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RETRACTS OF ABELIAN LATTICE ORDERED GROUPS

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Retracts of partially ordered sets were investigated in [1], [2], [3] and [4]. In particular, in [4] they were applied for defining the notion of a variety of partially ordered sets (namely, a nonempty class of partially ordered sets is said to be a variety if it is closed with respect to retracts and direct products).

The present paper deals with retracts of direct products of abelian lattice ordered groups.

Assume that an abelian lattice ordered group $H$ is an internal direct product of its $l$-subgroups $A$ and $B$. Let $G$ be a retract of $H$. It will be shown that $G$ need not be, in general, an internal direct product of a retract of $A$ and a retract of $B$. Nevertheless, it will be proved that there exist a retract $R_1$ of $A$ and a retract $R_2$ of $B$ such that $G$ is isomorphic to the direct product of $R_1$ and $R_2$.

The above result will be applied for investigating retracts of lattice ordered groups which can be represented as direct products of a finite number of linearly ordered groups. Further, complete retract mappings of a complete lattice ordered group are studied.

Retract varieties of abelian lattice ordered groups (defined analogously as varieties of partially ordered sets [4]) will be investigated in a subsequent paper by using the results established here.

PRELIMINARIES

We recall that a nonempty subset $Q$ of a partially ordered set $P$ is said to be a retract of $P$ if there is an isotone mapping $f$ of $P$ onto $Q$ such that $f(q) = q$ for each element $q$ of $Q$.

Let us remark that each nonempty subset of a partially ordered set is viewed as being partially ordered by the inherited relation of the partial order.

Let $H$ be an abelian lattice ordered group and let $G$ be an $l$-subgroup of $H$. If there exists a homomorphism $f$ of $H$ onto $G$ such that $f(g) = g$ for each element $g$ of $G$, then $G$ will be said to be a retract of $H$. Also, the mapping $f$ with the just mentioned properties will be called a retract mapping of $H$ onto $G$.

We shall apply the following notation which concerns direct products and lexicographic products.
Assume that $A$ and $B$ are $l$-subgroups of $H$ such that the following conditions are satisfied:

(i) for each $h \in H$ there exist uniquely determined elements $h(A) \in A$ and $h(B) \in B$ such that $h = h(A) + h(B)$;

(ii) for each $h, h' \in H$ and each $t \in \{+, \wedge, \vee\}$ we have

$$(hth')(A) = h(A) \ t \ h'(A),$$

and similarly for $B$.

Under these assumptions $H$ is said to be an internal direct product of $A$ and $B$; we shall express this fact by writing $H = (i) \ A \times B$.

If $A_1, A_2, \ldots, A_n$ are $l$-subgroups of $H$, then the relation

$$H = (i) \ A_1 \times A_2 \times \ldots \times A_n$$

is defined analogously as in the case of two internal direct factors.

If $A_1$ and $B_1$ are lattice ordered groups, then their direct product $A_1 \times B_1$ is defined in the usual way. Let $H = (i) \ A \times B$. The mapping $\varphi: H \rightarrow A \times B$ defined by $\varphi(h) = (h(A), h(B))$ for each $h \in H$ is an isomorphism of $H$ onto $A \times B$.

Let

$$H = (i) \ A \times B$$

be valid and let $X \subseteq H$. Then we put

$$X(A) = \{x(A): x \in X\};$$

$X(B)$ is defined analogously.

Now assume that $H$ is linearly ordered. Next, let $A$ and $B$ be $l$-subgroups of $H$ such that the condition (i) above is satisfied and that, moreover, the following conditions hold:

(iii) for each $h, h' \in H$ we have $(h + h')(A) = h(A) + h'(A)$, and similarly for $B$;

(iv) for each $h \in H$ we have $h \geq 0$ if and only if either $h(A) > 0$, or $h(A) = 0$ and $h(B) \geq 0$.

Under these assumptions $H$ will be said to be an internal lexicographic product of $A$ and $B$, and we write $H = (i) \ A \circ B$.

