Tsugunori Nogura
Products of sequential convergence properties


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PRODUCTS OF SEQUENTIAL CONVERGENCE PROPERTIES

TsuGUNORI NOGURA, Matsuyama

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1. INTRODUCTION

A topological space $X$ is said to be strongly Fréchet [15] (= countably bi-sequential, in the sense of E. Michael [8]) if for any decreasing sequence $\{A_n : n \in N\}$ of subsets of $X$ such that $x \in A_n$, it follows that there exists a point $x_n$ in $A_n$ such that $\{x_n : n \in N\}$ converges to $x$. If we put $A_n = A$ for every $n \in N$, then the resulting space is said to be Fréchet. In general Fréchet spaces behave very badly with respect to product operations. In fact P. Simon [14] showed that the product of two compact Fréchet spaces need not be Fréchet. Nevertheless it is known that there exist some subclasses of Fréchet spaces which behave nicely with respect to product operations; bi-sequential spaces [8], $W$-spaces [5] and $\langle \alpha_3-FU\rangle$-spaces [1, 2, 10, 11] form such nice subclasses. The purpose of this paper is to present such nice classes. Before we mention the purpose in detail we would like to note an example obtained by G. Gruenhage [6]: Assuming a Martin’s axiom there exists a space $X$ such that $X^n$ is strongly Fréchet for every $n \in N$, but $X^\omega$ is not Fréchet. In this paper we shall study the following problems:

Problem 1.1. Let $\prod_{i=1}^n X_i$ be Fréchet for each $n \in N$. Under what additional conditions for $X_i$, is $\prod_{i=1}^\infty X_i$ Fréchet?

Problem 1.2. Let $X$ and $Y$ be regular countably compact Fréchet spaces. Under what additional conditions for $X$, is $X \times Y$ Fréchet?

Since $W$-spaces and bi-sequential spaces are $\langle \alpha_3-FU\rangle$-spaces, so far as the author knows, the class $\langle \alpha_3-FU\rangle$ is the widest class which satisfies the problems 1.1 and 1.2 at the same time ([2], [10]). But recently the author showed that there exists a non-$\langle \alpha_3-FU\rangle$-space $X$ which gives an answer to the problem 1.2 ([12]). Therefore it is expected to find out new classes which contain $\langle \alpha_3-FU\rangle$-spaces and the above example $X$, and which give answers to problems 1.1 and 1.2 at the same time.

We shall introduce new classes $\langle \alpha_3-FU\rangle = \langle \alpha^0-St\rangle \subset \langle \alpha^1-St\rangle \subset \langle \alpha^2-St\rangle \subset \ldots \subset \langle \alpha^k-St\rangle \subset \ldots \subset \langle \alpha^\omega-St\rangle$ and show that problems 1.1 and 1.2 are true for the
class $\langle \alpha^k \rangle$, $k \in N$. We also show that the class $\langle \alpha^o \rangle$ satisfies the problem 1.2. We construct, under the continuum hypothesis (CH), an $\langle \alpha^o \rangle$-space which is not an $\langle \alpha^m \rangle$-space for $m < n$. In fact the example obtained in [12] is an $\langle \alpha^1 \rangle$-space which is not an $\langle \alpha^y-FU \rangle$-space.

In this paper all spaces are assumed to be Hausdorff topological spaces.

2. DEFINITIONS AND PRODUCTS OF $\langle \alpha^k \rangle$, $\langle \alpha^o \rangle$-SPACES

Let $X$ be a space. In this paper a sequence in $X$ means a map $f : M \to X$, where $M$ is a countable set. If $M$ is infinite, then the sequence is said to be infinite. As usual we also denote the sequence $f$ as $\{x_n : n \in M\}$. But we strictly distinguish the sequence $f$ from the image of $f$. If $M$ is infinite, then the sequence $\{x_n : n \in M\}$ is infinite but the set $\{x_n : n \in M\}$ may be finite. We denote by $\|A\|$ the cardinality of a set $A$. For a collection $\mathcal{A} = \{A_n : n \in \Omega\}$ of subsets of a set $X$ and a map $g : X \to Y$, we put $g(\mathcal{A}) = \{g(A_n) : n \in \Omega\}$.

A collection $\mathcal{A}$ of convergent sequences in $X$ is said to be a sheaf in $X$ if all members of $\mathcal{A}$ converge to the same point of $X$, which is said to be the vertex of the sheaf $\mathcal{A}$. In this paper all sheaves are assumed to be countable. We consider the following properties of $X$ which were introduced by A. V. Arhangel'skii [1, 2]:

Let $\mathcal{A} = \{A_n : n \in N\}$ be a sheaf in $X$ with vertex $x \in X$. Then there exists a sequence $B$ converging to $x$ such that:

\begin{align*}
(\alpha_3) & \{n \in N : A_n \cap B \text{ is an infinite subsequence of } A_n \text{ and } B \}\mid = \omega, \quad \text{where } \omega \text{ is the first infinite cardinal number.} \\
(\alpha_4) & \{n \in N : A_n \cap B = 0\} = \omega.
\end{align*}

We say $B$ satisfies $(\alpha_i)$ with respect to $\mathcal{A}$ if $B$ satisfies the property $(\alpha_i)$ for $i = 3, 4$.

The class of spaces satisfying the property $(\alpha_i)$ for every sheaf $\mathcal{A}$ and vertex $x \in X$ is denoted by $\langle \alpha_i \rangle$. We denote by $\langle \alpha_i-FU \rangle$ the intersection of the class $\langle \alpha_i \rangle$ and the class of Fréchet spaces. For a class $\mathcal{C}$ of spaces an element of $\mathcal{C}$ is said to be a $\mathcal{C}$-space. A. V. Arhangel'skii showed that the class $\langle \alpha_i-FU \rangle$ is precisely the class of strongly Fréchet spaces, and showed that every bi-sequential space is an $\langle \alpha_i-FU \rangle$-space. For further properties and applications of $\langle \alpha_i \rangle$-spaces ($i = 1, 2, 3, 4$), see [1], [2], [4], [10], [11] and [12]. We introduce new properties $(\alpha^k)$ and $(\alpha^o)$ as follows.

Let $\mathcal{A} = \{A_n : n \in N\}$ be a sheaf in $X$ with vertex $x \in X$. A sheaf $\mathcal{B}$ with the same vertex $x$ is said to be a cross-sheaf of $\mathcal{A}$ if $\{n \in N : A_n \cap B = 0\} = \omega$ for each $B \in \mathcal{B}$ and $\bigcup \mathcal{B} \subseteq \bigcup \mathcal{A}$. A sheaf $\mathcal{A}' = \{A'_m : m \in M\}$ is said to be a subsheaf of $\mathcal{A}$ if $M$ is an infinite subset of $N$ and for each $m \in M$, $A'_m$ is an infinite subsequence of $A_m$. A sheaf $\mathcal{A}$ is said to be 0-nice if it is convergent, $\mathcal{A}$ is said to be $(k + 1)$-nice if each cross-sheaf of $\mathcal{A}$ has a $k$-nice subsheaf.

$(\alpha^k)$ For each sheaf $\mathcal{A}$ in $X$, there exists a $k$-nice subsheaf of $\mathcal{A}$.

We denote by $\langle \alpha^k \rangle (\langle \alpha^o \rangle)$ the class of all spaces satisfying $(\alpha^k)$ $(\langle \alpha^o \rangle)$ for an $n \in N$, 263
respectively) for every sheaf $\mathcal{A}$ and vertex $x \in X$. Clearly $\langle \alpha^0 \rangle = \langle \alpha_3 \rangle$. If $0 \leq k \leq n \leq \omega$, then every $\langle \alpha^k \rangle$-space is an $\langle \alpha^n \rangle$-space.

Now we study products of $\langle \alpha^k \rangle$-spaces.

**Lemma 2.1.** Let $\mathcal{G}$ be a sheaf in $Z = X \times Y$ with vertex $z$. If each subsheaf of $\pi_X(\mathcal{G})$ and $\pi_Y(\mathcal{G})$ has a $k$-nice subsheaf, then there exists a subsheaf $\mathcal{G}'$ of $\mathcal{G}$ such that $\pi_X(\mathcal{G}')$ and $\pi_Y(\mathcal{G}')$ are $k$-nice subsheaves of $\pi_X(\mathcal{G})$ and $\pi_Y(\mathcal{G})$, respectively, where $\pi_X$ and $\pi_Y$ are the projections from $Z$ to $X$ and $Y$, respectively.

**Proof.** By the assumption, there exist $L \subseteq N$ and an infinite subsequence $\{C_n^0\}$ of $\{C_n^k\}$ for every $n \in L$ such that $\{\pi_X(C_n^k): n \in L\}$ is a $k$-nice subsheaf of $\pi_X(\mathcal{G})$. Put $\mathcal{G}' = \{C_n^k: n \in L\}$. Since $\pi_Y(\mathcal{G}')$ has a $k$-nice subsheaf, there exists $M \subseteq L$ and an infinite subsequence $\{C_n^k\}$ of $\{C_n^m\}$ for every $n \in M$ such that $\{\pi_Y(C_n^m): n \in M\}$ is a $k$-nice subsheaf of $\{\pi_Y(C_n^m): n \in L\}$. $\mathcal{G}' = \{C_n^k: n \in M\}$ is a desired subsheaf of $\mathcal{G}$. The proof is completed.

**Lemma 2.2.** Let $\mathcal{G}$ be a sheaf in $Z = X \times Y$ with vertex $z = (x, y)$ and $\mathcal{G}'$ be a subsheaf of $\mathcal{G}$ such that $\pi_X(\mathcal{G}')$ and $\pi_Y(\mathcal{G}')$ are $k$-nice subsheaves of $\pi_X(\mathcal{G})$ and $\pi_Y(\mathcal{G})$, respectively. Then $\mathcal{G}'$ has a $k$-nice subsheaf.

