Czechoslovak Mathematical Journal

Pedro José Paúl

New applications of Pták's extension theorem to weak compactness

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 3, 454–458

Persistent URL: http://dml.cz/dmlcz/102316

NEW APPLICATIONS OF PTÁK'S EXTENSION THEOREM TO WEAK COMPACTNESS

PEDRO J. PAÚL, Sevilla

(Received June 23, 1987)

Our purpose is to show how Pták's double sequence method in weak compactness, developed in [4, 5] (see also [2, § 24.6]), and his Extension Theorem [5, Thm. (2,2)] can be used to give some new Eberlein- and Krein-type Theorems, extending also some of Valdivia's results [6, Thms. 3 and 11].

The notation and the terminology will be standard.

Definition. Let (E, E') be a dual pair. We say that two sets $A \subset E$ and $M \subset E'$ *interchange double limits*, and we write $A \sim M$, if for every sequence x_n from A and every sequence u_m from M the existence of both the double limits $\lim_n \lim_m \langle x_n, u_m \rangle$ and $\lim_m \lim_n \langle x_n, u_m \rangle$ ensures that they are equal.

The proof of the following Krein-type theorem is only sketched in [5, Thm.(3,2)]; since it is somehow delicate and the result will be used in the sequel, we include a full detailed proof for the sake of completeness.

Theorem 1. *Let X be a Banach space (with unit ball U) and A a bounded set in X. Let M be a norm-generating subset of X' (i.e. $M \subset U^0$ and there exists $r > 0$ such that $p_M(x) \geqq r\|x\|$ for all x in X). Then A^{00} is weakly compact if (and only if) $A \sim M$.*

Proof. Let J be the natural injection from X into the Banach space $CB(M)$ of continuous and bounded functions on M given by $\langle J(x), u \rangle := \langle x, u \rangle$. Since $r\|x\| \leqq \|J(x)\| := p_M(x) \leqq \|x\|$, J is one-to-one, continuous and open. Then we may consider the adjoint J' from $CB(M)'$ onto X'. Define $B(x, u) := \langle x, u \rangle$ on $A \times M$; by using Pták's Extension Theorem [5, Thm. (2,2)], B can be extended to a separately continuous bilinear form B on $CB(A)' \times CB(M)'$ (considering on both spaces the weak-star topology). For $\hat{x}$ in $CB(A)'$ we define a linear form on X' in the following way:

$$P(\hat{x}) : v \in X' \to \langle P(\hat{x}), v \rangle := B(\hat{x}, J'^{-1}(v)) \in \mathbb{K}$$

This research was suggested by V. Pták and developed during the author's stay at the Mathematical Institute of the Czechoslovak Academy of Sciences supported by the Spanish Ministry of Education. Author is deeply indebted to V. Pták for his comments and to P. Vrbová, V. Müller, J. Fuka and K. Horák for their encouragement.

We must check that $B(\hat{x}, \hat{u})$ does not depend on the choice of $\hat{u}$ in $J'^{-1}(v)$. Indeed: since M is total in $CB(M)'$ (w^*), for $\hat{u}$ in $CB(M)'$ there exists a net u_α from lin (M) such that $\hat{u} = w^* - \lim_\alpha u_\alpha$, then for all x in A we have:

$$(*) \qquad B(x, \hat{u}) = \lim_\alpha B(x, u_\alpha) = \lim_\alpha \langle x, u_\alpha \rangle = \lim_\alpha \langle J(x), u_\alpha \rangle = \\ = \lim_\alpha \langle x, J'(u_\alpha) \rangle = \langle x, J'(\hat{u}) \rangle .$$