If $A_1$ and $B_1$ are linearly ordered groups, then we can define their (external) lexicographic product $A_1 \circ B_1$ (cf., e.g., [5]). If

$$(2) \quad H = (i) \ A \circ B$$

holds, then the mapping $\varphi: H \rightarrow A \circ B$ defined by $\varphi(h) = (h(A), h(B))$ for each $h \in H$ is an isomorphism of $H$ onto $A \circ B$.

If (2) holds and $X \subseteq H$, then $X(A)$ and $X(B)$ are defined analogously as in the case of the direct product.

The following two lemmas are easy to verify.

1.1. Lemma. Let (1) be valid and let $X$ be an $l$-subgroup of $H$. Then the following conditions are equivalent:

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(i) \( X(A) \subseteq X \) and \( X(B) \subseteq X \);
(ii) \( X = (i) X(A) \times X(B) \).

Moreover, if there are \( l \)-subgroups \( A_1, B_1 \) of \( A \) and \( B \), respectively, such that \( X = (i) A_1 \times B_1 \), then \( A_1 = X(A) \) and \( B_1 = X(B) \).

1.2. Lemma. Let \( H \) be linearly ordered and let (2) hold. Let \( X \) be an \( l \)-subgroup of \( H \). Then the assertion of 1.1 remains valid if the symbol \( \times \) is replaced by \( \circ \).

1.3. Lemma. Let \( f: H \to G \) be a retract mapping and let \( H_1 \) be the kernel of \( f \). Then the group \( H \) is a direct sum of its subgroups \( G \) and \( H_1 \).

Proof. Clearly we have \( G \cap H_1 = \{0\} \). Let \( z \in H \). Put \( g = f(z), h_1 = z - g \). Then \( h_1 \in H_1 \). Assume that \( g' \in G, h_1' \in H_1 \) and \( z = g' + h_1' \). Thus \( g = f(z) = f(g') + f(h_1') = g' \), and then \( h_1 = h_1' \). Therefore the group \( H \) is the direct sum of \( G \) and \( H_1 \).

DIRECT PRODUCT DECOMPOSITIONS

Again, let \( H \) be an abelian lattice ordered group. Assume that the relation (1) above is valid.

2.1. Lemma. Let \( A_1 \) be a retract of \( A \) (with the corresponding retract mapping \( g_1 \)) and let \( B_1 \) be a retract of \( B \) (with the corresponding retract mapping \( g_2 \)). Put

\[
 f(h) = g_1(h(A)) + g_2(h(B)) \quad \text{for each} \quad h \in H ,
\]

and \( f(H) = G \). Then \( G \) is a retract of \( H \) (with the corresponding retract mapping \( f \)).

Moreover, \( G = (i) A_1 \times B_1 \).

The proof is routine; it will be omitted.

Let \( f: H \to G \) be a retract mapping. Let \( x \in H, x(A) = a, x(B) = b \). Then \( f(x) = \)

\[
 f(a) (A) = a_1 , \quad f(a) (B) = b_1 , \quad f(b) (A) = a_2 , \quad f(b) (B) = b_2 .
\]

Hence

\[
 f(x) = a_1 + b_1 + a_2 + b_2 .
\]

2.2. Lemma. Under the above notation we have

\[
 f(a_1) = a_1 + b_1 , \quad f(b_2) = a_2 + b_2 ,
\]

\[
 f(b_1) = f(a_2) = 0 .
\]

Proof. First we assume that \( x \geq 0 \). Because of \( a_1 \geq 0 \) and \( b_1 \geq 0 \) we obtain \( f(a_1) \geq 0 \) and \( f(b_1) \geq 0 \). Since \( f(a) = a_1 + b_1 \), we have \( f(a) = f(a_1) + + f(b_1) \). Thus \( f(b_1) \leq f(a) \). But \( a \land b_1 = 0 \), whence \( f(a) \land f(b_1) = 0 \) and thus \( f(b_1) = 0 \). Therefore \( f(a_1) = a_1 + b_1 \).