**Proof.** We show the assertion by induction on $k$. For $k = 0$, put $\mathcal{G}' = \{C_n^m: m \in M\}$, $C_n^m = \{x_n^m, y_n^m\}: n \in N \}$. Without loss of generality we can assume $x_n^m \neq x$ and $y_n^m \neq y$ for every $(m, n) \in M \times N$. Put $D_m = C_n^m - \pi_X^{-1}\{\{x_i^j: i, j \leq m - 1\}\} \cup \pi_Y^{-1}\{\{y_i^j: i, j \leq m - 1\}\}$. Then $C_n^m - D_m$ is a finite set. Put $\mathcal{D} = \{D_m: m \in M\}$. We show $\mathcal{D}$ is a 0-nice subsheaf of $\mathcal{G}'$, i.e. $\bigcup \mathcal{D}$ is convergent sequence.

Let $U \times V$ be an open neighborhood of $(x, y)$ in $X \times Y$. Put $S = \{(x_n^m, y_n^m) \in \bigcup \mathcal{D}: x_n^m \notin U\}$, $T = \{(x_n^m, y_n^m) \in \bigcup \mathcal{D}: y_n^m \notin V\}$. We show $S \cup T$ is a finite set. Assume $S$ is infinite. Then, since $\pi_X(S)$ is finite, there exist $m_0, m_1 \in M$, $n_0, n_i \in N$ ($i \in N$) such that $(x_{m_0}^{m_i}, y_{m_0}^{m_i}) \in D_{m_i}$ for $i \in N$. But this is impossible since $D_{m_i} \cap \pi_X^{-1}(x_{n_0}^{m_0}) = 0$ for $n_i > n_0$. Similarly $T$ is finite.

Assume the assertion has been proved for $k (\geq 0)$. We show the case $k + 1$.

Let $\mathcal{B}$ be a cross-sheaf of $\mathcal{G}'$. Then $\pi_X(\mathcal{B})$ and $\pi_Y(\mathcal{B})$ are cross-sheaves of $\pi_X(\mathcal{G}')$ and $\pi_Y(\mathcal{G}')$, respectively. Since $\pi_X(\mathcal{G}')$ and $\pi_Y(\mathcal{G}')$ are $(k + 1)$-nice subsheaves of $\pi_X(\mathcal{G})$ and $\pi_Y(\mathcal{G})$, respectively, $\pi_X(\mathcal{B})$ and $\pi_Y(\mathcal{B})$ have $k$-nice subsheaves. By Lemma 2.1 and by the inductive assumption, $\mathcal{B}$ has a $k$-nice subsheaf. This shows the assertion is true for the $k + 1$. The proof is completed.
By Lemma 2.1 and 2.2 we have the following theorem.

**Theorem 2.3.** Let $X$ and $Y$ be $<\alpha^k>$-spaces ($<\alpha^\omega>$-spaces). Then $Z = X \times Y$ is an $<\alpha^k>$-space, ($<\alpha^\omega>$-space, respectively).

**Lemma 2.4.** Let $A$ be a sheaf in $X = \prod_{i=1}^{\infty} X_i$ with vertex $x \in X$. If each subsheaf of $\pi_n(A)$ has a $k$-nice subsheaf for every $n \in N$, then there exists a subsheaf $A'$ of $A$ such that $\pi_n(A')$ is a $k$-nice subsheaf of $\pi_n(A)$ for every $n \in N$, where $\pi_n: X \rightarrow \prod_{i=1}^{n} X_i$ is the projection for every $n \in N$.

**Proof.** We first construct a subsheaf $A_n$ of $\pi_n(A)$ and its $k$-nice subsheaf $A'_n$ such that $\pi_n(A'_m)$ is a $k$-nice subsheaf of $\pi_n(A)$ for $m \geq n$, where $\pi_n: \prod_{i=1}^{m} X_i \rightarrow \prod_{i=1}^{n} X_i$ is the projection. Put $A_1 = \{\pi_1(A_n): n \in N\}$ and $M_0 = N$. Let $A'_1 = \{A'_1: n \in M_1\}$ be a $k$-nice subsheaf of $A_1$ such that $A'_n$ is an infinite subsequence of $\pi_1(A_n)$ for every $n \in M_1$. Inductively we construct $A_j$ and $A'_j$ as follows: Let

$$A_j = \left\{(\pi_j^{i-1})(A_{n_j}^{i-1}) \cap \pi_j(A_n): n \in M_{j-1}\right\} \quad \text{for } j \geq 2 .$$

Then $A_j$ is a subsheaf of $\pi_j(A)$. Let $A'_j = \{A'_j: i \in M_j\}$ be a $k$-nice subsheaf of $A_j$, where $M_j \subset M_{j-1}$ and $A'_n$ is an infinite subsequence of $(\pi_j^{i-1})(A_{n_j}^{i-1}) \cap \pi_j(A_n)$ for $n \in M_j$. Note that $M_j$ is infinite by the definition of a subsheaf. Choose $n_j \in M_j$ such that $n_j < n_{j+1}$ for each $j \in N$.

Now we define

$$A' = \{\pi_n^{-1}(A'_n) \cap A_n: n \in N\} .$$

Then $A'$ is a subsheaf of $A$ and $\pi_n(A')$ is a subsheaf of $A'_n$. Clearly a subsheaf of a $k$-nice sheaf is $k$-nice. Therefore $\pi_n(A')$ is $k$-nice for every $n \in N$. The proof is completed.

**Lemma 2.5.** Let $A = \{A_n: n \in N\}$ be a sheaf in $X = \prod_{i=1}^{\infty} X_i$ with vertex $x$ such that each subsheaf of $\pi_n(A)$ has a $k$-nice subsheaf for every $n \in N$. Then $A$ has a $k$-nice subsheaf.

**Proof.** We prove the assertion by induction on $k$. Let $k = 0$. By Lemma 2.4, choose a subsheaf $A'$ of $A$ such that $\pi_n(A')$ is $0$-nice for every $n \in N$. To avoid the complexity of the notations we put $A' = A$. Put

$$A_n = \{x^m_n: m \in N\} ,$$

$$A'_n = A_n - \bigcup_{m=1}^{n} \pi_m^{-1}(\{x^j_m: i, j \leq m, \pi_m(x^j_i) \neq \pi_m(x^j_m)\}) .$$

Then $A' = \{A'_n: n \in N\}$ is a subsheaf of $A$. We show $A'$ is $0$-nice. Let $V = \bigcup_{i=1}^{\infty} U_i \times \prod_{i=n+1}^{\infty} X_i$ be an open neighborhood of $x$ in $X$. Since $\bigcup \pi_n(A')$ is a convergent se-
sequence, \( \bigcup_{i=1}^{n} U_i \) is a finite set. Put
\[
\pi_n\left( \{x_{i_1}^{j_1}, \ldots, x_{i_m}^{j_m} \} \right) = \bigcup_{i=1}^{n} U_i,
\]
\[
t = \max \{ n, i_1, \ldots, i_m, j_1, \ldots, j_m \}.
\]
Then
\[
A'_s \cap \pi_n^{-1}\left( \pi_n\left( \{x_{i_1}^{j_1}, \ldots, x_{i_m}^{j_m} \} \right) \right) = \emptyset \quad \text{for} \quad s \geq t
\]
and
\[
|A'_s \cap \pi_n^{-1}\left( \pi_n\left( \{x_{i_1}^{j_1}, \ldots, x_{i_m}^{j_m} \} \right) \right)| < \omega \quad \text{for} \quad s < t
\]
because \( \pi_n(A'_s) \) is a sequence converging to \( \pi_n(x) \) and \( \pi_n(x_{r_n}^r) \neq \pi_n(x) \) for every \( r = 1, 2, \ldots, m \). These show \( (X - V) \cap (\bigcup \mathcal{A}') \) is a finite set. The proof of the case \( k = 0 \) is completed.

Assume the assertion has been proved for the case \( k \geq 0 \). We show the case \( k + 1 \).

Let \( \mathcal{A}' \) be a sheaf with vertex \( x \) such that \( \pi_n(\mathcal{A}) \) is \((k + 1)\)-nice for every \( n \in \mathbb{N} \). Then, by Lemma 2.4, there exists a subsheaf \( \mathcal{A}' \) of \( \mathcal{A} \) such that \( \pi_n(\mathcal{A}') \) is a \((k + 1)\)-nice subsheaf of \( \pi_n(\mathcal{A}) \) for every \( n \in \mathbb{N} \). We show \( \mathcal{A}' \) is \((k + 1)\)-nice.

Let \( \mathcal{B} \) be a cross-sheaf of \( \mathcal{A}' \). Then \( \pi_n(\mathcal{B}) \) is a cross-sheaf of \( \pi_n(\mathcal{A}') \). Since \( \pi_n(\mathcal{A}') \) is \((k + 1)\)-nice, \( \pi_n(\mathcal{B}) \) has a \( k \)-nice subsheaf. By Lemma 2.4, \( \mathcal{B} \) has a subsheaf \( \mathcal{B}' \) such that \( \pi_n(\mathcal{B}') \) is \( k \)-nice for every \( n \in \mathbb{N} \). Therefore, by the inductive assumption \( \mathcal{B}' \) has a \( k \)-nice subsheaf. This shows \( \mathcal{A}' \) is \((k + 1)\)-nice. The proof is completed.

**Theorem 2.6.** Let \( X_i \) be an \( \langle x^k \rangle \)-space for every \( i \in \mathbb{N} \). Then \( \prod_{i=1}^{m} X_i \) is an \( \langle x^k \rangle \)-space.

