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SPANNING TREES OF LOCALLY FINITE GRAPHS

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We shall consider infinite undirected graphs without loops and multiple edges. A graph G will be called *locally finite*, if each vertex of G has a finite degree.

If R is a subset of the vertex set $V(G)$ of a graph G , then by $G - R$ we shall denote the graph obtained from G by deleting all vertices of the set R .

For locally finite graphs, R. Halin [1] introduced the concept of an end of a graph. Before giving the definition, we define some auxiliary concepts.

A rest of a one-way infinite path P is a one-way infinite path, all of whose vertices and edges belong to P . Two one-way infinite paths P_1, P_2 of a locally finite graph G are called *equivalent*, if there exists a one-way infinite path P_0 in G (which may coincide with P_1 or with P_2) with the property that every rest of P_0 has common vertices with both P_1 and P_2 . This relation defined on the set of all one-way infinite paths of G in this way is really an equivalence relation [1]. Its equivalence classes are called the *ends* of G .

An end \mathfrak{E} of G is called *free*, if there exists a finite subset R of the vertex set $V(G)$ of G such that in the graph $G - R$ there exists a connected component which contains paths from \mathfrak{E} , but no one-way infinite paths from any other ends of G . (We say that R separates \mathfrak{E} from the other ends.)

If a locally finite graph G has finitely many ends, then all of them are free [1]. Obviously an infinite locally finite graph G contains at least one end, because it contains at least one infinite path.

We shall study ends of spanning trees of a graph G . The following propositions are easy to prove.

Proposition 1. *Let T be an infinite locally finite tree. Then two one-way infinite paths of T belong to the same end of T if and only if their intersection is a rest of both of them.*

Proposition 2. *Let G be an infinite locally finite graph, let T be its spanning tree.*

If two one-way infinite paths in T belong to the same end of T , then they belong to the same end of G , but not vice versa.

Now we prove a lemma.

Lemma 1. *Let G be a connected infinite locally finite graph, let T be its spanning tree, let \mathfrak{E} be a free end of G . Then T contains at least one path from \mathfrak{E} .*

Proof: As \mathfrak{E} is free, there exists a finite set R of vertices of G which separates \mathfrak{E} from the other ends. Let G_0 be the connected component of $G - R$ containing paths from \mathfrak{E} , let H_0 be the subgraph of G induced by the union of R and the vertex set $V(G_0)$ of G_0 . Let T_0 be the subgraph of T induced by the vertex set $V(H_0)$ of H_0 ; T_0 is a forest. Each connected component of T_0 contains at least one vertex of R ; otherwise it would be also a connected component of T and T would not be a tree. This implies that the number of connected components of T_0 is at most $|R|$. As T_0 is infinite and has a finite number of connected components, at least one connected component of T_0 is infinite. As T_0 is locally finite, this connected component contains a one-way infinite path. This path is also in G ; as it is in H_0 , it belongs to \mathfrak{E} .

Now we shall define a concept which will be useful in the sequel.

Let A be a non-empty finite subset of $V(G)$, let \mathfrak{E} be an end of G . We say that a subset R of $V(G)$ separates A from \mathfrak{E} , if each path from \mathfrak{E} with the initial vertex in A contains a vertex of R and $A \cap R = \emptyset$. Evidently, each non-empty finite subset A of $V(G)$ is separated from each end \mathfrak{E} of G by a finite set; for example, we may choose R as the set of all vertices of G which do not belong to A and are adjacent to at least one vertex of A . The cardinality of R is less than or equal to the sum of degrees of vertices of A . As A is finite and G is locally finite, this sum is finite and so is the cardinality of R . Hence we may define $c(A, \mathfrak{E})$ as the minimum cardinality of a set separating A from \mathfrak{E} ; it is a positive integer. Now take the supremum of $c(A, \mathfrak{E})$ for all non-empty finite subsets A of $V(G)$. This supremum will be called the *degree* of \mathfrak{E} and denoted by $d(\mathfrak{E})$. It is either a positive integer, or \aleph_0 .

Lemma 2. *Let G be a connected infinite locally finite graph. Let \mathfrak{E} be a free end of G , let $d(\mathfrak{E})$ be finite. Let T be a spanning tree of G . Then the number of ends of T which are included in \mathfrak{E} is at most $d(\mathfrak{E})$.*

Proof. Let k be the number of ends of T which are contained in \mathfrak{E} . From the definition of the end and from the fact that T is a tree it follows that T contains k one-way infinite paths P_1, \dots, P_k , where $k = d(\mathfrak{E})$, which are pairwise vertex-disjoint and have the property that the initial vertex of any P_i ($i = 1, \dots, k$) separates all other vertices of P_i from all vertices of the paths P_j for $i \neq j$. All paths P_1, \dots, P_k belong to the end \mathfrak{E} of G , but to pairwise distinct ends of T . Let A be the set of initial vertices of the paths P_1, \dots, P_k . Any set R separating A from \mathfrak{E} in G must have at least k vertices, otherwise there would be a path among P_1, \dots, P_k which would contain a vertex of R . Thus $c(A, \mathfrak{E}) \leq k$ and also $d(\mathfrak{E}) \leq k$.