Similarly we can verify the validity of the relations \( f(a_2) = 0 \) and \( f(b_2) = a_2 + b_2 \).
Now let \( x \) be any element of \( H \). Since \( x \) can be expressed in the form \( x = x_1 - x_2 \) where \( x_1, x_2 \in H \) and \( 0 \leq x_1, 0 \leq x_2 \), we infer that the assertion of the lemma holds for \( x \) as well.

Denote \( K_1 = f(A), K_2 = f(B) \).

2.3. Lemma. \( K_1 \) and \( K_2 \) are \( l \)-subgroups of \( G \); moreover, \( G = (i)K_1 \times K_2 \).

Proof. Since \( A \) is an \( l \)-subgroup of \( H \) and \( f: H \to G \) is a homomorphism, \( K_1 \) is an \( l \)-subgroup of \( G \). Similarly, \( K_2 \) is an \( l \)-subgroup of \( G \).

Let \( 0 \leq x \in G \). Then under the above notation we have
\[
x = f(x) = f(a) + f(b), \quad 0 \leq f(a) \in K_1, \quad 0 \leq f(b) \in K_2.
\]
Since \( a \cap b = 0 \), we obtain \( f(a) \cap f(b) = 0 \), whence
\[
x = f(a) \vee f(b).
\]
If \( x' \) is another element of \( G^+ \), then (under analogous notation) we have
\[
x' = f(a') \vee f(b')
\]
because \( a \cap b' = b \cap a' = 0 \), we infer that
\[
x \cap x' = (f(a) \cap f(a')) \vee (f(b) \cap f(b')) = f(a \cap a') \vee f(b \cap b'),
\]
and clearly
\[
x \vee x' = f(a \vee a') \vee f(b \vee b').
\]
Hence the mapping \( x \to (f(a), f(b)) (x \in G^+) \) is an isomorphism of the lattice \( G^+ \) onto the direct product \( K_1^+ \times K_2^+ \) of the lattices \( K_1^+ \) and \( K_2^+ \). Thus in view of Theorem 2 in [6], the relation \( G = (i)K_1 \times K_2 \) is valid.

2.4. Lemma. Let \( x, a, a_1 \) and \( b_1 \) be as above. Next, let \( x', a', a'_1 \) and \( b'_1 \) have an analogous meaning. If \( a_1 = a'_1 \), then \( b_1 = b'_1 \).

Proof. Assume that \( a_1 = a'_1 \). According to 2.2 we have
\[
a_1 + b_1 = f(a_1) = f(a'_1) = a'_1 + b'_1,
\]
hence \( b_1 = b'_1 \).

2.5. Lemma. For \( k_1 \in K_1 \) put \( f_1(k_1) = k_1(A) \). Then \( f_1 \) is an isomorphism of \( K_1 \) into \( A \).

Proof. The mapping \( f_1 \) is a homomorphism of \( K_1 \) into \( A \). Let \( k_1 \in K_1 \) such that \( f_1(k_1) = 0 \). Thus (under the notation as above) \( a_1 = 0 \) and then in view of 2.4 we have also \( b_1 = 0 \), hence \( k_1 = 0 \). Thus the kernel of \( f_1 \) is a one-element set and therefore \( f_1 \) is an isomorphism of \( K_1 \) into \( A \).

Similarly we define the mapping \( f_2 \) of \( K_2 \) into \( B \); the results are analogous.

2.6. Lemma. For each \( a \in A \) we put \( \varphi_1(a) = f_1(f(a)) \). Then \( \varphi_1 \) is a homomorphism of \( A \) onto \( f(K_1) \) and \( \varphi_1(a_1) = a_1 \) for each \( a_1 \in f(K_1) \).

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Proof. According to the definition of $K_1$, the mapping $f$ reduced to $A$ is a homomorphism of $A$ onto $K_1$. Hence in view of 2.5, $\varphi_1$ is a homomorphism of $A$ onto $f_1(K_1)$.