**Proof.** Let \( \mathcal{A} \) be a sheaf in \( X \) with vertex \( x \). By Theorem 2.3, \( \prod_{i=1}^{m} X_i \) is an \( \langle x^k \rangle \)-space for every \( n \in \mathbb{N} \). Therefore each subsheaf of \( \pi_n(\mathcal{A}) \) has a \( k \)-nice subsheaf. By Lemma 2.4 and 2.5, we can choose a \( k \)-nice subsheaf \( \mathcal{A}' \) of \( \mathcal{A} \). The proof is completed.

In § 3 we shall show that the countable product of \( \langle x^o \rangle \)-spaces need not be an \( \langle x^o \rangle \)-space under (CH).

### 3. \( \langle x^k \text{-St} \rangle \)-SPACES AND FINITE PRODUCTS

According to E. Michael [8], the product \( X \times C \) is Fréchet if and only if \( X \) is strongly Fréchet, where \( C = \{0\} \cup \{1/n: n \in \mathbb{N}\} \) with the usual topology. Hence if we expect the Fréchetness of products, it is natural to require that each factor space is strongly Fréchet. We denote by \( \langle x^k \text{-St} \rangle \) (\( \langle x^o \text{-St} \rangle \)), the class of all strongly Fréchet spaces with the property \( (x^k) ((x^o)) \), respectively).

**Lemma 3.1.** Let \( X \) be a strongly Fréchet space and \( Y \) be a regular countably compact Fréchet space. Let \( C_n = \{(x_m^n, y_m^n): m \in \mathbb{N}\}, n \in \mathbb{N} \), be convergent sequences in \( Z = X \times Y \) satisfying the following conditions:

1. Each \( C_n \) converges to a point \( (x, y^n) \) for \( n \in \mathbb{N} \).
(2) \( \{ y^n: n \in \mathbb{N} \} \) is a sequence in \( Y \) converging to a point \( y \), where \( y^n \neq y \) for every \( n \in \mathbb{N} \).

Put \( A_n = \{ x_m^n: m \in \mathbb{N} \} \). Then

(3) \( \mathcal{A} = \{ A_n: n \in \mathbb{N} \} \) is a \( k \)-nice sheaf with vertex \( x \).

Then there exists a sequence \( \{(x_{m(j)}^n, y_{m(j)}^n): j \in \mathbb{N} \} \) converging to the point \( (x, y) \).

Proof. We prove the assertion by induction on \( k \). Let \( k = 0 \). Then \( \mathcal{A} \) is a 0-nice sheaf, therefore \( \bigcup \mathcal{A} \) is a sequence converging to \( x \). Put \( A = \bigcup \{ C_n: n \in \mathbb{N} \} \). Then \( A \subseteq (\bigcup \mathcal{A} \cup \{ x \}) \times Y \) and \( (x, y) \in \widetilde{A} \). Since a regular countably compact Fréchet space is strongly Fréchet [13], \( (\bigcup \mathcal{A} \cup \{ x \}) \times Y \) is Fréchet by Michael’s theorem (see introduction of this section). Therefore there exists a sequence \( B \) in \( A \) converging to the point \( (x, y) \). If \( B \cap C_n \neq \emptyset \) and choose \( (x_{m(j)}^n, y_{m(j)}^n) \in B \cap C_n \).

Then \( \{(x_{m(j)}^n, y_{m(j)}^n): j \in \mathbb{N} \} \) converges to the point \( (x, y) \). The proof for the case \( k = 0 \) is completed.

Assume the assertion has been proved for \( k \geq 0 \). We show the case \( k + 1 \).

Put \( \mathcal{B} = \{ B \subseteq \bigcup \{ C_n: n \in \mathbb{N} \}: |B \cap C_n| < \omega \) for every \( n \in \mathbb{N} \), \( \pi_X(B) \) converges to the point \( x \) and \( \pi_Y(B) \) converges to a point \( y_B \in Y \).

We show

\( \ast \) \( y \in \text{cl} \{ y_B: B \in \mathcal{B} \} \).

Let \( V \) be an open neighborhood of \( y \) in \( Y \) and \( W \) be an open set such that \( y \in W \subset \overline{W} \subset V \). Since \( \{ y^n: n \in \mathbb{N} \} \) converges to \( y \), there exist \( n_0, n_m \in \mathbb{N} \) such that

\[ \{ y^n_m: i \geq n_m, m \geq n_0 \} \subseteq W. \]

Since \( \{ x^n_m: i \geq n_m \} \) converges to \( x \) for each \( m \in \mathbb{N} \), by using the strongly Fréchetness of \( X \), there exist \( x^n_{m(i)} \in A_{m(i)}, m_0 < m_j < m_{j+1}, n_{m(j)} < i_j \) such that \( \{ x^n_{m(i)}: j \in \mathbb{N} \} \) converges to the point \( x \). Since \( Y \) is countably compact Fréchet, there exist a convergent subsequence \( \{ y^n_{m(i):r(i)}: r(i) \in \mathbb{N} \} \) of \( \{ y^n_{m(i)}: j \in \mathbb{N} \} \) and its limit point \( y(W) \in W \subseteq V \). Put

\[ B = \{ (x^n_{m(i):r(i)}, y^n_{m(i):r(i)}): r(i) \in \mathbb{N} \}. \]

Then clearly \( B \in \mathcal{B} \) and \( y_B = y(W) \in \overline{W} \subseteq V \). The proof of \( \ast \) is completed.

Since \( Y \) is Fréchet, there exists \( B_n \in \mathcal{B} \) such that \( \{ y_{B_n}: n \in \mathbb{N} \} \) converges to the point \( y \). Without loss of generality we can assume \( y_{B_n} \neq y \). Put \( B'_n = B_n - \bigcup \{ C_i: i \leq n \} \). Then \( B'_n \) is an infinite subsequence of \( B_n \). Clearly \( \{ \pi_X(B'_n): n \in \mathbb{N} \} \) is a cross-sheaf of \( \{ \pi_X(C_n): n \in \mathbb{N} \} \). Since \( \{ \pi_X(C_n): n \in \mathbb{N} \} \) is \( (k + 1) \)-nice, \( \pi_X(B'_n): n \in \mathbb{N} \) has a \( k \)-nice subsheaf. Therefore we can choose an infinite subset \( M \) of \( N \) and an infinite subsequence \( B'_m \) of \( B'_n \) for each \( m \in M \) such that \( \{ \pi_X(B'_m): m \in M \} \) is a \( k \)-nice subsheaf of \( \{ \pi_X(B_n): n \in \mathbb{N} \} \). By the inductive assumption for \( k \), there exists a sequence \( A = \{ (x^n_{m(i):r(i)}, y^n_{m(i):r(i)}): i \in \mathbb{N} \} \) converging to the point \( x \). By the definition of \( B'_n \), \( A \cap C_n \) is a finite set for every \( n \in \mathbb{N} \). Hence, by taking a subsequence of \( A \), we can assume each \( (x^n_{m(i):r(i)}, y^n_{m(i):r(i)}) \) is contained in different \( C_n \). The proof is completed.
Since each \(<\alpha_3-FU>-space\) is an \(<\alpha^\alpha-S\>-space\), the following theorem gives a generalization of Theorem 5.16 of [2].

**Theorem 3.2.** Let X be an \(<\alpha^\alpha-S\>-space\) and Y be a regular countably compact Fréchet space. Then Z = X \times Y is Fréchet.

**Proof.** Let A \subseteq Z, z = (x, y) \in A. If z \in cl(A \cap \{x\} \times Y), then the arguments are completed trivially. Therefore without loss of generality we can assume A \cap \{x\} \times Y = \emptyset. Let \mathcal{V}_y be a neighborhood base of y in Y. Let V \in \mathcal{V}_y. We first show that there exist a convergent sequence A(V) in A \cap (X \times V) and its limit point (x, y(V)) such that y(V) \in V. Clearly x \in cl(\pi_X(A \cap X \times V)), so by using the Fréchetness of X, we can choose x_n \in \pi_X(A \cap X \times V) such that \lim x_n = x. Choose (x_n, y_n) \in \pi_X^{-1}(x_n) \cap A \cap X \times V. Then, since Y is countably compact Fréchet, there exist a convergent subsequence \{y_{n_k}: k \in N\} of \{y_n: n \in N\} and its limit point y(V). Put A(V) = \{(x_{n_k}, y_{n_k}): k \in N\}. Clearly A(V) converges to (x, y(V)) and y(V) \in V. By using the regularity of Y, it is easy to see y \in cl\{y(V): V \in \mathcal{V}_y\}. Now the Fréchetness of Y implies that there exists \{V_n: n \in N\} such that \lim y(V_n) = y. Let C_n = A(V_n) be a convergent sequence constructed for V_n in the above arguments. Put

\[ C_n = \{(x^n_k, y^n_k): k \in N\}, \quad A_n = \{x^n_k: k \in N\}, \quad \mathcal{A} = \{A_n: n \in N\}, \quad y(V_n) = y^n. \]

Note that \mathcal{A} has a k-nice subsheaf in X with vertex x for some k. By Lemma 3.1, there exists a sequence in A converging the point (x, y). The proof is completed.

It is easy to see that an \(<\alpha_3-FU>-space\) is an \(<\alpha^\alpha-FU>-space\) (hence strongly Fréchet space) by the definition. But as is shown by the following example, Fréchet space with the property \((\alpha^k)\) need not be strongly Fréchet.

**Example 3.3.** Let \(S^\omega\) be the sequential fan, i.e. \(S^\omega = \{\infty\} \cup N \times N\), where each point of \(N \times N\) is an isolated point, and the collection \(\{\infty\} \cup \{(m, n): n > f(m)\}: f\) is a function from \(N\) into \(N\)\} forms a neighborhood base of the point \(\infty\). Then as is well-known \(S^\omega\) is a Fréchet space with the property \((\alpha^k)\), but it is not strongly Fréchet.

