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SURFACES IN GENERAL AFFINE SPACE

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The theory of surface in the equiaffine 3-dimensional space is well (?) developed. On the other hand, little is known about the theory of submanifols of the space with the general affine group; compare the contributions of S. Gigena, K. Nomizu, U. Pinkall and U. Simon in [1]. In the present paper, I am going to study surfaces in the 3-dimensional general affine space and show some global characterizations of quadratic surfaces.

To each point m of an *elliptic* surface M^2 in the general affine space A^3 let us associate a frame $\{m; v_1, v_2, v_3\}$ such that $v_1, v_2 \in T_m(M^2)$. Then we have the fundamental equations

$$\mathrm{d}m = \omega^1 v_1 + \omega^2 v_2\,, \quad \mathrm{d}v_i = \omega_i^j v_j \quad (i, j, \ldots = 1, 2, 3) \tag{1}$$

with the usual integrability conditions

$$\mathrm{d}\omega^i = \omega^j \wedge \omega_j^i\,, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,. \tag{2}$$

It is easy to see that we may choose the frames in such a way that

$$\omega_1^3 = \omega^1\,, \quad \omega_2^3 = \omega^2\,; \tag{3}$$

the differential consequences are

$$\begin{aligned} &(2\omega_1^1 - \omega_3^3) \wedge \omega^1 + (\omega_1^2 + \omega_2^1) \wedge \omega^2 = 0\,, \\ &(\omega_1^2 + \omega_2^1) \wedge \omega^1 + (2\omega_2^2 - \omega_3^3) \wedge \omega^2 = 0\,, \end{aligned} \tag{4}$$

and we have

$$\begin{gathered} 2\omega_1^1 - \omega_3^3 = a_1\omega^1 + a_2\omega^2\,, \quad \omega_1^2 + \omega_2^1 = a_2\omega^1 + a_3\omega^2\,, \\ 2\omega_2^2 - \omega_3^3 = a_3\omega^1 + a_4\omega^2\,. \end{gathered} \tag{5}$$

Let the auxiliary 1-form φ be defined by

$$\varphi := \tfrac{1}{2}(\omega_1^2 - \omega_2^1)\,, \tag{6}$$

i.e.,

$$\omega_1^2 = \tfrac{1}{2}(a_2\omega^1 + a_3\omega^2) + \varphi\,, \quad \omega_2^1 = \tfrac{1}{2}(a_2\omega^1 + a_3\omega^2) - \varphi\,. \tag{7}$$

The integrability conditions of (5) are

$$(8)\qquad \begin{aligned}&(\mathrm{d}a_1 - \tfrac{1}{2}a_1\omega_3^3 - 3a_2\varphi + 3\omega_3^1) \wedge \omega^1 +\\ &+ [\mathrm{d}a_2 - \tfrac{1}{2}a_2\omega_3^3 + (a_1 - 2a_3)\varphi + \omega_3^2] \wedge \omega^2 = 0\,,\\ &[\mathrm{d}a_2 - \tfrac{1}{2}a_2\omega_3^3 + (a_1 - 2a_3)\varphi + \omega_3^2] \wedge \omega^1 +\\ &+ [\mathrm{d}a_3 - \tfrac{1}{2}a_3\omega_3^3 + (2a_2 - a_4)\varphi + \omega_3^1] \wedge \omega^2 = 0\,,\\ &[\mathrm{d}a_3 - \tfrac{1}{2}a_3\omega_3^3 + (2a_2 - a_4)\varphi + \omega_3^1] \wedge \omega^1 +\\ &+ (\mathrm{d}a_4 - \tfrac{1}{2}a_4\omega_3^3 + 3a_3\varphi + 3\omega_3^2) \wedge \omega^2 = 0\,.\end{aligned}$$

Let $\{m; w_1, w_2, w_3\}$ be another field of frames associated to our surface. Then we have the equations

$$(9)\qquad \mathrm{d}m = \tau^1 w_1 + \tau^2 w_2\,,\quad \mathrm{d}w_i = \tau_i^j w_j$$

analogous to (1); let us suppose the conditions of the type (3), i.e.,

$$(10)\qquad \tau_1^3 = \tau^1\,,\quad \tau_2^3 = \tau^2\,.$$

Let the relation between our two fields of frames be given by

$$(11)\qquad \begin{aligned}&w_1 = \alpha_{11}v_1 + \alpha_{12}v_2\,,\quad w_2 = \alpha_{21}v_1 + \alpha_{22}v_2\,,\\ &w_3 = \alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3\,.\end{aligned}$$