Let $a_1 \in f_1(K_1)$. There exists $k_1 \in K_1$ such that (under the above notation) we have $k_1 = a_1 + b_1$. Next, according to 2.2 the relation $f(a_1) = a_1 + b_1$ is valid, whence $\varphi_1(a_1) = a_1$.

2.7. Corollary. $f_1(K_1)$ is a retract of $A$.

Similarly we can define the mapping $\varphi_2$ of $K_2$ into $B$; we obtain analogous results. From 2.1, 2.3, 2.5 and 2.7 we infer:

2.8. Theorem. Let $H$ and $H'$ be lattice ordered groups, $H = (i) A \times B$. Then the following conditions are equivalent:

(i) $H'$ is isomorphic to a retract of $H$.

(ii) There are retracts $A_1$ of $A$ and $B_1$ of $B$ such that $H'$ is isomorphic to $A_1 \times B_1$.

This result can be generalized by an obvious induction for the case when $H$ is an internal direct product of a finite number of lattice ordered groups.

Again, let $G = (i) A \times B$ and let $R$ be a retract of $G$. The following example shows that $R$ need not be a direct product of a retract of $A$ and a retract of $B$.

2.9. Example. Let $H$ be the set of all pairs $(x, y)$ of integers; the operations $+, \wedge, \vee$ on $H$ are defined componentwise. Put

$$A = \{(x, y) \in H: y = 0\}, \quad B = \{(x, y) \in H: x = 0\}.$$ 

We have $H = (i) A \times B$. For each $(x, y) \in H$ let $f((x, y)) = (x, x)$. Then $f$ is a retract mapping and $f(H)$ is a retract of $H$.

Since $f(H)(A) = A$ and $A$ fails to be a subset of $f(H)$, in view of 1.1 the lattice ordered group $f(H)$ cannot be represented as an internal direct product $A_1 \times B_1$, where $A_1$ and $B_1$ are $l$-subgroups of $A$ or $B$, respectively.

3. LINEARLY ORDERED GROUPS

In this section we assume that $H$ is an abelian linearly ordered group.

3.1. Lemma. Assume that $H = (i) A \circ B$. For each $h \in H$ put $f(h) = h(A)$. Then $f$ is a retract mapping of $H$ onto $A$.

Proof. This is an immediate consequence of the definition of the lexicographic product.

Assume that $f: H \to G$ is a retract mapping and let $H_1$ be the kernel of $f$.

3.2. Lemma. Let $h_1 \in H_1$ and $0 < g \in G$. Then $h_1 < g$.

Proof. By way of contradiction, assume that there are $0 < g \in G$ and $h_1 \in H_1$ such that $g \leq h_1$. Since $H_1$ is an $l$-ideal in $H$ we obtain $g \in H_1$, which is impossible.
3.3. Lemma. Let \( h \in H, f(h) = g, h - g = h_1 \). Then the following conditions are equivalent:

(i) \( h \geq 0 \);
(ii) \( g > 0 \), or \( g = 0 \) and \( h_1 \geq 0 \).

Proof. Let (i) be valid. If \( g < 0 \), then \( h_1 \geq -g > 0 \); in view of 3.2, this is a contradiction. Thus \( g \geq 0 \). If \( g = 0 \), then clearly \( h_1 \geq 0 \). Hence (ii) holds. The implication (ii) \( \Rightarrow \) (i) is a consequence of 3.2.

An \( l \)-subgroup \( G \) of \( H \) will be called a large lexicographic factor of \( H \) if there is an \( l \)-subgroup \( G_1 \) of \( H \) such that \( H = (i) \) \( G \circ G_1 \).

From 1.3, 3.1 and 3.3 we obtain:

3.4. Theorem. Let \( H \) be an abelian linearly ordered group and let \( G \) be an \( l \)-subgroup of \( H \). Then the following conditions are equivalent:

(i) \( G \) is a retract of \( H \);
(ii) either \( G = \{0\} \) or \( G \) is a large lexicographic factor of \( H \).