Clearly an \(<\alpha_3-FU>-space\) is \(<\alpha^k-S\>-space\), but the converse need not be true under (CH). In fact we construct an \(<\alpha^{k+1}>-space X_{k+1}\) which is not an \(<\alpha^k>-space\) for each \(k = 0, 1, \ldots\) under (CH).

We denote by \(\beta N\) the Stone-Čech compactification of \(N\). For a subset \(A\) of \(N\), \(A^* = \text{Cl}_{\beta N} A\). In the arguments below of this section, Int A and Bdy A denote respectively the interior of \(A\) in \(N^*\) and the boundary of \(A\) in \(N^*\) for a set \(A \subseteq N^*\).

We recall the following well-known facts we shall use later.

**Fact 1.** Two disjoint cozero sets in \(N^*\) have disjoint closures.

**Fact 2.** Let \(Z\) be a non-empty zero set in \(N^*\). Then \(\text{Int} Z = \emptyset\).

**Lemma 3.4(CH).** Let \(Z\) be a zero set in \(N^*\) with non-empty boundary Bdy Z
in \( N^* \). Let \( H \) be a non-empty closed subset of \( \text{Bdy} \ Z \) and \( U \) be a clopen subset of \( N^* \) with \( H \subset U \). Then there exists a regular closed subset \( E_1 \) in \( N^* \) such that:

(i-1) \( E_1 \subset Z \cap U \),
(ii-1) \( H = \text{Bdy} \ E_1 \),
(iii-1) \( H \subset \text{Bdy} (U - E_1) \).

Proof. Without loss of generality we can assume \( U = N^* \). We construct \( F_1 \) by transfinite induction. Note that the cardinality of the set of all zero sets in \( N^* \) equals the cardinality of the continuum. Let \( \{ Z_\alpha : \alpha < \omega_1 \} \) be the family of all zero sets in \( Z \) such that \( H \cap Z_\alpha \neq \emptyset \) for \( \alpha < \omega_1 \), where \( \omega_1 \) is the first uncountable ordinal. Let \( \{ K_\alpha : \alpha < \omega_1 \} \) be a family of zero sets in \( Z \) such that \( K_\alpha \subset K_\beta \) for \( \alpha > \beta \), \( K_\alpha = \bigcap \{ K_\beta : \beta < \alpha \} \) if \( \alpha \) is a limit ordinal and \( H = \bigcap \{ K_\alpha : \alpha < \omega_1 \} \).

Let \( O_1 \) and \( V_1 \) be non-empty disjoint clopen subsets of \( N^* \) contained in \( Z_1 \cap K_1 \). Inductively we assume that we have defined, for each \( \beta < \alpha \), non-empty clopen subsets \( O_\beta \) and \( V_\beta \) of \( N^* \) such that:

1. \( \bigcup \{ O_\gamma : \gamma < \beta \} \subset O_\beta \), \( \bigcup \{ V_\gamma : \gamma < \beta \} \subset V_\beta \),
2. \( (O_\beta - \bigcup \{ O_\gamma : \gamma < \beta \}) \cup (V_\beta - \bigcup \{ V_\gamma : \gamma < \beta \}) \subset K_\beta \),
3. \( O_\beta \cap Z_\beta \neq \emptyset \), \( V_\beta \cap Z_\beta \neq \emptyset \),
4. \( O_\gamma \cap V_\delta = \emptyset \) for \( \gamma, \delta < \alpha \).

We define \( O_\alpha \) and \( V_\alpha \). We first define \( O'_\alpha \) and \( V'_\alpha \) as follows: Put \( O'_\alpha = \bigcup \{ O_\beta : \beta < \alpha \} \). If \( \alpha \) is isolated, we put \( O'_\alpha = O_\alpha \setminus O_{\alpha-1} \) and \( V'_\alpha = V_\alpha \setminus V_{\alpha-1} \). Assume \( \alpha \) is a limit ordinal. Then the relation

\[
\bigcup \{ (N^* - K_\beta) - O_\beta : \beta < \alpha \} = (N^* - \bigcap \{ K_\beta : \beta < \alpha \}) - O_\alpha
\]

shows that \( (N^* - \bigcap \{ K_\beta : \beta < \alpha \}) - O_\alpha \) is a cozero set in \( N^* \). Thus it follows that \( O_\alpha \) and \( (N^* - \bigcap \{ K_\beta : \beta < \alpha \}) - O_\alpha \) are disjoint cozero sets in \( N^* \). By Fact 1 we can choose disjoint clopen subsets \( O'_\alpha \) and \( V'_\alpha \) which contain \( O_\alpha \) and \( (N^* - \bigcap \{ K_\beta : \beta < \alpha \}) - O_\alpha \), respectively.

Choose non-empty disjoint clopen subsets \( S_\alpha \) and \( T_\alpha \) such that

\( S_\alpha \cup T_\alpha \subset K_\alpha \cap \text{Int} Z - O'_\alpha \cup V'_\alpha \).

Let \( O_\alpha = O'_\alpha \cup S_\alpha \) and \( V_\alpha = V'_\alpha \cup T_\alpha \). We have chosen \( O_\alpha \) and \( V_\alpha \) \( (\alpha < \omega_1) \) satisfying the conditions (1)-(4).

Put

\[
E_1 = \text{Cl}(\bigcup \{ O_\alpha : \alpha < \omega_1 \})
\]

Then clearly \( E_1 \) is a regular closed set in \( N^* \) which satisfies (i-1). We show (ii-1) and (iii-1). Let \( U \) be any clopen subset of \( N^* \) with \( U \cap H \neq \emptyset \). Then \( U \cap Z \neq \emptyset \) by Fact 2. Since \( U \cap Z \) is a zero set in \( N^* \) which has non-empty intersection with \( H \), there exists \( \alpha < \omega_1 \) such that \( Z_\alpha = U \cap Z \). Then \( \emptyset \neq Z_\alpha \cap E_1 \subset U \cap E_1 \) and \( \emptyset \neq Z_\alpha \cap (N^* - E_1) \subset U \cap (N^* - E_1) \) by (3). Therefore \( H \subset \text{Bdy} E_1 \) and \( H \subset \text{Bdy} (N^* - E_1) \). We show \( \text{Bdy} E_1 \subset H \). By (2) \( \text{Bdy} E_1 \subset K_\alpha \) for every \( \alpha < \omega_1 \), therefore \( \text{Bdy} E_1 \subset \bigcap \{ K_\alpha : \alpha < \omega_1 \} = H \). The proof of Lemma 3.4 is completed.
We call a regular closed set $E_1$ constructed by the above method a \textit{1-nice set with respect to $(Z, H, U)$}. We shall define a $k$-nice set $(k \geq 1)$ with respect to $(Z, H, U)$ by induction on $k$. The definition depends heavily on the above construction. Assume a $k$-nice set $E_k$ with respect to $(Z', H', U')$ has been defined for a zero set $Z'$, a non-empty closed subset $H' \subset \text{Bdy} Z'$ and a clopen set $U'(\supset H')$.

$(k+1)$-construction: Let $Z, H, U$ be the same as the ones in Lemma 3.4. We shall define a $(k+1)$-nice set $E_{k+1}$ with respect to $(Z, H, U)$ as follows: The notation used below is the same as in the proof of Lemma 3.4. We first define a regular closed set \( L_a \) by transfinite induction on $a$. Put $L_0 = 0$. Assume $L_\beta$, $\beta < \omega_1$ has been defined satisfying

(5) $L_\beta \subset U_\beta$ for $\beta < \alpha$,

(6) $L_\gamma \subset L_\beta$ for $\gamma < \beta < \alpha$,

(7) $S_\beta \subset L_\beta$ for $\beta < \alpha$,

(8) $L_\gamma \cap O_\gamma = L_\gamma$ for $\gamma < \beta < \alpha$,

(9) $\text{Cl} \hat{L}_\beta - \hat{L}_\beta \subset N^* - \hat{O}_\beta$, where $\hat{L}_\beta = \bigcup \{ L_\gamma : \gamma < \beta \}$.

If $\alpha$ is isolated, then put $L_\alpha = L_{\alpha-1} \cup S_\alpha$. Let $\alpha$ be a limit ordinal. Put

(10) $H_\alpha = \text{Cl} \hat{L}_\alpha \cap (N^* - \hat{O}_\alpha)$.

Then

(11) $H_\alpha = \text{Cl} \hat{L}_\alpha - \hat{L}_\alpha$.

We show (11). Since $\hat{L}_\alpha \subset \hat{O}_\alpha$ by (5), $H_\alpha \cap \hat{L}_\alpha \subset (N^* - \hat{O}_\alpha) \cap \hat{L}_\alpha = 0$. This shows $H_\alpha \subset \text{Cl} \hat{L}_\alpha - \hat{L}_\alpha$. We show the converse implication. Let $p \in \text{Cl} \hat{L}_\alpha - \hat{L}_\alpha$. It is enough to show $p \in N^* - \hat{O}_\alpha$. If $p \in \hat{O}_\alpha$, then $p \in O_\beta$ for some $\beta < \alpha$. By (8) $O_\beta - \hat{L}_\beta = \hat{O}_\beta - L_\beta$, therefore $O_\beta - L_\beta$ is an open neighborhood of $p$ which is disjoint from $\hat{L}_\alpha$. This is a contradiction. The proof of the equality (11) is completed. By (10) and (11), $H_\alpha$ is a non-empty closed subset of the boundary of a zero set $N^* - \hat{O}_\alpha$; we can choose a $k$-nice set $L_\alpha$ with respect to $(N^* - \hat{O}_\alpha, H_\alpha, O_\alpha)$ by the inductive assumption for $k$. Put $L_\alpha = L_\alpha \cup \hat{L}_\alpha \cup S_\alpha$. We have defined $L_\alpha (\alpha < \omega_1)$ satisfying (5)-(9).

