, the conditions (1.4) and (1.5) automatically hold, and Theorem 3 reduces to the following known result.

Corollary. (Theorem 6 in [3]): *Let H and K be h.p.s.d. matrices partitioned in the form* (1.3) *then* $H + K/H_{11} + K_{11} \geqq H/H_{11} + K/K_{11}$.

APPLICATION TO LINEAR ELECTROMECHANICAL SYSTEMS

Let us consider two linear electromechanical systems with constitutive operators H and K and having the same structure operator A. For terminology and notation the reader may refer Duffin and Morley [4]. If H and K are a.d operators then their transfer impedances $\psi(H)$ and $\psi(K)$ exist and both are a.d. If H and K are partitioned in the form (1.3) then by Theorem 7 of [4],

$$\psi(H) = (A^{+})^{*}(H_{22} - H_{21}H_{11}^{+}H_{12})A^{+} = (A^{+})^{*}(H/H_{11})A^{+}$$

$$\psi(K) = (A^{+})^{*}(K_{22} - K_{21}K_{11}^{+}K_{12})A^{+} = (A^{+})^{*}(K/K_{11})A^{+} .$$

If we assume that the constitutive operators H and K satisfies (1.4) and (1.5) respectively and $H \geqq -K$, then by Theorem 3,

$$H/H_{11} \geqq -K/K_{11} \Rightarrow (A^+)^* (H/H_{11}) A^+ \geqq -(A^+)^* (K/K_{11}) A^+ \Rightarrow$$
$$\Rightarrow \psi(H) \geqq -\psi(K) .$$

Thus the monotonicity of the constitutive operators is preserved for the corresponding transfer impedances.

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Author's address: Department of Mathematics, Annamalai University, Annamalainagar, 608 002, India.