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CONDITIONS FOR TRANSITIVE PRINCIPAL TOLERANCES

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By a *principal tolerance* on an algebra $\mathfrak{A} = (A, F)$ we mean the least compatible symmetric reflexive relation on $\mathfrak{A}$ containing a given pair $[a, b] \in A \times A$. Such a relation exists for any pair $[a, b] \in A \times A$.

An algebra $\mathfrak{A}$ is said to have *transitive* (alias *trivial*) *principal tolerances* if each principal tolerance on $\mathfrak{A}$ is transitive, i.e. it is a principal congruence. A class of algebras $\mathscr{V}$ is said to have *transitive principal tolerances* if any algebra in $\mathscr{V}$ has transitive principal tolerances.

Let $(*)$ and $(**)$ denote the following systems of identities:

$$(*)\begin{cases} f_1(s(x_1, \ldots, x_n), t(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\ f_2(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g(v(x_1, \ldots, x_n), u(x_1, \ldots, x_n), x_1, \ldots, x_n)\,, \end{cases}$$

$$(**)\begin{cases} f_1(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g_1(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\ f(s(x_1, \ldots, x_n), t(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g_1(v(x_1, \ldots, x_n), u(x_1, \ldots, x_n), x_1, \ldots, x_n) \\ f(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g_2(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\ f_2(s(x_1, \ldots, x_n), t(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\ = g_2(v(x_1, \ldots, x_n), u(x_1, \ldots, x_n), x_1, \ldots, x_n)\,. \end{cases}$$

Theorem. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent:*

(*A*) *$\mathscr{V}$ has transitive principal tolerances.*

(*E*) *For every natural number n, every $(n + 2)$-ary $\mathscr{V}$-polynomials f_1, g, f_2 and every n-ary $\mathscr{V}$-polynomials s, t, u, v such that $(*)$ holds in $\mathscr{V}$ there exist $(n + 2)$-ary $\mathscr{V}$-polynomials g_1, f, g_2 such that $(**)$ holds in $\mathscr{V}$.*

(*F*) *For every natural number n, every $(n + 2)$-ary $\mathscr{V}$-polynomials f_1, f_2 and every n-ary $\mathscr{V}$-polynomials s, t there exist $(n + 2)$-ary $\mathscr{V}$-polynomials g_1, f, g_2*

such that $(**)$ *holds in* $\mathscr{V}$, *where*

$$(***)\ \begin{cases} u(x_1, \ldots, x_n) \equiv f_1(s(x_1, \ldots, x_n), t(x_1, \ldots, x_n), x_1, \ldots, x_n) \\ v(x_1, \ldots, x_n) \equiv f_2(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) . \end{cases}$$

Proof. For $(\boldsymbol{A}) \Leftrightarrow (\boldsymbol{E})$ see [1].

Troughout the proof, $\boldsymbol{x}$ is a concise form for $x_1, \ldots, x_n$.

$(\boldsymbol{E}) \Rightarrow (\boldsymbol{F})$: Let $(\boldsymbol{E})$ be true in $\mathscr{V}$. Let n be a natural number and f_1, f_2 arbitrary $(n + 2)$-ary $\mathscr{V}$-polynomials, s, t arbitrary n-ary $\mathscr{V}$-polynomials. Take $g(y, z, \boldsymbol{x}) \equiv y$ and u, v as in $(***)$. Since $(*)$ is satisfied in $\mathscr{V}$, there exist $(n + 2)$-ary $\mathscr{V}$-polynomials g_1, f, g_2 such that $(**)$ holds in $\mathscr{V}$. This proves statement $(\boldsymbol{F})$.

$(\boldsymbol{F}) \Rightarrow (\boldsymbol{E})$: Let $(\boldsymbol{F})$ be true in $\mathscr{V}$. Let $n, f'_1, g', f'_2, s', t', u', v'$ satisfy the assumptions of statement $(\boldsymbol{E})$. Inasmuch as $n, f_1 \equiv f'_1, f_2 \equiv f'_2$, $s \equiv s'$, $t \equiv t'$ also satisfy the assumptions of statement $(\boldsymbol{F})$, there exist $(n + 2)$-ary $\mathscr{V}$-polynomials g_1, f, g_2 such that $(**)$ holds in $\mathscr{V}$ for u, v defined by $(***)$. Put

$$g'_1(y, z, \boldsymbol{x}) \equiv g_1(g'(y, z, \boldsymbol{x}), g'(z, y, \boldsymbol{x}), \boldsymbol{x}), \quad g'_2(y, z, \boldsymbol{x}) \equiv g_2(g'(y, z, \boldsymbol{x}), g'(z, y, \boldsymbol{x}), \boldsymbol{x})$$

and $f' \equiv f$. Since

$$u(\boldsymbol{x}) = g'(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \quad v(\boldsymbol{x}) = g'(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x})$$

are $\mathscr{V}$-identities, we obtain $\mathscr{V}$-identities

$$\begin{aligned}
& f'_1(t'(\boldsymbol{x}), s'(\boldsymbol{x}), \boldsymbol{x}) = g_1(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) = g_1(g'(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \\
& g'(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}) \equiv g'_1(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \\
& f'(s'(\boldsymbol{x}), t'(\boldsymbol{x}), \boldsymbol{x}) = g_1(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) = g_1(g'(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}), \\
& g'(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}) \equiv g'_1(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}), \\
& f'(t'(\boldsymbol{x}), s'(\boldsymbol{x}), \boldsymbol{x}) = g_2(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) = g_2(g'(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \\
& g'(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}) \equiv g'_2(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \\
& f'_2(s'(\boldsymbol{x}), t'(\boldsymbol{x}), \boldsymbol{x}) = g_2(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x}) = g_2(g'(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}), \\
& g'(u'(\boldsymbol{x}), v'(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}) \equiv g'_2(v'(\boldsymbol{x}), u'(\boldsymbol{x}), \boldsymbol{x}),
\end{aligned}$$

proving $(\boldsymbol{E})$. Q.E.D.

In the case $(\boldsymbol{F})$, $(**)$ with $(***)$ should be read

$$\begin{aligned}
f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) &= g_1(f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}), f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}), \\
f(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) &= g_1(f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}), f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}), \\
f(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) &= g_2(f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}), f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}), \\
f_2(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) &= g_2(f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}), f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}), \boldsymbol{x}) .
\end{aligned}$$

Reference

[1] *Niederle, J.:* Conditions for trivial principal tolerances. Arch. Math. (Brno) *19* (1983), 145–152.

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