Put

\[ E_{k+1} = \text{Cl} \bigcup \{ L_\alpha : \alpha < \omega_1 \}. \]

It is easy to prove that $E_{k+1}$ is a regular closed set. We call a regular closed set $E_{k+1}$ \textbf{a $(k+1)$-nice with respect to $(Z, H, U)$}. For the sake of the convenience, we call a set $E_0$ 0-nice with respect to $(Z, H, U)$ if $E_0 = Z \cap U$.

**Remark 3.5.** Let $W$ be a clopen set with $W \cap H \neq \emptyset$. Then clearly $W \cap E_{k+1}$ is a $(k+1)$-nice set with respect to $(W \cap Z, W \cap H, W \cap U)$.

The properties of a $k$-nice set $E_k$ are summarized by the following lemma.

**Lemma 3.6(CH).** Let $Z, H,$ and $U$ be the same as in Lemma 3.4. Then there exists a regular closed subset $E_k$ in $N^*$ such that:

(i-k) $E_k \subset Z \cap U$, 

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(ii-k) \( H = \text{Bdy } Z \cap \text{Bdy } E_k \),

(iii-k) \( H \subseteq \text{Bdy } (U - E_k) \),

(iv-k) Let \( J \) be a zero set in \( N^* \) such that \( J \cap E_k \neq \emptyset \). Then \( \text{Int } J \cap E_k \neq \emptyset \).

(v-k) Let \( J \) be a zero set in \( N^* \) such that \( J \cap H \neq \emptyset \). Then there exist non-empty clopen subsets \( P_i \subset Z \cap E_k \) and a clopen subset \( W \) of \( N^* \) such that \( W \cap E_k - P \) is a \((k-1)\)-nice set with respect to \((W - P, \text{Bdy } P, W)\), where \( P = \bigcup \{ P_i : i \in N \} \).

(vi-k) Let \( \{ P_i : i \in N \} \) be a collection of non-empty clopen subsets of \( N^* \) such that \( P_i \cap E_k \neq \emptyset \) for every \( n \in N \). Then there exist an infinite subset \( M \subset N \), a clopen set \( \beta \subset P \) for every \( m \in M \) and \( P \) such that \( \text{Wn } E_k - \beta \) is a \( k \)-nice set with respect to \((\text{Wn } E_k - \beta, \text{Bdy } \beta, \text{Wn } E_k)\), where \( \beta = \bigcup \{ Q_m : m \in M \} \).

Proof. It is easy to prove (i-k) – (iii-k) by the construction and the arguments in the case \( k = 1 \). Since the boundary of \( E_k \) is the boundary of some \( i \)-nice set \((1 \leq i \leq k-1)\) by the inductive construction, it is easy to prove (iv-k) by the construction. We prove (v-k). Since \( J \cap H \neq \emptyset \), by (5) and (7), we can choose \( \alpha_1 < \alpha_2 < \ldots \) such that \( P_{\alpha_1} \subset J \cap E_k \). Put \( P_i = P_{\alpha_i} \) and \( \alpha = \sup \{ \alpha_i : i \in N \} \). Then the relation
\[
(N^* - K_\alpha) - P = \bigcup \{ (N^* - K_m) : m \in M \} - \bigcup \{ P_i : i = 1, 2, \ldots, n \} : n \in N
\]
shows that \( (N^* - K_\alpha) - P \) is a cozero set in \( N^* \). Let \( W \subset O \) be a clopen set which contains \( P \) and disjoint from \((N^* - K_\alpha) - P) \). Then
\[
W \cap \text{Bdy } E_k = W \cap \text{Bdy } L_\alpha = \text{Bdy } P
\]
and \( W \cap \bar{L}_\alpha = P \). By Remark 3.5, \( W \cap E_k - P \) is a \((k-1)\)-nice subset with respect to \((W - P, \text{Bdy } P, W)\). The proof of (v-k) is completed. The proof of (vi-k) can be done using a similar method as in the proof of (v-k) and the inductive hypothesis but it is routine. We left it to the reader. The proof of Lemma 3.6 is completed.

Let \( F \) be a closed subset of \( N^* \). Put \( X = N \cup \{ F \} \) and topologize \( X \) as follows: each point of \( N \) is isolated and the family \( \{ U \cup \{ F \} : U \subset N \text{ and } F \subset U^* \} \) is a neighborhood base of the point \( F \) in \( X \). The following facts are well known.

Fact 3. Let \( A \subset N \). Then \( A \) converges to the point \( F \) in \( X \) if and only if \( A^* \subset F \).

Fact 4. ([7, Theorem 1]). \( X = N \cup \{ F \} \) is strongly Fréchet if and only if \( F \) is regular closed in \( N^* \) and, for each zero set \( K \) in \( N^* \) such that \( K \cap F \neq \emptyset \), \( \text{Int } K \cap F \neq \emptyset \).

Now we construct an \(<\alpha^{k+1}-\text{St}>-\text{space} \) \( X_{k+1} \) which is not an \(<\alpha^k-\text{St}>-\text{space} \).

Example 3.7 (CH). Let \( E_{k+1} \) be a \((k+1)\)-nice set with respect to \((Z, \text{Bdy } Z, N^*)\). Put \( F_{k+1} = (N^* - Z) \cup E_{k+1} \). Then \( X_{k+1} = N \cup \{ F_{k+1} \} \) is an \(<\alpha^{k+1}-\text{St}>-\text{space} \) which is not an \(<\alpha^k-\text{St}>-\text{space} \) for \( k = 0, 1, \ldots \).

Proof. It is easy to see that the zero set \( Z \) can be expressed in the form \( Z = \bigcup \{ T_n^* : n \in N \} \), where \( \{ T_n : n \in N \} \) is a family of pairwise disjoint infinite subsets of \( N \) and \( \bigcup \{ T_n : n \in N \} = N \). Clearly \( \mathcal{T} = \{ T_n : n \in N \} \) is a sheaf in \( X \) with vertex \( F_{k+1} \) by Fact 3. We call such a sheaf \( \mathcal{T} \) a principal sheaf in \( X_{k+1} \).
Assertion 1. The space $X_t$ does not satisfy $(\alpha^0)$.

Proof. Let $\mathcal{T} = \{T_n: n \in N\}$ be a principal sheaf in $X_t$. Let $\mathcal{T}' = \{T'_m: n \in M\}$ be any sub-sheaf of $\mathcal{T}$. Put $T = \bigcup \mathcal{T}'$. Then $T^* \cap \text{Bdy} Z \neq \emptyset$. Therefore there exists an infinite subset $A$ of $T$ such that $A^* \subset Z$ and $A^* \cap E_1 = \emptyset$ by (iii-1) By Fact 3 $T$ is not a convergent sequence.

Assertion 2. The space $X_t$ is an $\langle x^1\text{-St}\rangle$-space.

Proof. By fact 4 $X_t$ is strongly Fréchet. We show $X_t$ satisfies $(\alpha^1)$. Let $\mathcal{A} = \{A_n: n \in N\}$ be a sheaf with vertex $x$. If $x \neq F_1$, then the arguments are completed trivially. So we assume $x = F_1$. By Fact 3, $A_n^* \subset F_1$. Note that the sets $A_n^* \cap (N^* - Z)$ and $A_n^* \cap E_1$ are clopen subsets of $N^*$ for each $n \in N$. Indeed, if $A_n^* \cap (N^* - Z)$ is not clopen in $N^*$, then $\text{Bdy} H \cap \text{Cl} \left( A_n^* \cap (N^* - Z) \right) = \emptyset$, hence $A_n^* \cap \text{Bdy} Z = \emptyset$. By (iii-1), $A_n^* \cap (Z - E_1) = \emptyset$, which contradicts $A_n^* \subset F_1$.

Similarly we can show $A_n^* \cap E_1$ is a clopen subset in $N^*$.

Case I. $\left| \{ n \in N: A_n^* \neq 0 \} \right| = \omega$.

Put $M = \{ n \in N: A_n^* \neq 0 \}$. By (vi-1) in Lemma 3.6 there exist an infinite subset $L$ of $M$ and an infinite subset $W$ of $N$ such that $A_n^* \cap E_1 - P$ is a $O$-nice set with respect to $(W^* - P, \text{Bdy} P, W^*)$, where $P = \bigcup \{ C_n^*: n \in N\}$. Therefore by the definition of a $O$-nice set with respect to $(W^* - P, \text{Bdy} P, W^*)$, $W^* \cap E_1 - P = W^* \cap (W^* - P) = W^* - P$. Since $W^* \cap E_1 \subset E_1$ and $P \subset E_1$, $W^* \subset E_1$.

Let $\mathcal{A}' = \{ A_m \cap W: m \in M\}$. Since $A_m \cap W$ contains $A_m \cap C_m$ except finitely many elements of $N$ for each $m \in M$, it follows that $\mathcal{A}'$ is a subsheaf of $\mathcal{A}$. Let $\mathcal{D} = \{ D_i: i \in N\}$ be a cross-sheaf of $\mathcal{A}'$. Then $D_i \subset W$ for any $i \in N$. The implication $W^* \subset E_1 \subset F_1$ shows $W$ converges to $F_1$ in $X_t$ by Fact 1. Since $\bigcup \mathcal{D} \subset W$, $\mathcal{D}$ is $O$-nice. This implies $\mathcal{A}'$ is $1$-nice.

Case II. $\left| \{ n \in N: A_n^* \neq 0 \} \right| < \omega$.

In this case $\left| \{ n \in N: B_n^* \neq 0 \} \right| = \omega$. We prove this case dividing it into two cases.