From this and from (1) + (9), we get

$$(12)\qquad \begin{aligned}\mathrm{d}m &= \omega^1 v_1 + \omega^2 v_2 = \tau^1(\alpha_{11}v_1 + \alpha_{12}v_2) + \tau^2(\alpha_{21}v_1 + \alpha_{22}v_2)\,,\\ \mathrm{d}w_1 &= \mathrm{d}\alpha_{11}.v_1 + \mathrm{d}\alpha_{12}.v_2 + \alpha_{11}(\omega_1^1 v_1 + \omega_1^2 v_2 + \omega^1 v_3) +\\ &\quad + \alpha_{12}(\omega_2^1 v_1 + \omega_2^2 v_2 + \omega^2 v_3) =\\ &= \tau_1^1(\alpha_{11}v_1 + \alpha_{12}v_2) + \tau_1^2(\alpha_{21}v_1 + \alpha_{22}v_2) + \tau^1(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3)\,,\\ \mathrm{d}w_2 &= \mathrm{d}\alpha_{21}.v_1 + \mathrm{d}\alpha_{22}.v_2 + \alpha_{21}(\omega_1^1 v_1 + \omega_1^2 v_2 + \omega^1 v_3) +\\ &\quad + \alpha_{22}(\omega_2^1 v_1 + \omega_2^2 v_2 + \omega^2 v_3) =\\ &= \tau_2^1(\alpha_{11}v_1 + \alpha_{12}v_2) + \tau_2^2(\alpha_{21}v_1 + \alpha_{22}v_2) + \tau^2(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3)\,,\\ \mathrm{d}w_3 &= \mathrm{d}\alpha_{31}.v_1 + \mathrm{d}\alpha_{32}.v_2 + \mathrm{d}\alpha_{33}.v_3 + \alpha_{31}(\omega_1^1 v_1 + \omega_1^2 v_2 + \omega^1 v_3) +\\ &\quad + \alpha_{32}(\omega_2^1 v_1 + \omega_2^2 v_2 + \omega^2 v_3) + \alpha_{33}(\omega_3^1 v_1 + \omega_3^2 v_2 + \omega_3^3 v_3) =\\ &= \tau_3^1(\alpha_{11}v_1 + \alpha_{12}v_2) + \tau_3^2(\alpha_{21}v_1 + \alpha_{22}v_2) + \tau_3^3(\alpha_{31}v_1 + \alpha_{32}v_2 + \alpha_{33}v_3)\,.\end{aligned}$$

From (12_1),

$$(13)\qquad \omega^1 = \alpha_{11}\tau^1 + \alpha_{21}\tau^2\,,\quad \omega^2 = \alpha_{12}\tau^1 + \alpha_{22}\tau^2\,.$$

Comparing the terms at v_3 in $(12_{2,3})$ and using (13), we get

$$(14)\qquad \alpha_{11}^2 + \alpha_{12}^2 = \alpha_{21}^2 + \alpha_{22}^2 = \alpha_{33}\,,\quad \alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0\,.$$

Thus $\alpha_{33} > 0$, and there are functions α, β such that

$$(15)\qquad \begin{aligned}&\alpha_{11} = \alpha\cos\beta\,,\quad \alpha_{12} = -\alpha\sin\beta\,,\quad \alpha_{21} = \varrho\alpha\sin\beta\,,\quad \alpha_{22} = \varrho\alpha\cos\beta\,,\\ &\alpha_{33} = \alpha^2\,;\quad \alpha > 0\,,\quad \varrho = \pm 1\,.\end{aligned}$$

Comparing the terms at v_1, v_2 in $(12_{2,3})$ and at v_3 in (12_4) and using (15), we get after elementary calculations