Now, if $J'(\hat{u}_1) = J'(\hat{u}_2)$ then $B(\cdot, \hat{u}_1) = B(\cdot, \hat{u}_2)$ in A and, since A is total in $CB(A)'$ (w^*), therefore in $CB(A)'$. Now we shall see that, acting on X', $P(\hat{x})$ can be identified with an element of X. Indeed: $B(\hat{x}, \cdot)$ is continuous on $CB(T)'$ (w^*), hence $B(\hat{x}, \hat{u}) = \langle f, \hat{u} \rangle$ for some f in $CB(M)$. Suppose that f is not in $J(X)$; since $J(X)$ is closed in $CB(M)$ (because J is open), by Hahn-Banach theorem there exists $\hat{u}_0$ in $CB(M)'$ such that $\langle J(x), \hat{u}_0 \rangle = 0$ for all x in X and $\langle f, \hat{u}_0 \rangle = 1$. But if we take a net x_α in lin (A) such that $\hat{x} = w^* - \lim x_\alpha$, then

$$1 = \langle f, \hat{u}_0 \rangle = B(\hat{x}, \hat{u}_0) = \lim_\alpha B(x_\alpha, \hat{u}_0) = (\text{by}(*)) = \\ = \lim_\alpha \langle x_\alpha, J'(\hat{u}_0) \rangle = \lim_\alpha \langle J(x_\alpha), \hat{u}_0 \rangle = 0$$

a contradiction. Thus $f = J(x)$ for some x in X and, in this case, for all v in X' we have:

$$\langle P(\hat{x}), v \rangle = B(\hat{x}, J'^{-1}(v)) = \langle f, J'^{-1}(v) \rangle = \langle J(x), J'^{-1}(v) \rangle = \langle x, v \rangle .$$

In this way, we can define the linear map $P\colon CB(A)' \to X$ by setting $\langle P(\hat{x}), v \rangle = B(\hat{x}, J'^{-1}(v))$ for v in X'. P is weak*-to-weak continuous since $J'^{-1}(v)$ is in $CB(M)'$ and B is separately continuous. On the other hand, for x in A and every v in X' we have:

$$\langle P(x), v \rangle = B(x, J'^{-1}(v)) = (\text{by } (*)) = \langle x, v \rangle$$

hence P acts on A, and being linear also on acx(A) (the absolutely convex hull of A), as the identity. Finally, since A is contained in the unit ball of $CB(A)'$ which is weakly-star compact and absolutely convex, acx(A) is contained in a weakly compact set, namely, the image under P of the unit ball of $CB(A)'$. This implies that A^{oo} is weakly compact. The "only if" part is immediate. QED

In what follows, (E, E') stands for a dual pair. $\mathcal{M}$ will be a family of weakly relatively countably compact sets of E', whose union is $\sigma(E', E)$-total in E'. τ will denote the topology on E of uniform convergence on $\mathcal{M}$. We recall that τ can be strictly finer than the Mackey topology (e.g. if E and E' are both the space φ of finitely non-zero sequences and e_n is the sequence of unit vectors, then $M = \{e_n\}$ is $\sigma(\varphi, \varphi)$-relatively compact but acx(M) is not $\sigma(\varphi, \varphi)$-relatively compact since it contains the finite sections of any vector in the unit ball of ℓ^1 and these sections converge to the vector in the $\sigma(\ell^1, \varphi)$-topology).

The following is an extension of Krein Theorem [2, § 24.5.(4) and 6.(1)].

Theorem 2. *Assume that* $E(\tau)$ *is quasi-complete. Then for a* τ*-bounded set* $A \subset E$ *the following are equivalent*:

(1) A *is* $\sigma(E, E')$*-relatively compact*,

(2) $A \sim M$ *for all* M *in* $\mathcal{M}$,

(3) $A^{oo} \sim M^{oo}$ *for all* M *in* $\mathcal{M}$,

(4) A^{oo} *is* $\sigma(E, E')$*-compact*.