The above theorem and Theorem 2.8 (generalized to a finite number of factors) yield

3.5. Theorem. Let \( H \) be a lattice ordered group, \( H = (i) A_1 \times A_2 \times \ldots \times A_n \), where \( A_1, A_2, \ldots, A_n \) are linearly ordered groups. Let \( H' \) be a lattice ordered group. Then the following conditions are equivalent:

(i) \( H' \) is isomorphic to a retract of \( H \).
(ii) There are linearly ordered groups \( B_1, B_2, \ldots, B_n \) such that \( H' \) is isomorphic to \( B_1 \times B_2 \times \ldots \times B_n \), and for each \( i \in \{1, 2, \ldots, n\} \), either \( B_i = \{0\} \) or \( B_i \) is a large lexicographic factor of \( A_i \).

4. COMPLETE RETRACT MAPPINGS

A retract mapping \( f: H \rightarrow G \) of a lattice ordered group \( H \) onto a lattice ordered group \( G \) will be called complete if \( f \) is a complete homomorphism, i.e., if, whenever \( \{h_{i,f}\}_{i \leq \lambda} \subseteq H \) and \( \bigwedge_{i \leq \lambda} h_i = h \), then \( \bigwedge_{i \leq \lambda} f(h_i) = f(h) \) (and dually).

In this section we assume that \( H \) is a complete lattice ordered group and that \( f: H \rightarrow G \) is a complete retract mapping.

It will be shown that if

\[
(1) \quad H = (i) A \times B
\]

is valid and if \( f \) is a complete retract mapping on \( H \), then there is a refinement

\[
(2) \quad H = (i) A_1 \times A_2 \times B_1 \times B_2
\]

of the internal direct product decomposition of (1) such that \( f \) can be constructed by means of certain complete retract mappings on \( A_1 \) and \( B_1 \), and complete homo-
morphisms
(3) \( \varphi_1: A_2 \to B_1, \ \varphi_2: A_2 \to A_1, \ \psi_1: B_2 \to A_1, \ \psi_2: B_2 \to B_1 \).

The following result is easy to verify.

4.1. Lemma. Let \( H \) be a lattice ordered group, \( H = (i) A_1 \times A_2 \times B_1 \times B_2 \). Assume that mappings \( f_1: A_1 \to A_1, f_2: B_1 \to B_1 \) and mappings (3) are given such that the following conditions are satisfied:

(i) \( f_1 \) and \( f_2 \) are retract mappings;
(ii) \( \varphi_1, \varphi_2, \psi_1 \) and \( \psi_2 \) are homomorphisms;
(iii) for each \( a_2 \in A_2 \) and each \( b_2 \in B_2 \) the relations \( f_2(\varphi_1(a_2)) = 0 = f_1(\varphi_2(a_2)) \) and \( f_1(\psi_1(b_2)) = 0 = f_2(\psi_2(b_2)) \) are valid.

For each \( h \in H \) put

\[ (*) \quad f(h) = f_1(h(A_1)) + \varphi_2(h(A_2)) + h(A_2) + \varphi_1(h(A_2)) + \]
\[ + f_2(h(B_1)) + \psi_2(h(B_2)) + h(B_2) + \psi_1(h(B_2)). \]

Then \( f \) is a retract mapping of \( H \). If, moreover, all the homomorphisms \( f_1, f_2, \varphi_1, \varphi_2, \psi_1 \) and \( \psi_2 \) are complete, then \( f \) is a complete retract mapping.

Below it will be proved that if (1) holds and if \( f \) is a complete retract mapping of \( H \), then there are direct decompositions

\[ A = (i) A_1 \times A_2, \quad B = (i) B_1 \times B_2 \]

and mappings \( f_1, f_2, \varphi_1, \varphi_2, \psi_1 \) and \( \psi_2 \) with the properties as in 4.1 such that all these mappings are complete homomorphisms and for each \( h \in H \) the relation (*) holds.