Lemma 3. *Let G be a connected infinite locally finite graph. Let \mathfrak{E} be a free end*

of G , let $d(\mathfrak{C})$ be finite. Let k be an integer, $1 \leq k \leq d(\mathfrak{C})$. Then there exists a spanning tree T of G which has exactly k ends contained in \mathfrak{C} .

Proof. First, consider $k = d(\mathfrak{C})$. As \mathfrak{C} is free, there exists a finite set $R_0 \subset V(G)$ separating \mathfrak{C} from all other ends. Let H be the subgraph of G induced by the union of R_0 and the vertex set of the connected component of $G - R_0$ containing paths from \mathfrak{C} . The graph H evidently has exactly one end \mathfrak{C}_0 which is a subset of \mathfrak{C} . Take a set R_1 of the least cardinality which separates R_0 from \mathfrak{C}_0 . We have $|R_1| \leq d(\mathfrak{C})$. If $|R_1| < d(\mathfrak{C})$, we find a set R_2 separating R_1 from \mathfrak{C} ; if $|R_2| < d(\mathfrak{C})$, we continue by finding R_3 separating R_2 from \mathfrak{C} , etc. If we can proceed to infinity in this way, then every subset of $V(G)$ can be separated from \mathfrak{C} by less than $d(\mathfrak{C})$ vertices, which is a contradiction. Thus there exists a positive integer m such that R_m is separated from \mathfrak{C} by a set S_0 such that $|S_0| = d(\mathfrak{C})$ and by no set of a lesser cardinality. Now we can construct an infinite sequence $(S_i)_{i=0}^\infty$ recurrently. If S_j is constructed for some j , then we can find a set S_{j+1} such that $|S_{j+1}| = d(\mathfrak{C})$ and S_{j+1} separates S_j from \mathfrak{C}_0 . Now for each positive integer i let F_i be the subgraph of G induced by the union of $S_i \cup S_{i+1}$ and the vertex set of the connected component of $G - (S_i \cup S_{i+1})$ which contains paths from S_i to S_{i+1} . The sets S_i, S_{i+1} in F_i are separated by not less than $d(\mathfrak{C})$ vertices. Thus according to Menger's Theorem there exist $k = d(\mathfrak{C})$ pairwise vertex-disjoint paths from S_i to S_{i+1} . If the vertices of S_i are denoted by $a_1^{(i)}, \dots, a_k^{(i)}$, then we denote these paths by $P_1^{(i)}, \dots, P_k^{(i)}$ in such a way that $a_j^{(i)}$ is a terminal vertex of $P_j^{(i)}$ for $j = 1, \dots, k$. Then the terminal vertex of $P_j^{(i)}$ in S_{i+1} will be denoted by $a_j^{(i+1)}$. We proceed in this way for all i 's. The union of $P_j^{(i)}$ for all i 's is a one-way infinite path P_j for each $j = 1, \dots, k$. Thus we have constructed k pairwise vertex-disjoint one-way infinite paths P_1, \dots, P_k . Now let H' be the connected component of $G - (R_0 \cup S_0)$ which contains paths from R_0 to S_0 ; it is a finite graph, because it contains no infinite path. Let H'' be the graph by the union of R_0, S_0 and the vertex set of H' . Let T_0' be a spanning tree of H'' . If we add the paths P_1, \dots, P_k to it, we obtain a spanning tree T_0 of H . We can construct a spanning tree T of G having T_0 as a subtree and this is the required spanning tree for $k = d(\mathfrak{C})$.

Now suppose $1 \leq d(\mathfrak{C}) < k$. We proceed by induction. Suppose that there exists a spanning tree T_1 of G having $k + 1$ ends contained in \mathfrak{C} . Then T_1 contains $k + 1$ pairwise vertex-disjoint one-way infinite paths P_1, \dots, P_{k+1} belonging to \mathfrak{C} . There exists a subgraph G^* of G such that the subgraph T_1^* of T_1 induced by $V(G_0)$ consists of the rests P_1^*, \dots, P_{k+1}^* of P_1, \dots, P_{k+1} . As P_1^*, P_{k+1}^* belong to the same end of G , there exists a one-way infinite path Q in G having infinitely many common vertices with both P_1^* and P_{k+1}^* . We traverse Q starting at its initial vertex. Whenever we enter a vertex v of P_{k+1}^* by an edge e not belonging to P_{k+1} , we add e to T_1^* (previously e was not in T_1^*). Simultaneously we delete the edge of P_{k+1}^* ending at v (when traversing P_{k+1}^* from its initial vertex). As Q has infinitely many common vertices with P_{k+1}^* , in this way we delete infinitely many edges of P_{k+1}^* , thus none of its rests is in the resulting graph. Evidently neither a new one-way infinite path, nor a circuit is obtained; thus we have constructed the required tree.