(16)
$$d\alpha + \alpha\cos^2\beta.\omega_1^1 - \alpha\sin\beta\cos\beta.(\omega_2^1 + \omega_1^2) + \alpha\sin^2\beta.\omega_2^2 = = \alpha\tau_1^1 + (\alpha_{31}\cos\beta - \alpha_{32}\sin\beta)\,\tau^1\,,$$
$$-\alpha\,d\beta + \alpha\sin\beta\cos\beta.(\omega_1^1 - \omega_2^2) - \alpha\sin^2\beta.\omega_2^1 + \alpha\cos^2\beta.\omega_1^2 = = \varrho\alpha\tau_1^2 + (\alpha_{31}\sin\beta + \alpha_{32}\cos\beta)\,\tau^1\,,$$
$$\varrho\alpha\,d\beta + \varrho\alpha\sin\beta\cos\beta.(\omega_1^1 - \omega_2^2) + \varrho\alpha\cos^2\beta.\omega_2^1 - \varrho\alpha\sin^2\beta.\omega_1^2 = = \alpha\tau_2^1 + (\alpha_{31}\cos\beta - \alpha_{32}\sin\beta)\,\tau^2\,,$$
$$\varrho\,d\alpha + \varrho\alpha\sin^2\beta.\omega_1^1 + \varrho\alpha\sin\beta\cos\beta.(\omega_2^1 + \omega_1^2) + \varrho\alpha\cos^2\beta.\omega_2^2 = = \varrho\alpha\tau_2^2 + (\alpha_{31}\sin\beta + \alpha_{32}\cos\beta)\,\tau^2\,,$$
$$2\alpha\,d\alpha + \alpha_{31}\omega^1 + \alpha_{32}\omega^2 + \alpha^2\omega_3^3 = \alpha^2\tau_3^3\,.$$

Let

(17)
$$2\tau_1^1 - \tau_3^3 = a_1'\tau^1 + a_2'\tau^2\,,\quad \tau_1^2 + \tau_2^1 = a_2'\tau^1 + a_3'\tau^2\,,$$
$$2\tau_2^2 - \tau_3^3 = a_3'\tau^1 + a_4'\tau^2$$

be equations analogous to (5). Using (16), we obtain

(18)
$$a_1' = \alpha(\cos^3\beta.a_1 - 3\sin\beta\cos^2\beta.a_2 + 3\sin^2\beta\cos\beta.a_3 - \sin^3\beta.a_4) - - 3\alpha^{-1}(\alpha_{31}\cos\beta - \alpha_{32}\sin\beta)\,,$$
$$a_2' = \varrho\alpha[\sin\beta\cos^2\beta.a_1 + \cos\beta(\cos^2\beta - 2\sin^2\beta)\,a_2 + + \sin\beta(\sin^2\beta - 2\cos^2\beta)\,a_3 + + \sin^2\beta\cos\beta.a_4] - - \varrho\alpha^{-1}(\alpha_{31}\sin\beta + \alpha_{32}\cos\beta)\,,$$
$$a_3' = \alpha[\sin^2\beta\cos\beta.a_1 + \sin\beta(2\cos^2\beta - \sin^2\beta)\,a_2 + + \cos\beta(\cos^2\beta - 2\sin^2\beta)\,a_3 - \sin\beta\cos^2\beta.a_4] - - \alpha^{-1}(\alpha_{31}\cos\beta - \alpha_{32}\sin\beta)\,,$$
$$a_4' = \varrho\alpha(\sin^3\beta.a_1 + 3\sin^2\beta\cos\beta.a_2 + 3\sin\beta\cos^2\beta.a_3 + \cos^3\alpha.a_4) - - 3\varrho\alpha^{-1}(\alpha_{31}\sin\beta + \alpha_{32}\cos\beta)$$

and

(19)
$$a_1' + a_3' = \alpha\cos\beta.(a_1 + a_3) - \alpha\sin\beta.(a_2 + a_4) - 4\alpha^{-1}(\alpha_{31}\cos\beta - \alpha_{32}\sin\beta)\,,$$
$$a_2' + a_4' = \varrho\alpha\sin\beta.(a_1 + a_3) + \varrho\alpha\cos\beta.(a_2 + a_4) - 4\varrho\alpha^{-1}(\alpha_{31}\sin\beta + \alpha_{32}\cos\beta)\,.$$