Thus let us suppose that (1) holds and that \( f \) is a complete retract mapping of \( H \). Denote

\[ A_1 = \{ a \in A : f(a) \in A \}. \]

From the definition of \( A_1 \) we immediately obtain:

4.2. Lemma. \( A_1 \) is a convex \( l \)-subgroup of \( A \).

For \( X \subseteq A \) we set

\[ X^0 = \{ a \in A : |a| \wedge |x| = 0 \text{ for each } x \in X \}. \]

4.3. Lemma. \( A_1^{\#} = A_1 \).

Proof. Clearly \( A_1 \subseteq A_1^{\#} \). Hence we have to verify that \( A_1^{\#} \subseteq A_1 \). By way of contradiction, suppose that \( A_1^{\#} \) fails to be a subset of \( A_1 \). Then there is \( 0 < a \in A_1^{\#} \) such that \( a \) does not belong to \( A_1 \). Hence \( f(a) \notin A \). Thus there are \( 0 \leq a' \in A \) and \( 0 < b \in B \) with \( f(a) = a' + b \). Then according to 2.2,

\[ f(a') = a' + b, \quad f(b) = 0. \]

Let \( \{ a_i : i \in I \} \) be the set of all elements \( a_i \in A_1 \) with \( 0 \leq a_i \leq a' \). Since \( H \) is a complete lattice ordered group, there exists

\[ a_1 = \bigvee_{i \in I} a_i \]

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in \( A \). Because \( f \) is a complete retract mapping, we have

\[
f(a_1) = \bigvee_i f(a_i).
\]

Since \( f(a_i) \in A \) for each \( i \in I \), we infer that \( f(a_i) \land b' = 0 \) whenever \( 0 \leq b' \leq B \); hence \( f(a_i) \land b' = 0 \) for each \( 0 < b' \leq B \). Thus \( f(a_1) \in A \) and hence \( a_1 \in A_1 \). Next,

\[
0 \leq a' - a_1. \quad \text{If there existed } 0 < a'' \leq a' - a_1 \text{ with } a'' \in A_1, \text{ then we would have } a_1 + a'' \in A_1 \text{ and }
\]

\[
a_1 < a_1 + a'' \leq a_1 + (a' - a_1) = a'.
\]

which is a contradiction. Therefore \( a' - a_1 \in A_1^d \). At the same time, \( 0 \leq a' - a_1 \leq a' \in A_1^d \); we obtain \( a' - a_1 = 0 \), whence \( a' = a_1 \in A_1 \).

From 4.3 and from the completeness of \( H \) we get

**4.4. Lemma.** \( A = (i) A_1 \times A_1^d. \)

For \( a \in A_1^d \) with \( f(a) = a_2 + b, a_2 \in A, b \in B \) put \( \varphi_1(a) = b \) and \( \varphi_2(a) = a_2(A_1) \).

**4.5. Lemma.** \( \varphi_1 \) is an isomorphism of \( A_1^d \) into \( B \).

**Proof.** Let \( a \in A_1^d \). Let \( a_2 \) and \( b \) be as above.

Since \( f \) is an endomorphism of \( H \) and since \( A_1 \) is an \( l \)-subgroup of \( H \), the mapping \( \varphi_1 \) is a homomorphism of \( A_1^d \) into \( B \). Assume that \( \varphi_1(a) = 0 \). Then \( a \in A_1 \), hence \( a = 0 \). Thus the kernel of \( \varphi_1 \) is \( \{0\} \), and therefore \( \varphi_1 \) is an isomorphism.

**4.6. Lemma.** Let \( a \in A_1^d \). Then \( f(a)(A_1^d) = a \).

**Proof.** There are \( a' \in A \) and \( b \in B \) with \( f(a) = a' + b \). According to 2.2 we have \( f(a') = a' + b \), whence \( f(a' - a) = 0 \) and thus \( a' - a \in A_1 \). Put \( a' - a = a_1 \).

Hence \( a'(A_1) = a_1 \) and \( a'(A_1^d) = a \). Next, \( a' = f(a)(A) \), thus

\[
a'(A_1) = f(a)(A)(A_1^d) = f(a)(A_1^d).
\]

We obtain \( f(a)(A_1^d) = a \).