Proof. (1) implies (2) is exactly as in [2, § 24.6.(1)]. (2) implies (3) (this does not depend on the quasi-completeness hypothesis): Let $p_M(x) := \sup \{|\langle x, u\rangle| : u \in M\}$ be the polar seminorm associated to M. Consider the null space $N := \{x : p_M(x) = 0\}$. The quotient E/N is normed space under the norm $\|\hat{x}\| := p_M(x)$ where $\hat{x}$ is the coset of x in E/N. Let X be the completion of $E/N(\|\cdot\|)$. Then $\hat{A}$ is a bounded subset of the Banach space X, M is a norm-generating subset of $X' \cong N^{\perp}$ (whose unit ball is M^{oo}) and, since $\langle \hat{x}, u\rangle_{(X,X')} = \langle x, u\rangle_{(E,E')}$, we have that $A \sim M$. By using Theorem 1, we obtain: $(\hat{A})^{oo}$ (bipolar in X) is $\sigma(X, X')$-compact, then $(\hat{A})^{oo} \sim M^{oo}$ (again by Theorem 1) and, since the coset of A^{oo} is contained in $(\hat{A})^{oo}$, it follows that $A^{oo} \sim M^{oo}$. (3) implies (4): Again, exactly as in [2, § 24.6.(1)] we have that if z is a $\sigma(E'^*, E')$-closure point of A^{oo} in E'^* (the algebraic dual of E') then, under (3), z must be weakly continuous on M^{oo} for every M in $\mathcal{M}$. By the completion theorem [2, § 21.9.(2)] we have that z is in the completion $F(\tilde{\tau})$ of $E(\tau)$. If F' denotes the dual of $F(\tilde{\tau})$, we have that B, the closure of A^{oo} in $F(\sigma(F, F'))$ is $\sigma(F, F')$-compact. Now we have that:

$$B = \overline{A^{oo}}^{F(\tilde{\tau})} = \overline{A^{oo}}^{E(\tau)} = A^{oo} \subset E$$

(first equality: A^{oo} is convex; second equality: $E(\tau)$ is quasi-complete). Thus $A^{oo} = B$ is, indeed, $\sigma(E, F')$-compact and, since $E' \subset F'$, therefore $\sigma(E, E')$-compact. (4) implies (1) is immediate. QED

Definition. A $\sigma(E, E')$-bounded subset A of E is said to be *weakly* (*relatively*) *q-partially compact* if the following holds: Whenever z is a $\sigma(E'^*, E')$-adherent point of A in E'^* then for each weakly relatively compact sequence u_n from E' there exists x_0 in A (in E) with

$$(**) \qquad \langle z - x_0, u_n\rangle = 0 \quad n = 1, 2, \ldots .$$

Remark. By replacing such a given sequence u_n for the sequence $v_1, v_2, v_3 \ldots :=$ $:= u_1, u_2, u_1, u_3, u_2, u_1, u_4, u_3, u_2, u_1, \ldots$ one can see that $(**)$ is equivalent to the (formally) weaker condition $\lim_n \langle z - x_0, u_n\rangle = 0$. This also applies to the concept of weak (r) partial compactness introduced by Day (in which z is assumed to be a $\sigma(E'^*, E')$-adherent point of a countable subset of A, but the sequences u_n are taken from absolutely convex, $\sigma(E', E)$-compact sets, see [2, § 24.3]). Bearing this in mind, we have [2, § 24.3.(2), (3), (4)] that if A is weakly (r.) countably-, pseudo- or convex-compact then A is weakly (relatively) q-partially compact. This is also the case of the sets studied by Valdivia [6]. A stronger condition is true, indeed:

Proposition. *Let A be a subset of E such that every $\sigma(E, E')$-continuous real-valued function on E is bounded on A. Then for every sequence u_n from E' and every z, $\sigma(E'^*, E')$-adherent point of A in E'^*, there exists x_0 in E such that $(**)$ holds.*

Proof. The functions $f_n(x) := |\langle x - z, u_n\rangle|$ $n = 1, 2, \dots$ are weakly continuous on E, therefore $f(x) := \sum_{n=1}^{\infty} f_n(x)/2^n(1 + f_n(x))$ is also weakly continuous (Weierstrass M-test). Since we may choose x such that $f_1(x), \dots, f_m(x)$ are small enough for every fixed m, it is clear that $\inf\{f(x): x \in A\} = 0$. If there exists x_0 in E such that $f(x_0) = 0$, we are done. If $f(x) > 0$ for all x in E, then $g(x) := 1/f(x)$ would be a weakly continuous real valued function on E, but unbounded on A: a contradiction. QED