From the last equations we see that *we may choose the frames in such a way that*

(20)
$$a_1 + a_3 = a_2 + a_4 = 0\,.$$

Let $\{m; v_i\}$, $\{m; w_i\}$ be such two fields of frames; then

(21)
$$\alpha_{31} = \alpha_{32} = 0\,.$$

In what follows, let us suppose (20). From $(5_{1,3})$ we obtain

(22)
$$\omega_1^1 = \tfrac{1}{2}(\omega_3^3 - a_3\omega^1 + a_2\omega^2)\,,\quad \omega_2^2 = \tfrac{1}{2}(\omega_3^3 + a_3\omega^1 - a_2\omega^2)\,.$$

Adding $(8_{1,3})$, we get

(23) $$\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0\,,$$

i.e.,

(24) $$\omega_3^1 = b_1\omega^1 + b_2\omega^2\,, \quad \omega_3^2 = b_2\omega^1 + b_3\omega^2\,,$$

and the equations (8) reduce to

(25)
$$(\mathrm{d}a_2 - \tfrac{1}{2}a_2\omega_3^3 - 3a_3\varphi) \wedge \omega^1 + (\mathrm{d}a_3 - \tfrac{1}{2}a_3\omega_3^3 + 3a_2\varphi) \wedge \omega^2 = (b_3 - b_1)\,\omega^1 \wedge \omega^2,$$
$$(\mathrm{d}a_3 - \tfrac{1}{2}a_3\omega_3^3 + 3a_2\varphi) \wedge \omega^1 - (\mathrm{d}a_2 - \tfrac{1}{2}a_2\omega_3^3 - 3a_3\varphi) \wedge \omega^2 = -2b_2\omega^1 \wedge \omega^2.$$

The differentiation of (24) yields

(26) $$(\mathrm{d}b_1 - b_1\omega_3^3 - 2b_2\varphi) \wedge \omega^1 + [\mathrm{d}b_2 - b_2\omega_3^3 + (b_1 - b_3)\,\varphi] \wedge \omega^2 = $$
$$= [\tfrac{1}{2}a_2(b_1 - b_3) + a_3b_2]\,\omega^1 \wedge \omega^2\,,$$
$$[\mathrm{d}b_2 - b_2\omega_3^3 + (b_1 - b_3)\,\varphi] \wedge \omega^1 + (\mathrm{d}b_3 - b_3\omega_3^3 + 2b_2\varphi) \wedge \omega^2 =$$
$$= [\tfrac{1}{2}a_3(b_1 - b_3) - a_2b_2]\,\omega^1 \wedge \omega^2\,.$$

Comparing the terms at v_1, v_2 in (12_4) and using (15) + (21), we obtain

(27) $$\alpha\omega_3^1 = \cos\beta\,.\,\tau_3^1 + \varrho\sin\beta\,.\,\tau_3^2\,, \quad \alpha\omega_3^2 = -\sin\beta\,.\,\tau_3^1 + \varrho\cos\beta\,.\,\tau_3^2\,.$$

If we write

(28) $$\tau_3^1 = b_1'\tau^1 + b_2'\tau^2\,, \quad \tau_3^2 = b_2'\tau^1 + b_3'\tau^2\,,$$

elementary calculations yield

(29) $$b_1' = \alpha^2(\cos^2\beta\,.\,b_1 - 2\sin\beta\cos\beta\,.\,b_2 + \sin^2\beta\,.\,b_3)\,,$$
$$b_2' = \varrho\alpha^2[\sin\beta\cos\beta\,.\,(b_1 - b_3) + (\cos^2\beta - \sin^2\beta)\,.\,b_2]\,,$$
$$b_3' = \alpha^2(\sin^2\beta\,.\,b_1 + 2\sin\beta\cos\beta\,.\,b_2 + \cos^2\beta\,.\,b_3)$$

and

(30) $$b_1' + b_3' = \alpha^2(b_1 + b_3)\,.$$

Thus we are able to choose the frames $\{m; v_i\}$ such that $b_1 + b_3 = 0$ or ± 1, respectively.

