Josef Niederle
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CONDITIONS FOR TRANSITIVE PRINCIPAL TOLERANCES

JOSEF NIEDERLE, Brno

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By a principal tolerance on an algebra \( \mathcal{A} = (A, F) \) we mean the least compatible symmetric reflexive relation on \( \mathcal{A} \) containing a given pair \([a, b] \in A \times A\). Such a relation exists for any pair \([a, b] \in A \times A\).

An algebra \( \mathcal{A} \) is said to have transitive (alias trivial) principal tolerances if each principal tolerance on \( \mathcal{A} \) is transitive, i.e. it is a principal congruence. A class of algebras \( \mathcal{V} \) is said to have transitive principal tolerances if any algebra in \( \mathcal{V} \) has transitive principal tolerances.

Let \((*)\) and \((**\) denote the following systems of identities:

\[
(*) \quad \begin{cases} 
    f_1(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\
    = g_1(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\
    f_2(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\
    = g_2(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\
    \end{cases}
\]

\[
(**) \quad \begin{cases} 
    f_1(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\
    = g_1(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\
    f_2(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) = \\
    = g_2(u(x_1, \ldots, x_n), v(x_1, \ldots, x_n), x_1, \ldots, x_n) \\
    \end{cases}
\]

Theorem. Let \( \mathcal{V} \) be a variety of algebras. The following conditions are equivalent:

(A) \( \mathcal{V} \) has transitive principal tolerances.

(E) For every natural number \( n \), every \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( f_1, g, f_2 \) and every \( n \)-ary \( \mathcal{V} \)-polynomials \( s, t, u, v \) such that \((*)\) holds in \( \mathcal{V} \) there exist \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( g_1, f, g_2 \) such that \((**\) holds in \( \mathcal{V} \).

(F) For every natural number \( n \), every \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( f_1, f_2 \) and every \( n \)-ary \( \mathcal{V} \)-polynomials \( s, t \) there exist \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( g_1, f, g_2 \)
such that (**) holds in \( \mathcal{V} \), where

\[
\begin{align*}
(\ast\ast) & \quad \{ u(x_1, \ldots, x_n) \equiv f_1(s(x_1, \ldots, x_n), t(x_1, \ldots, x_n), x_1, \ldots, x_n) \\
& \quad \{ v(x_1, \ldots, x_n) \equiv f_2(t(x_1, \ldots, x_n), s(x_1, \ldots, x_n), x_1, \ldots, x_n) .
\end{align*}
\]

Proof. For \((A) \iff (E)\) see [1].

Throughout the proof, \( x \) is a concise form for \( x_1, \ldots, x_n \).

\((E) \Rightarrow (F)\): Let \((E)\) be true in \( \mathcal{V} \). Let \( n \) be a natural number and \( f_1, f_2 \) arbitrary \((n + 2)\)-ary \( \mathcal{V} \)-polynomials, \( s, t \) arbitrary \( n \)-ary \( \mathcal{V} \)-polynomials. Take \( g(y, z, x) \equiv y \) and \( u, v \) as in \((\ast\ast)\). Since (\ast) is satisfied in \( \mathcal{V} \), there exist \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( g_1, f, g_2 \) such that (**) holds in \( \mathcal{V} \). This proves statement \((F)\).

\((F) \Rightarrow (E)\): Let \((F)\) be true in \( \mathcal{V} \). Let \( n, f'_1, f'_2, s', t' \), \( u', v' \) satisfy the assumptions of statement \((E)\). Inasmuch as \( n, f_1 \equiv f'_1, f_2 \equiv f'_2, s \equiv s', t \equiv t' \) also satisfy the assumptions of statement \((F)\), there exist \((n + 2)\)-ary \( \mathcal{V} \)-polynomials \( g_1, f, g_2 \) such that (**) holds in \( \mathcal{V} \) for \( u, v \) defined by \((\ast\ast)\). Put

\[
g'_1(y, z, x) \equiv g_1(g'(y, z, x), g'(y, z, x), x), \quad g'_2(y, z, x) \equiv g_2(g'(y, z, x), g'(z, y, x), x)
\]

and \( f' \equiv f \). Since \( u(x) = g'(u'(x), v'(x), x) \), \( v(x) = g'(v(x), u'(x), x) \) are \( \mathcal{V} \)-identities, we obtain \( \mathcal{V} \)-identities

\[
\begin{align*}
f'_1(t'(x), s'(x), x) & = g_1(u(x), v(x), x) = g_1(g'(u'(x), v'(x), x), g'(v'(x), u'(x), x), g'(u'(x), v'(x), x) = g'_1(u'(x), v'(x), x),
\end{align*}
\]

proving \((E)\). Q.E.D.

In the case \((F)\), (**) with (***) should be read

\[
\begin{align*}
f_1(t(x), s(x), x) & = g_1(f_1(s(x), t(x), x), f_2(t(x), s(x), x), x) ,
\end{align*}
\]

References


Author’s address: Viniční 60, 615 00 Brno 15, Czechoslovakia.