Our Eberlein-type theorem for weak relative q-partial compactness is the following:

Theorem 3. *Let $E(\tau)$ be quasi-complete and A a τ-bounded weakly relatively q-partially compact subset of E, then A is weakly relatively compact.*

Proof. According to Theorem 2, we have to prove that $A \sim M$ for every $M \in \mathcal{M}$. Assume that x_n from A and u_m are sequences such that $\langle x_n, u_m\rangle \to_m \alpha_n \to_n \alpha$ and $\langle x_n, u_m\rangle \to_n \beta_m \to_m \beta$. Since A is $\sigma(E, E')$-bounded, the sequence x_n has a $\sigma(E'^*, E')$-adherent point z in E'^* and $\beta_m = \langle z, u_m\rangle$ for all m. Since M is $\sigma(E', E)$-relatively countably compact, u_m is $\sigma(E', E)$-relatively compact and $\alpha_n = \langle x_n, u_0\rangle$ for all n and some u_0 $\sigma(E', E)$-adherent point of u_m in E'. Now take the point x_0 in E corresponding to $(**)$ for the sequence $u_0, u_1, u_2, \dots$ then we have

$$|\alpha - \beta| \leqq |\alpha - \langle x_n, u_0\rangle| + |\langle x_n - z, u_0\rangle| + |\langle x_0, u_0 - u_m\rangle| + |\langle x_0, u_m\rangle - \beta|$$

and these summands can be made arbitrarily small for suitable m and n. QED

Corollary 1. *A Banach space E is reflexive if and only if every $\sigma(E'^*, E')$-closure point of its unit ball is $\sigma(E', E)$-continuous when restricted to separable subspaces of E'.*

Our Theorem extends results previously given by Pták [3], Dieudonné [1] and, by using the Proposition above, Valdivia [6, Thm. 3]. Also note that the Theorem remains valid if we assume Day's weak relative partial compactness, since we use only that z is $\sigma(E'^*, E')$-adherent to x_n. In this way we also extend [2, § 24.3.(6)]. The purpose of keeping the condition $(**)$ for every adherent z is to give, finally, an extension of [6, Thm. 11].

Corollary 2. *Let $E(\tau)$ be quasi-complete and A a convex weakly relatively q-partially compact subset of E, then A is weakly relatively compact.*

Proof. We have to prove that A is τ-bounded. We shall make use of an idea of [6, Thm. 8]: If $M \in \mathcal{M}$, let G be the subspace of E' of elements bounded on M. Let A^* be the $\sigma(E'^*, E')$-closure of A in E'^*. If A^* is not contained in G we can find z in A^* and u_n from M such that $\langle z, u_n\rangle \to_n \infty$, take the point x_0 in E satisfying

$(**)$ to obtain a contradiction with the fact that M is $\sigma(E', E)$-bounded. Thus A^* is contained in G and, since A^* is $\sigma(G, E')$-compact and convex and M^0 (the polar in G) is a barrel, M^0 absorbs A^* and this implies that $\sup\{p_M(x): x \in A\} < +\infty$. QED

References

[1] *J. Dieudonné:* Sur un théorème de Šmulian. Arch. d. Math. *3* (1953), 436—440.
[2] *G. Köthe:* Topological Vector Spaces I. Springer-Verlag (1969), Berlin, Heidelberg, New York.
[3] *V. Pták:* On a theorem of W. F. Eberlein. Studia Math. *14* (1954), 276—284.
[4] *V. Pták:* A combinatorial lemma on the existence of convex means and its applications to weak compactness. Proc. Symp. Pure Math. *VII* (1963), 437—450.
[5] *V. Pták:* Extension theorem for separately continuous functions and its application to functional analysis. Czech. Math. Jour. *14* (1964), 562—581.
[6] *M. Valdivia:* Some new results on weak compactness. Jour. Funct. Anal. *24* (1977), 1—10.

Author's address: E.S. Ingenieros Industriales, Av. Reina Mercedes s/n, 41012 - Sevilla, Spain.