From (21) we see that the straight line $n = \{m + tv_3;\ t \in \mathbb{R}\}$ is an invariant of our surface; it is the so-called *affine normal*. Let us look at the foci of the congruence of affine normals associated to our surface. Let

(31) $$F = m + xv_3$$

be a focus of the normal congruence. Then

(32) $$\mathrm{d}F = (\omega^1 + x\omega_3^1)\,v_1 + (\omega^2 + x\omega_3^2)\,v_2 + (\mathrm{d}x + x\omega_3^3)\,v_3\,;$$

eliminating ω^1, ω^2 from $\omega^1 + x\omega_3^1 = \omega^2 + x\omega_3^2 = 0$, we get

(33) $$1 + (b_1 + b_3)\,x + (b_1b_3 - b_2^2)\,x^2 = 0\,.$$

Thus $b_1 + b_3 = 0$ (at the point m) if and only if the foci F_1, F_2 do not exist (in the case $b_1b_3 - b_2^2 = 0$ at m) or the point m is the center of the interval F_1F_2. In what follows, *let us consider surfaces with $b_1 + b_3 \neq 0$ at each point.* Points with $b_1 + b_3 = 0$ may be called *maximal*; this follows from the fact that each surface with $b_1 + b_3 = 0$ at each point is maximal in the terminology of E. Calabi.

Consequently, let us consider just the fields of frames $\{m; v_i\}$ satisfying

$$b_1 + b_3 = -2\varepsilon\,, \quad \varepsilon = \pm 1\,. \tag{34}$$

Let $\{m; w_i\}$ be another field of frames satisfying $b_1' + b_3' = -2\varepsilon$; then

$$\alpha = 1\,. \tag{35}$$

From (13) and (15) we see that

$$ds^2 := (\omega^1)^2 + (\omega^2)^2 \tag{36}$$

is an affine invariant of our surface; it is the so-called *affine metric*. Because of (34), let the function b_0 be introduced by

$$b_1 = b_0 - \varepsilon\,, \quad b_3 = -(b_0 + \varepsilon)\,. \tag{37}$$

Using (20) + (37) and (35), the equations (18) and (29) reduce to

$$\begin{aligned} a_2' &= \varrho \cos 3\beta\,.\,a_2 - \varrho \sin 3\beta\,.\,a_3\,, & a_3' &= \sin 3\beta\,.\,a_2 + \cos 3\beta\,.\,a_3\,,\\ b_0' &= \cos 2\beta\,.\,b_0 - \sin 2\beta\,.\,b_2\,, & b_2' &= \varrho \sin 2\beta\,.\,b_0 + \varrho \cos 2\beta\,.\,b_2\,. \end{aligned} \tag{38}$$

Thus *the functions*

$$a_2^2 + a_3^2\,, \quad b_0^2 + b_2^2 \tag{39}$$

are affine invariants of our surface.

Because of (37), the equations (26) reduce to

$$\begin{aligned} &(db_0 - b_0\omega_3^3 - 2b_2\varphi + \varepsilon\omega_3^3) \wedge \omega^1 + (db_2 - b_2\omega_3^3 + 2b_0\varphi) \wedge \omega^2 =\\ &\qquad = (a_2b_0 + a_3b_2)\,\omega^1 \wedge \omega^2\,,\\ &(db_2 - b_2\omega_3^3 + 2b_0\varphi) \wedge \omega^1 - (db_0 - b_0\omega_3^3 - 2b_2\varphi - \varepsilon\omega_3^3) \wedge \omega^2 =\\ &\qquad = (a_3b_0 - a_2b_2)\,\omega^1 \wedge \omega^2\,, \end{aligned} \tag{40}$$

and we get the existence of functions c_1, c_2 such that

$$\omega_3^3 = c_1\omega^1 + c_2\omega^2\,. \tag{41}$$

Let the 1-form ω be defined by

$$\omega := \varphi + \tfrac{1}{2}(c_2\omega^1 - c_1\omega^2)\,; \tag{42}$$

then it is easy to see that

$$d\omega^1 = -\omega^2 \wedge \omega\,, \quad d\omega^2 = \omega^1 \wedge \omega\,. \tag{43}$$