Case II-i. There exist infinitely many $B_n(\neq 0)$ and their infinite subsets $B_{n_j}$ such that $\text{Bdy} Z \cap \text{Cl} \bigcup \{ B_{n_j}^*: j \in N \} = \emptyset$.

Choose a subset $C$ of $N$ such that

$$
\text{Cl} \bigcup \{ B_{n_j}^*: j \in N \} \subset C^* \subset N^* - Z .
$$

Put $\mathcal{A}' = \{ B_{n_j} \cap U: j \in N\}$.

If $B_{n_j}^* \neq 0$, then $A_{n_j} \cap C$ contains $A_{n_j} \cap B_{n_j}$ except for finitely many elements of $N$. Therefore $\mathcal{A}'$ is a subsheaf of $\mathcal{A}$. Let $\mathcal{D} = \{ D_j: j \in N\}$ be a cross-sheaf of $\mathcal{A}'$. Then $D_j \subset C$ for any $j \in \mathcal{D}$. The implication $C^* \subset N^* - Z \subset F$ shows $C$ converges to $F_1$ by Fact 3. Hence $\mathcal{D}$ is $O$-nice. Therefore $\mathcal{A}'$ is $1$-nice.
Case II-ii. For each infinite subset \( B_{n_j} \), and its infinite subset \( B_{n_j} \), \( \text{Bdy} \ Z \cap \cap \text{Cl} \bigcup \{ B_{n_j}^*: n \in N \} = \emptyset \).

We first show \( (N^* - Z) - \bigcup \{ B_{n}^*: n \in N \} \) is a cozero set in \( N^* \). Let \( \mathcal{F} = \{ T_i: i \in N \} \) be a principal sheaf in \( X_1 \). Since \( N^* - Z = \bigcup \{ T_i^*: i \in N \} \), it is enough to show \( T_i^* - \bigcup \{ B_{n}^*: n \in N \} \) is a cozero set in \( N^* \). In fact, put \( M_i = \{ n \in N: T_i^* \cap B_{n}^* = \emptyset \} \), then by our assumption, \( M_i \) is a finite set. Therefore \( T_i^* - \bigcup \{ B_{n}^*: n \in N \} \) is a cozero set in \( N^* \).

Since \( (N^* - Z) - \bigcup \{ B_{n}^*: n \in N \} \) and \( \bigcup \{ B_{n}^*: n \in N \} \) are disjoint cozero sets in \( N^* \), it follows by Fact 1 that these have disjoint closures. Let \( C \) be a subset of \( N \) such that
\[
\bigcup \{ B_{n}^*: n \in N \} \subseteq C^* \subseteq (N^* - Z) - \bigcup \{ B_{n}^*: n \in N \}.
\]
Put \( \mathcal{A}' = \{ B_n \cap U: B_n \) is an infinite subset of \( N \} \). Then \( \mathcal{A}' \) is a subsheaf of \( \mathcal{A} \). Let \( \mathcal{D} = \{ D_n: n \in N \} \) be a cross-sheaf of \( \mathcal{A}' \). Then \( D_n^* \subseteq C^* \) and \( B_n^* \cap D_n^* = \emptyset \) for each \( m \in N \). These show \( D_n^* \cap (N^* - Z) = \emptyset \) for each \( n \in N \). Therefore \( D_n^* \subseteq E_1 \). Since \( \bigcup \{ D_n^*: n \in N \} \) and \( N^* - Z \) are disjoint cozero sets in \( N^* \), they have disjoint closures in \( N^* \). Choose a subset \( D \) of \( N \) such that
\[
\bigcup \{ D_n^*: n \in N \} \subseteq D^* \subseteq E_1.
\]

Then \( D \) converges to \( F_1 \). Therefore \( \mathcal{D} \) is 0-nice which implies \( \mathcal{A}' \) is 1-nice.

To prove that \( X_{k+1} \) is not an \( \langle \alpha^k \rangle \)-space, it is enough to show the following assertion.

**Assertion 3.** Assume a principal sheaf of \( X \) does not have any i-nice subsheaf for \( i \leq k \). We show that a principal sheaf \( \mathcal{F} = \{ T_n: n \in N \} \) in \( X_{k+1} \) doesn't have a \( (k + 1) \)-nice subsheaf.

**Proof.** Let \( \mathcal{F}' = \{ T_m^*: m \in M \} \) be an arbitrary subsheaf of \( \mathcal{F} \). Then \( Z' = N^* - \bigcup \{ T_m^*: n \in N \} \) is a zero set in \( N^* \) such that \( Z' \cap \text{Bdy} \ Z \neq \emptyset \). By \((v-k+1)\) in Lemma 3.6, we can choose non-empty clopen sets \( P_i \subseteq Z \cap E_{k+1} \) and a clopen set \( W \) in \( N^* \) such that \( W \cap E_{k+1} \subseteq P \) is a k-nice set with respect to \( (P \cap E_{k+1}, \text{Bdy} P, W) \), where \( P = \bigcup \{ P_i: i \in N \} \). Let \( \mathcal{F} = \{ S_i: i \in N \} \) be a disjoint family of subsets of \( N \) such that \( S_i^* = P_i \) and \( W = (\bigcup \mathcal{F})^* \). Then \( \mathcal{F} \) is a principal sheaf of a subspace \( Y = \bigcup \mathcal{F} \cup \{ F_k \} = \bigcup \mathcal{F} \cup \{ P \cap F_k \} = \bigcup \mathcal{F} \cup \{ (N^* - P) \cup (\text{W} \cap E_{k+1} - P) \} \).

We can consider \( Y \) as \( X_k \). Therefore the principal sheaf \( \mathcal{F} \) of \( Y \) does not have a k-nice subsheaf by the inductive assumption. Since \( \mathcal{F} \) is a cross-sheaf of \( \mathcal{F}' \) and \( \mathcal{F}' \) is an arbitrary subsheaf of \( \mathcal{F} \), \( \mathcal{F} \) does not have a \( (k+1) \)-nice subsheaf. The proof of Assertion 3 is completed.

The proof that \( X_{k+1} \) is an \( \langle \alpha^{k+1} \rangle \)-space can be done by similar arguments of Assertion 2 using \((v\cdot i-k+1)\), but it is routine. We left it to the reader.

**Example 3.8.** Let \( X_k \) be an \( \langle \alpha^k \rangle \)-space constructed in the above example 3.7. Then \( \prod_{i=1}^{k} X_i \) is an \( \langle \alpha^k \rangle \)-space (hence \( \langle \alpha^w \rangle \)-space) for every \( k \in N \) by Theorem 2.3, but \( X = \prod_{i=1}^{\infty} X_i \) is not an \( \langle \alpha^w \rangle \)-space.
Proof. Let $T_k^n = \{T^*$: $n \in N\}$ be a principal sheaf in $X_k$. Put $T_k = \{x_k^m(m): m \in N\}$ and 
\[ T(n) = \{(x_n^1(m), x_n^2(m), \ldots, x_n^{n+1}(m), F_{n+2}, \ldots): m \in N\}. \]
Then $T(n)$ is a convergent sequence in $X$ with limit point $(F_1, F_2, \ldots)$. Let $\mathcal{A} = \{T(n): n \in N\}$. Then $\mathcal{A}$ is a sheaf in $X$ with vertex $(F_1, F_2, \ldots)$. We show that $\mathcal{A}$ does not have a $k$-nice subsheaf for every $k \in N$. Let $\mathcal{A}'$ be a $k$-nice subsheaf of $\mathcal{A}$. Then $\pi_n(\mathcal{A}')$ is $k$-nice for every $n \in N$, where $\pi_n: X \to X_n$ is the projection. Since $\pi_n(\mathcal{A}')$ is a subsheaf of the principal sheaf $\mathcal{F}_n$, it is not $n$-nice, therefore it is not $k$-nice for $n \geq k$. This is a contradiction. The proof is completed.

4. INVERSE LIMITS AND INFINITE PRODUCTS

The following lemma is well known but we give its proof for the sake of completeness.

**Lemma 4.1.** A Fréchet space is strongly Fréchet if and only if it contains no subspace which is homeomorphic to $S_\omega$ (see Example 3.2).

**Proof.** Assume $X$ is Fréchet but not strongly Fréchet (hence $X$ is not an $\langle a_4$-FU$\rangle$-space). Let $\mathcal{A} = \{A_n: n \in N\}$ be a sheaf with vertex $x$ which does not satisfy $(\sigma_4)$. Without loss of generality we assume $A_n \cap A_m = \emptyset$ for $n \neq m$. Then it is easy to show $\{x\} \cup \{A_n: n \in N\}$ is homeomorphic to $S_\omega$. The proof is completed.

Let $\{X_m: \pi_m^n\}$ be an inverse limiting system of a sequence of spaces $X_m$ with the onto projections $\pi_m^n: X_m \to X_n (m \geq n)$. Let $X = \lim \{X_m: \pi_m^n\}$ be the inverse limit of this system and $\pi_m: X \to X_m$ the projections.

The following theorem has been proved by the author and Y. Tanaka. We present it here with the kind permission of Y. Tanaka.

**Theorem 4.2.** Let $X = \lim \{X_m: \pi_m^n\}$ and let $X_m$ be strongly Fréchet for every $n \in N$. Then the following conditions are equivalent.

(i) $X$ is a Fréchet space.

(ii) $X$ is a strongly Fréchet space.

**Proof.** Obviously (ii) $\implies$ (i). We shall show (i) $\implies$ (ii). Assume $X$ is Fréchet but not strongly Fréchet. Then by Lemma 4.1 $X$ contains a copy of $S_\omega = \{\infty\} \cup N \times N$.

Put $T_n = \{n\} \times N$. In the arguments below we assume $S_\omega = \{\infty\} \cup N \times N$.