Let us define \( B_1 \) and \( B_1^d \) analogously as we did for \( A_1 \) and \( A_1^d \). Lemmas 4.1–4.6 are valid for \( B_1 \) and \( B_1^d \) as well; instead of \( \varphi_1 \) and \( \varphi_2 \) we now apply the notation \( \psi_1 \) and \( \psi_2 \).

For \( a \in A_1 \) let us put \( f_1(a) = f(a) \). We have \( f(f(a)) = f(a) \in A \), thus \( f(a) \in A_1 \). Hence we obtain:

**4.7. Lemma.** \( f_1 \) is a retract mapping on \( A_1 \).

If \( f_2 \) has an analogous meaning with respect to \( B_1 \), then 4.7 remains valid with \( f_1 \) replaced by \( f_2 \) and \( A_1 \) by \( B_1 \), respectively.

From the definitions of the mappings under consideration we immediately obtain:

**4.8. Lemma.** All the mappings \( f_1, f_2, \varphi_1, \varphi_2, \psi_1 \) and \( \psi_2 \) are complete homomorphisms.

**4.9. Lemma.** Let \( a \in A_1 \). Then \( \varphi_1(a)(B_1^d) = 0 \).
Proof. Let \( a \in A_1^0 \). Put \( f(a)(A) = a',\; f(a)(B) = b \). Then \( \varphi_1(a) = b \). According to 2.2 we have \( f(b) = 0 \), thus \( b \in B_1 \). From \( \varphi_1(a) \in B_1 \) we obtain \( \varphi_1(a)(B_1^0) = 0 \). Analogously, \( \psi_1(b)(A_1^0) = 0 \) for each \( b \in B_1 \).

Lemmas 4.5 and 4.9 yield:

4.10. Lemma. \( \varphi_1 \) is an isomorphism of \( A_1^0 \) into \( B_1 \); similarly, \( \psi_1 \) is an isomorphism of \( B_1^0 \) into \( A_1 \).

4.11. Lemma. For each \( a \in A_1^0 \) and each \( b \in B_1^0 \) we have \( f_2(\varphi_1(a)) = 0 \) and \( f_3(\psi_1(b)) = 0 \).

Proof. In view of 4.10, \( \varphi_1(a) = f(a)(B_1) \). Thus \( \varphi_1(a) \in B_1 \). Hence \( f_2(\varphi_1(a)) = f(\varphi_1(a)) \). On the other hand, under the notation as in the proof of 4.9 we have \( f(\varphi_1(a)) = f(b) = 0 \). Therefore \( f_2(\varphi_1(a)) = 0 \). Analogously we can verify the validity of the relation \( f_3(\psi_1(b)) = 0 \).

4.12. Lemma. For each \( a \in A_1^0 \) and each \( b \in B_1^0 \) we have \( f_3(\varphi_2(a)) = 0 \) and \( f_2(\psi_2(b)) = 0 \).

Proof. Let \( a \in A_1^0 \). Put \( f(a) = a' + b, \; a' \in A, \; b \in B \). Next, let \( a_1 = a'(A_1) \). In view of 4.2, \( a'(A_1^0) = a_1 \). Denote \( a'(A_1) = a_1 \). Hence \( f(a) = a_1 + a + b \), whence \( f(a) = f(f(a)) = f(a_1) + f(a) + f(b) \). According to 2.2, \( f(b) = 0 \). Therefore \( 0 = f(a_1) = f_1(a_1) = f_1(\varphi_2(a)) \). Analogously we can verify that \( f_2(\psi_2(b)) = 0 \).

The above results can be summarized as follows.

4.13. Theorem. Let \( f: H \to G \) be a complete retract mapping of a complete lattice ordered group \( H \). Let \( H = (i) A \times B \). Let us apply the notation introduced above and put \( A_1^0 = A_2, \; B_1^0 = B_2 \). Then the conditions (i), (ii) and (iii) from 4.1 are valid. For each \( h \in H \), the relation (s) from 4.1 holds.

References


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