Because of (7) and (22), we have

$$\begin{aligned} \omega_1^1 &= \tfrac{1}{2}(c_1 - a_3)\,\omega^1 + \tfrac{1}{2}(c_2 + a_2)\,\omega^2\,,\\ \omega_2^2 &= \tfrac{1}{2}(c_1 + a_3)\,\omega^1 + \tfrac{1}{2}(c_2 - a_2)\,\omega^2\,,\\ \omega_1^2 &= \tfrac{1}{2}(a_2 - c_2)\,\omega^1 + \tfrac{1}{2}(a_3 + c_1)\,\omega^2 + \omega\,,\\ \omega_2^1 &= \tfrac{1}{2}(a_2 + c_2)\,\omega^1 + \tfrac{1}{2}(a_3 - c_1)\,\omega^2 - \omega\,. \end{aligned} \tag{44}$$

The differential consequences are (25) and (40), i.e.,

(45)
$$\begin{aligned}(\mathrm{d}a_2 - 3a_3\omega) \wedge \omega^1 + (\mathrm{d}a_3 + 3a_2\omega) \wedge \omega^2 &= (-2b_0 - a_3c_1 + a_2c_2)\,\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}a_3 + 3a_2\omega) \wedge \omega^1 - (\mathrm{d}a_2 - 3a_3\omega) \wedge \omega^2 &= (-2b_2 + a_2c_1 + a_3c_2)\,\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}b_0 - 2b_2\omega) \wedge \omega^1 + (\mathrm{d}b_2 + 2b_0\omega) \wedge \omega^2 &= (a_2b_0 + a_3b_2 + \varepsilon c_2)\,\omega^1 \wedge \omega^2\,,\\ (\mathrm{d}b_2 + 2b_0\omega) \wedge \omega^1 - (\mathrm{d}b_0 - 2b_2\omega) \wedge \omega^2 &= (a_3b_0 - a_2b_2 - \varepsilon c_1)\,\omega^1 \wedge \omega^2\,.\end{aligned}$$

From (41) we get

(46) $$(\mathrm{d}c_1 - c_2\omega) \wedge \omega^1 + (\mathrm{d}c_2 + c_1\omega) \wedge \omega^2 = 0\,,$$

i.e.,

(47) $$\mathrm{d}c_1 - c_2\omega = c_{11}\omega^1 + c_{12}\omega^2\,, \quad \mathrm{d}c_2 + c_1\omega = c_{12}\omega^1 + c_{22}\omega^2\,.$$

From $(16_5) + (35) + (13) + (15)$ we see that *the function*

(48) $$c_1^2 + c_2^2$$

is an affine invariant of our surface.

The *Gauss curvature* $\varkappa$ of the affine metric (36) is given, because of (43), by

(49) $$\mathrm{d}\omega = -\varkappa\omega^1 \wedge \omega^2\,,$$

this being well known. The differential consequence of (42) yields the following

Lemma. (Theorema egregium.) *We have*

(50) $$2\varkappa = c_{11} + c_{22} + a_2^2 + a_3^2 + 2\varepsilon\,.$$

Theorem 1. *Let $M^2 \subset A^3$ be an analytic elliptic surface each point of which is non-maximal; suppose*

(51) $$\varkappa = \varepsilon = \pm 1 \quad \text{on} \quad M^2\,.$$

Then M^2 is part of a quadric (an ellipsoid for $\varkappa = 1$ and a hyperboloid for $\varkappa = -1$), or the set

(52) $$N := \{m \in M^2;\ a_2^2 + a_3^2 = b_0^2 + b_2^2 = c_1^2 + c_2^2 = 0 \text{ at } m\}$$

consists of isolated points.

Proof. Let m_0 be a non-isolated point of N; let $D \subset M^2$ be a bounded coordinate neighborhood of m_0. In D, take local coordinates (x, y) such that

(53) $$\omega^1 = r(x, y)\,\mathrm{d}x\,, \quad \omega^2 = s(x, y)\,\mathrm{d}y\,; \quad r(x, y)\,s(x, y) \neq 0\,.$$

From (43), we get

(54) $$\omega = -s^{-1}\frac{\partial r}{\partial y}\,\mathrm{d}x + r^{-1}\frac{\partial s}{\partial x}\,\mathrm{d}y\,.$$