Put $\mathcal{B}_n = \{B: B$ is open in $X_n$ and $\pi_n(\infty) \in B\}$,

$\mathcal{B}_n' = \{S_\omega \cap \pi^{-1}_n(B): B \in \mathcal{B}_n\}$,

$\mathcal{B} = \cup \{\mathcal{B}_n: n \in N\}$.

Then clearly $\mathcal{B}_n \subset \mathcal{B}_n' + 1$ for each $n \in N$ and $\mathcal{B}'$ is a neighborhood base of $\{\infty\}$ in $S_\omega$.

**Assertion.** For each $n \in N$, there exists an open neighborhood $W_n$ of $\infty$ in $S_\omega$ such that

$\bullet$ For any $B' \in \mathcal{B}_n$, $|\{i \in N: B' \cap T_i - W_n \neq \emptyset\}| = \omega$. We shall prove the above assertion in two steps.

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Case I. For each \( n \in \mathbb{N} \), there exists a finite set \( M_n \subseteq \mathbb{N} \) such that 
\[
|\pi_n(T_j) - \bigcup \{\pi_n(T_i) : i \in M_n\}| < \omega \quad \text{for each} \quad j \in \mathbb{N}.
\]
In this case there exist \( i_0 \in M_n \), an infinite subset \( N_n \) of \( \mathbb{N} \) and an infinite subsequence \( T_k^\prime \) of \( T_k \) for each \( k \in N_n \) such that 
\[
\pi_n(T_k^\prime) = \pi_n(T_{i_0}) \quad \text{for each} \quad k \in N_n.
\]

Case I-i. \( \pi_n(T_{i_0}) \) is a finite set.

Since \( \pi_n(T_{i_0}) \) is a convergent sequence with limit point \( \pi_n(\infty) \), there exists an infinite subset \( T_k^\prime \subseteq T_k \) with \( \pi_n(\infty) = \pi_n(T_k^\prime) \) for \( k \in N_n \). Choose \( x_{m_k}^k \in T_k^\prime \). Then \( \pi_n(x_{m_k}^k) = \pi_n(\infty) \). Put 
\[
W_n = \{\infty\} \cup \bigcup \{x_{m_k}^k : m > n_k\} \cup \{T_j : i \in N - N_n\}.
\]
We show \( W_n \) satisfies (*). Let \( B' \in B_n \). Then \( B' = S_\omega \cap \pi_n^{-1}(B) \) for some \( B \in B_n \). We show \( B' \cap T_k - W_n = \emptyset \) for \( k \in N_n \). The relation \( \pi_n(\infty) \in B \) implies \( T_k^\prime \subseteq B' \), hence \( x_{m_k}^k \in B' \). However \( x_{m_k}^k \notin W_n \). This shows \( B' \cap T_k - W_n = \emptyset \). The proof of Case I-i is completed.

Case I-ii. \( \pi_n(T_{i_0}) \) is an infinite set.

Put 
\[
W_n = \{\infty\} \cup \{T_k - \pi_n^{-1}(\{y_j : j \leq k\}) : k \in N_n\} \cup \{T_k : k \in N - N_n\}.
\]
We show \( W_n \) satisfies (*). Let \( B' \in B_n \), \( B' = S_\omega \cap \pi_n^{-1}(B) \) for some \( B \in B_n \). Then \( \pi_n(\infty) \in B \) and \( \lim_{k \to \infty} y_k = \pi_n(\infty) \) imply that there exists \( k_0 \in N \) such that \( y_k \in B \) for \( k \geq k_0 \). For \( k \geq k_0 \) and \( k \in N_n \), \( S_\omega \cap \pi_n^{-1}(y_k) \subseteq B' \) but \( \pi_n^{-1}(y_k) \cap T_k = \emptyset \). This shows \( B' \cap T_k - W_n = \emptyset \). The proof of Assertion is completed in this case.

Case II. There exists \( n \in \mathbb{N} \) such that: for each finite subset \( M \) of \( \mathbb{N} \), there exists \( j \in \mathbb{N} \) such that 
\[
|\pi_n(T_j) - \bigcup \{\pi_n(T_i) : i \in M\}| = \omega.
\]
By induction on \( n \) it is easy to choose an infinite subset \( N_n \) of \( \mathbb{N} \) and an infinite subset \( T_k^\prime \) of \( T_k \) for each \( k \in N_n \) such that
\[
\pi_n|T_k^\prime : T_k^\prime \to \pi_n(T_k^\prime) \text{ is one-to-one},
\]
\[
\pi_n(T_k^\prime) \cap \pi_n(T_m^\prime) = \emptyset \text{ for } k, m \in N_n \text{ and } k \neq m.
\]
Since \( X_\omega \) is strongly Fréchet (hence \( \langle \alpha_4, FU \rangle \)), there exists a convergent sequence \( K \) in \( X_\omega \) which satisfies \( \langle \alpha_4 \rangle \) with respect to \( \{\pi_n(T_k^\prime) : k \in N_n\} \). Put \( K = \{\pi_n(x_{m_i}^i) : i \in \mathbb{N}\} \). By taking a subsequence of \( K \) we can assume \( n_i \neq n_j \) if \( i \neq j \). Put \( L = \{n_i : i \in \mathbb{N}\} \) and put 
\[
W_n = \{\infty\} \cup \bigcup \{x_{m_i}^i : m > m_i\} \cup \{T_j : j \in N - L\},
\]
it is a neighborhood of \( \infty \) in \( S_\omega \). We show \( W_n \) satisfies (*).
Let $B' \in \mathcal{B}_n$, $B' = S_n \cap \pi_n^{-1}(B)$ for some $B \in \mathcal{B}_n$. Since $\{\pi_n(x_{n_i}^m) : m \geq n_i\}$ converges to $\pi_n(\infty)$, $B$ contains $\{\pi_n(x_{n_i}^m) : m \geq n_i\}$ except for finitely many elements. Therefore $B'$ contains $\{x_{n_i}^m : m \geq n\}$ except for finitely many elements. On the other hand $W_n \cap \{x_{n_i}^m : i \in \mathbb{N}\} = \emptyset$. These show that $B' - W_n \neq \emptyset$. The proof of the Assertion is completed.

Let $\{W_n : n \in \mathbb{N}\}$ be a sequence of open neighborhoods of $\infty$ in $S_w$ the existence of which is guaranteed by the Assertion. We can assume $W_n \supset W_{n+1}$ for each $n \in \mathbb{N}$. Put

$$T_n = T_n \cap W_n,$$

$$W = \{\infty\} \cup \{T_n : n \in \mathbb{N}\}.$$ 

Then $W$ is a neighborhood of $\infty$ in $S_w$. We shall show $B' - W \neq \emptyset$ for each $B' \in \mathcal{B}$. This is a contradiction because $\mathcal{B}$ is a neighborhood base of $\infty$ in $S_w$. Note that

$$W - W_n \subset \bigcup \{T_i : i = 1, 2, \ldots, n\}.$$ 

Therefore

$$B' - W \supset B' - (W_n \cup \{T_i : i = 1, 2, \ldots, n\}).$$ 

By Assertion, there exists $T_k$, $k > n$ such that

$$B' \cap T_k - W_n \neq \emptyset.$$ 

This shows

$$B' - W \supset B' \cap T_k - (W_n \cup \{T_i : i = 1, 2, \ldots, n\}) \cap T_k \supset$$

$$\supset B' \cap T_k - W_n \cap T_k \neq \emptyset.$$ 

The proof is completed.

**Corollary 4.3.** Let $|X| \geq 2$ for infinitely many $n \in \mathbb{N}$. Then the following (i) and (ii) are equivalent.

(i) $\prod_{i=1}^{\infty} X_i$ is Fréchet.

(ii) $\prod_{i=1}^{\infty} X_i$ is strongly Fréchet.

**Proof.** The implication (ii) $\rightarrow$ (i) is trivial. We show (i) $\rightarrow$ (ii). It is enough to show that $\prod_{i=1}^{n} X_i$ is strongly Fréchet for each $n \in \mathbb{N}$. In fact $\prod_{i=n+1}^{\infty} X_i$ is a non-discrete Fréchet space. Choose an infinite set $\{x_i : i \in \mathbb{N}\}$ and a point $x_0$ in $\prod_{i=n+1}^{\infty} X_i$ such that $\lim_{i \to \infty} x_i = x_0$. Then $\prod_{i=1}^{n} X_i \times (\{x_0\} \cup \{x_i : i \in \mathbb{N}\})$ is Fréchet. By E. Michael's theorem (see § 3, introduction), $\prod_{i=1}^{n} X_i$ is strongly Fréchet. The proof is completed.

**Lemma 4.4.** Let $X = \lim_{n \to \infty} \{X_n : \pi_n^m\}$ and each $X_n$ be an $\langle \alpha^k-St \rangle$-space. Let $A \subset X$ and $x \in \bar{A} - A$. Then there exists a $k$-nice sheaf $\mathcal{A}_n$ in $X_n$ with vertex $\pi_n(x)$ satisfying:

(i) $\bigcup \mathcal{A}_n \subset \pi_n(A),$

(ii) $\pi_n^m(\mathcal{A}_n)$ is a subsheaf of $\mathcal{A}_n$ for $m > n$. 

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Proof. Since \( \pi_n(x) \in \text{Cl} \pi_0(A) \), the Fréchetness of \( X_n \) implies that there exists an infinite sequence \( A_n \subset \pi_0(A) \) converging to \( \pi_n(x) \). Without loss of generality, we can choose the sequence satisfying:

\[ (*) \quad \pi_i(A_n) = \pi_i(x) \quad \text{or otherwise} \quad \pi_i(A_n) \text{ is an infinite set which does not contain} \quad \pi_i(x) \quad \text{for each} \quad i \leq n. \]

Put \( \mathcal{A}^1 = \{ \pi_i(A_n) : n \in \mathbb{N} \} \). Then \( \mathcal{A}^1 \) is a sheaf with vertex \( \pi_1(x) \). As \( X_1 \) is an \( \langle \alpha^2\text{-St} \rangle \text{-space} \), choose a \( k \)-nice subsheaf \( \mathcal{A}_1 = \{ A_m^1 : m \in N_1 \} \) of \( \mathcal{A}^1 \), where \( A_m^1 \) is an infinite subsequence of \( \pi_m^1(A_m) \) for \( m \in N_1 \).