These lemmas imply a theorem.

Theorem 1. *Let G be a connected infinite locally finite graph, let \mathfrak{E} be its free end, let $d(\mathfrak{E})$ be finite. Let k be an integer. Then the following two assertions are equivalent:*

- (i) $1 \leq k \leq d(\mathfrak{E})$.
- (ii) *There exists a spanning tree of G having exactly k ends included in \mathfrak{E} .*

Corollary 1. *Let G be a connected infinite locally finite graph with finitely many ends $\mathfrak{E}_1, \dots, \mathfrak{E}_m$ of finite degrees. Let k be an integer. Then the following two assertions are equivalent:*

- (i) $m \leq k \leq \sum_{i=1}^m d(\mathfrak{E}_i)$.
- (ii) *There exists a spanning tree of G having exactly k ends.*

Now we shall consider the case when $d(\mathfrak{E})$ is infinite.

Theorem 2. *Let G be a connected infinite locally finite graph, let \mathfrak{E} be its free end, let $d(\mathfrak{E}) = \aleph_0$. Then there exists a spanning tree T of G having infinitely many ends belonging to \mathfrak{E} .*

Proof. The construction is similar to that from the proof of Lemma 3. However, here the cardinalities of the sets S_i are not equal; they form a non-decreasing sequence tending to infinity. If $|S_i| = |S_{i+1}| = k$, we construct the paths $P_1^{(i)}, \dots, P_k^{(i)}$ in the same way as in the proof of Lemma 3. If $k = |S_i| < |S_{i+1}| = l$, we construct again $P_1^{(i)}, \dots, P_k^{(i)}$ and denote their terminal vertices in S_{i+1} by $a_1^{(i+1)}, \dots, a_k^{(i+1)}$. The remaining vertices in S_{i+1} will be denoted arbitrarily by $a_{k+1}^{(i+1)}, \dots, a_l^{(i+1)}$. Now for each positive integer j the path P_j is the union of paths $P_j^{(i)}$ for all i 's for which such a path exists. Thus we have infinitely many pairwise vertex-disjoint one-way infinite paths P_1, P_2, \dots . From G we delete all edges which join a non-initial vertex of one of these paths with a vertex of another one; then we construct a spanning tree of the graph thus obtained. This tree is the required tree T .

Now we propose two conjectures.

Conjecture 1. *Let G be a connected infinite locally finite graph, let \mathfrak{E} be its free end, let $d(\mathfrak{E}) = \aleph_0$. Then for each positive integer k there exists a spanning tree T of G having exactly k ends included in \mathfrak{E} .*

Conjecture 2. *The assertion of Lemma 1 holds even without the assumption that \mathfrak{E} is free.*

We present a partial result concerning these conjectures.

Theorem 3. *There exists a connected infinite locally finite graph G with one end \mathfrak{E} such that $d(\mathfrak{E}) = \aleph_0$ and with property that for each positive integer k there exists a spanning tree T_k of G having exactly k ends.*

Proof. We will construct the graph G . Its vertex set is the set of all ordered pairs

(i, j) of positive integers. Two vertices $(i_1, j_1), (i_2, j_2)$ are adjacent if and only if either $i_1 = i_2$ and $|j_1 - j_2| = 1$, or $j_1 = j_2$ and $|i_1 - i_2| = 1$. Let N denote the set of all positive integers. Let P_0 be the one-way infinite path with the vertex set $\{(i, 0) \mid i \in N\}$. For each $k \in N$ let P_k be the one-way infinite path with the vertex set $\{(k, j) \mid j \in N\}$. Further, for positive integers i, k let $Q_i^{(k)}$ be the finite path which is the union of the path with the vertex set $\{(k + i, j) \mid j \leq i + 1\}$ and the path with the vertex set $\{(j, i + 1) \mid k \leq j \leq k + i\}$. Now the tree T_k is the union of the paths $P_0, P_1, \dots, P_{k-1}, Q_1^{(k)}, Q_2^{(k)}, \dots$.

Reference

[1] *Halin, R.*: Über unendliche Wege in Graphen: Math. Annalen 157 (1964), 125–137.

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