Because of (50) and (51),

(55) $$c_{11} + c_{22} + a_2^2 + a_3^2 = 0$$

on M^2. From (45)–(47) and (55), we get, on D, the following system of partial differential equations for $a_2, a_3, b_0, b_2, c_1, c_2$:

$$
(56) \quad s\frac{\partial a_2}{\partial x} + r\frac{\partial a_3}{\partial y} = -\left(3\frac{\partial s}{\partial x} + rsc_1\right)a_2 - \left(3\frac{\partial r}{\partial y} + rsc_2\right)a_3 + 2rsb_2\,,
$$

$$
r\frac{\partial a_2}{\partial y} - s\frac{\partial a_3}{\partial x} = -\left(3\frac{\partial r}{\partial y} + rsc_2\right)a_2 + \left(3\frac{\partial s}{\partial x} + rsc_1\right)a_3 + 2rsb_0\,,
$$

$$
s\frac{\partial b_0}{\partial x} + r\frac{\partial b_2}{\partial y} = -\left(2\frac{\partial s}{\partial x} + rsa_3\right)b_0 - \left(2\frac{\partial r}{\partial y} - rsa_2\right)b_2 + \varepsilon rsc_1\,,
$$

$$
r\frac{\partial b_0}{\partial y} - s\frac{\partial b_2}{\partial x} = -\left(2\frac{\partial r}{\partial y} + rsa_2\right)b_0 + \left(2\frac{\partial s}{\partial x} - rsa_3\right)b_2 - \varepsilon rsc_2\,,
$$

$$
s\frac{\partial c_1}{\partial x} + r\frac{\partial c_2}{\partial y} = -rs(a_2^2 + a_3^2) - \frac{\partial s}{\partial x}c_1 - \frac{\partial r}{\partial y}c_2\,,
$$

$$
r\frac{\partial c_1}{\partial y} - s\frac{\partial c_2}{\partial x} = -\frac{\partial r}{\partial y}c_1 + \frac{\partial s}{\partial x}c_2\,.
$$

Obviously, this is an elliptic system; see [2], p. 76. The zero points of its solution being not isolated, we have (see [2], Theorem 5.4.1 and p. 76)

$$
(57) \qquad a_2 = a_3 = b_0 = b_2 = c_1 = c_2 = 0 \quad \text{on} \quad D
$$

and, by analyticity, on M^2. Thus we get

$$
(58) \qquad \omega^3 = 0\,, \quad \omega_1^3 = \omega^1\,, \quad \omega_2^3 = \omega^2\,, \qquad \omega_1^1 = \omega_2^2 = \omega_3^3 = 0\,,
$$

$$
\omega_1^2 = \omega,\quad \omega_2^1 = -\omega\,, \quad \omega_3^1 = -\varepsilon\omega^1\,, \quad \omega_3^2 = -\varepsilon\omega^2
$$

from (3), (44), (41), (24), (37) and (57). The rest of our assertion may be proved easily. QED.

Theorem 2. *Let $M \equiv M^2 \subset A^3$ be an elliptic surface each point of which is non-maximal; let ∂M be its boundary. Suppose*

$$
(59) \qquad \varepsilon = 1 \quad \text{and} \quad \varkappa \leqq 1 \quad \text{on} \quad M\,; \quad c_1^2 + c_2^2 = 0 \quad \text{on} \quad \partial M\,.
$$

Then M is (part of) an ellipsoid.

Proof. Consider the 1-form

$$
(60) \qquad \Omega := -c_2\omega^1 + c_1\omega^2
$$

on M; it is easy to show that it is an affine invariant of our surface. The Stokes theorem reads

$$
(61) \qquad \int_{\partial M}\Omega = \int_M (c_{11} + c_{22})\,\omega^1 \wedge \omega^2\,.
$$

Because of $\Omega = 0$ on ∂M and (50), (61) turns out to be

$$
(62) \qquad \int_M [2(1 - \varkappa) + a_2^2 + a_3^2]\,\omega^1 \wedge \omega^2 = 0\,.
$$

From (59) and (62), $a_2 = a_3 = 0$ on M. The system (45) implies $b_0 = b_2 = 0$ and $c_1 = c_2 = 0$ on M, and we get (58) with $\varepsilon = 1$. QED.

References

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