Assume a \( k \)-nice sheaf \( \mathcal{A}_i = \{ A_m^i : m \in N_i \} \) in \( X_i \) has been defined for \( i \leq n \). We define a \( k \)-nice sheaf \( \mathcal{A}_{n+1} \) in \( X_{n+1} \) as follows:

Put

\[ \mathcal{A}^{n+1} = \{(\pi_n^{n+1})^{-1}(A_m^n) \cap \pi_m^{n+1}(A_m) : m \geq n + 1, m \in N_n \}. \]

Then \( \mathcal{A}^{n+1} \) is a sheaf in \( X_{n+1} \) with vertex \( \pi_n(x) \). Choose a \( k \)-nice subsheaf \( \mathcal{A}_{n+1} = \{ A_m^{n+1} : m \in N_{n+1} \} \) of \( \mathcal{A}^{n+1} \), where \( A_m^{n+1} \) is an infinite subsequence of \( (\pi_n^{n+1})^{-1}(A_m^n) \cap \pi_m^{n+1}(A_m) \) for every \( m \in N_{n+1} \). We have defined a \( (k + 1) \)-nice sheaf \( \mathcal{A}_n \) with vertex \( \pi_n(x) \) for every \( n \in \mathbb{N} \) satisfying the conditions (i) and (ii). The proof is completed.

Lemma 4.5. Let \( X = \lim \{ X_m : \pi_m^n \} \) and \( X_n \) be a strongly Fréchet space for every \( n \in \mathbb{N} \). Let \( x \in X \) and \( \mathcal{A}_n \) be a \( k \)-nice sheaf in \( X_n \) with vertex \( \pi_n(x) \) such that \( \pi_m^n(\mathcal{A}_m) \) is a subsheaf of \( \mathcal{A}_n \) for every \( n \leq m \). Then there exists a convergent sequence \( K_n \) in \( X_n \) with limit point \( \pi_n(x) \) satisfying:

1. \( K_n \subset \bigcup \mathcal{A}_n \),
2. \( \pi^n_n(K_n) \subset K_n \) for \( n \leq m \).

Proof. We show the assertion by induction on \( k \). For \( k = 0 \), the assertion is trivial since a 0-nice sheaf is nothing but a convergent sequence. Assume the assertion has been proved for \( k (\geq 0) \). We show the case \( k + 1 \).

Since \( X_n \) is a strongly Fréchet space, there exists a convergent sequence \( B_n \) which satisfies \( \langle \alpha \rangle \) with respect to \( \mathcal{A}_n \). Put

\[ \mathcal{B}_1 = \{ \pi_1^n(B_n) : n \in \mathbb{N} \}. \]

Then \( \mathcal{B}_1 \) is a cross-sheaf of a \( (k + 1) \)-nice sheaf \( \mathcal{A}_1 \). Choose a \( k \)-nice subsheaf \( \mathcal{B}_i = \{ B_m^i : m \in N_i \} \) of \( \mathcal{B}_1 \), where \( N_i \subset \mathbb{N} \) and \( B_m^i \) is an infinite subsequence of \( \pi_i^1(B_m) \) for \( m \in N_i \). Assume that a \( k \)-nice sheaf \( \mathcal{B}_i = \{ B_m^i : m \in N_i \} \) which is a cross-sheaf of \( \mathcal{A}_i \) has been defined for \( i \leq n \) such that \( \pi_i^j(\mathcal{B}_i) \) is a subsheaf of \( \mathcal{B}_j \). We construct \( \mathcal{B}_{n+1} \) as follows: Put

\[ \mathcal{B}_{n+1} = \{(\pi_n^{n+1})^{-1}(B_m^n) \cap \pi_m^{n+1}(B_m) : m \geq n + 1 \text{ and } m \in N_n \}. \]

Then \( \mathcal{B}_{n+1} \) is a cross-sheaf of \( \mathcal{A}_{n+1} \). Choose a \( k \)-nice subsheaf \( \mathcal{B}_{n+1} = \{ B_m^{n+1} : m \in N_{n+1} \} \) of \( \mathcal{B}_{n+1} \), where \( N_{n+1} \subset \mathbb{N} \) and \( B_m^{n+1} \) is an infinite subsequence of \( (\pi_n^{n+1})^{-1}(B_m^n) \cap \pi_m^{n+1}(B_m) \) for \( m \in N_{n+1} \). Since \( \pi_i^j(\mathcal{B}_i) \) is a \( k \)-nice subsheaf of \( \mathcal{B}_j \), by the inductive assumption for \( k \), there exists \( K_n \subset \bigcup \mathcal{B}_n \subset \bigcup \mathcal{A}_n \) satisfying \( \pi^n_n(K_n) \subset K_n \) for \( n \leq m \). The proof is completed.
Lemma 4.6. Let \( X = \lim \{ X_m, \pi_m^n \} \). Let \( A \subset X \) and \( x \in A - A \). If there exists a convergent sequence \( K_n \) in \( X_n \) with limit point \( \pi_n(x) \) satisfying:

1. \( K_n \subset \pi_n(A) \),
2. \( \pi_n^n(K_m) \subset K_n \) for \( n \leq m \),

then there exists a sequence \( \{ a_n; n \in \mathbb{N} \} \) in \( A \) converging to the point \( x \).

Proof. Put \( K_n = \{ b_m^n; m \in \mathbb{N} \} \) and choose \( a_1 \in \pi_1^{\sim}(b_1^1) \cap A \). Assume \( \{ a_1, a_2, \ldots, a_n \} \) has been chosen. Let

\[
K_{n+1} = K_n + 1 - \bigcup \{ (\pi_j)^{-1}(b_j^i); n \geq i, b_j^i + \pi_j(x) \}.
\]

Since \( (\pi_j)^{-1}(b_j^i) \cap K_i \) is a finite set for \( b_j^i + \pi_j(x) \), we have \( K_{n+1} \neq 0 \). Choose \( b \in K_{n+1} \) and \( a_{n+1} \in \pi_n^{-1}(b) \cap A \). We have chosen \( a_n \) for every \( n \in \mathbb{N} \). We show \( \lim a_n = x \). Let \( W \) be any neighborhood of \( x \) in \( X \). Then \( x \in \pi_n^{-1}(U) \subset W \) for some \( n \rightarrow \infty \).

The following theorem follows from Theorem 4.2, Lemma 4.4, 4.5 and 4.6.

Theorem 4.7. Let \( X = \lim \{ X_m; \pi_m^n \} \). If \( X_n \) is an \( \langle \alpha^k \text{-St} \rangle \)-space for every \( n \in \mathbb{N} \), then \( X \) is an \( \langle \alpha^k \text{-St} \rangle \)-space.

Corollary 4.8. If \( \prod_{i=1}^{n} X_i \) is an \( \langle \alpha^k \text{-St} \rangle \)-space for every \( n \in \mathbb{N} \), then

\[
X = \prod_{i=1}^{\infty} X_i \text{ is an } \langle \alpha^k \text{-St} \rangle \text{-space}.
\]

Corollary 4.9. Let \( X_i \) be a regular countably compact Fréchet space with the property \( (\alpha^k) \) for every \( i \in \mathbb{N} \). Then \( X = \prod_{i=1}^{n} X_i \) is countably compact Fréchet.

Proof. According to \([9, \text{Theorem 4.5}]\), \( X \) is countably compact. We show \( \prod_{i=1}^{n} X_i \) is \( \langle \alpha^k \text{-St} \rangle \) for every \( n \in \mathbb{N} \). Since a regular countably compact Fréchet space is strongly Fréchet \([13, \text{Corollary 5.2}]\), we have \( X_i \) is an \( \langle \alpha^k \text{-St} \rangle \)-space for every \( i \in \mathbb{N} \). By Theorem 2.3, \( \prod_{i=1}^{n} X_i \) is an \( \langle \alpha^k \text{-St} \rangle \)-space for each \( n \in \mathbb{N} \). Therefore \( X \) is Fréchet.

The proof is completed.

Problems 4.10. Let \( \mathcal{P} \) be a class of spaces. Let \( \mathcal{F}(\mathcal{P}) = \{ X; X \times Y \text{ is Fréchet for any } Y \in \mathcal{P} \} \). We use the following notations:

\( C \) = the class of compact Fréchet spaces,
CC = the class of countably compact Fréchet spaces,
B = the class of bi-sequential spaces (see [8] for the definition),
St = the class of strongly Fréchet spaces.

(1) Is \( \mathcal{F}(C) = \mathcal{F}(CC) \)?

(2) Give inner characterizations of classes \( \mathcal{F}(C) \), \( \mathcal{F}(CC) \) and \( \mathcal{F}(St) \).

(3) Is there a "naive" example of a \( \mathcal{F}(CC) \)-space which is not bi-sequential?

(4) Is there an example of a \( \mathcal{F}(CC) \)-space which is not an \( \langle \alpha_\omega \rangle \)-FU-space?

(5) Is \( B = \mathcal{F}(St) \)?

(6) Let \( X_i \) be an \( \langle \alpha_\omega \rangle \)-St-space for every \( i \in \mathbb{N} \). Then is \( \prod_{i=1}^{\infty} X_i \) Fréchet?

References


Author's address: Department of Mathematics, Faculty of Science, Ehime University, Matsuyama Japan.