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$B(X)$ is generated in strong operator topology by two of its elements
B(X) IS GENERATED IN STRONG OPERATOR TOPOLOGY
BY TWO OF ITS ELEMENTS

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Let X be a real or complex Banach space and τ a topology on B(X). We say that
B(X) is τ-generated by its subset G if it coincides with the smallest τ-closed sub-
algebra of B(X) containing G. In particular we say that B(X) is strongly generated
by G if τ is the strong operator topology. Similarly we say that B(X) is generated
by G if τ is the norm topology and B(X) is algebraically generated by G if τ is the
discrete topology.

As a motivation for the result presented here we mention a similar theorem known
for X being a separable Hilbert space (see [1], [2], cf also [3] and the references
therein) and the results given in [6]–[8]. In [6] it is shown that for any Hilbert
space H there are two subalgebras A₁, A₂ \subset B(H) with square zero and therefore
commutative such that the set G = A₁ ∪ A₂ algebraically generates B(H). In [7]
this result is extended to those Banach spaces X which are \(n\)-th powers”, \(n > 1,\)
i.e. which can be decomposed into direct sums \(X = X_1 \oplus \ldots \oplus X_n\) of closed sub-
spaces \(X_1\) which are mutually isomorphic. In [8] it is shown that for every Banach
space X the algebra B(X) is strongly generated by two commutative subalgebras
(in fact by two subalgebras of square zero) if \(\dim X > 1\). Our present result improves
the last one for separable Banach space so that instead of commutativity we have
a single generation. Our proof is based upon the following result due to Ovsepian
and Pelczyński ([4], Theorem 1) on the existence of total bounded biorthogonal
systems in separable Banach spaces:

**Theorem [O–P].** Let \(x\) be a separable Banach space. Then there is a sequence
\((x_i)\) of elements in \(X\) and a sequence \((f_i)\) of functionals in \(X\) such that
(1) \(f_m(x_n) = \delta_{m,n}\) (the Kronecker symbol) for \(m, n = 1, 2, \ldots\).
(2) The linear span of \((x_i)\) is dense in \(X\) in the norm topology.
(3) If \(f_n(x) = 0\) for all \(n\) then \(x = 0\).
(4) \(\sup_n \|x_n\| / \|f_n\| = M < \infty\).

By [5] we can assume \(M < 1 + \varepsilon\) for a given positive \(\varepsilon\). In what follows we shall
assume that the sequences \((x_i)\) and \((f_i)\) satisfying (1)–(4) are normalized so that they
satisfy
\[(5) \quad \|x_i\| = 1 \quad \text{and} \quad \|f_i\| \leq M \quad \text{for all} \quad i.
\]

We can now formulate our result.

**Theorem.** Let $X$ be a real or complex Banach space. There exist operators $R$ and $S$ in $B(X)$ such that $B(X)$ is strongly generated by $\{R, S\}$.

**Proof.** For operators $R$ and $S$ in $B(X)$ denote by $\text{Alg}(R, S)$ the subalgebra of $B(X)$ algebraically generated by $R$ and $S$. It consists of linear combinations of the products of powers of $R$ and $S$. We have to construct two operators $R$ and $S$ so that $\text{Alg}(R, S)$ is strongly dense in $B(X)$. This means that for a given $T$ in $B(X)$, $y_1, \ldots, y_n$ linearly independent elements in $X$ and a positive $\varepsilon$, there is an operator $B$ in $\text{Alg}(R, S)$ such that
\[(6) \quad \|By_i - Ty_i\| < \varepsilon \quad \text{for} \quad 1 \leq i \leq n.
\]

In order to have (6) it is sufficient to find operators $B_i$ in $\text{Alg}(R, S)$, $1 \leq i \leq n$, such that $\|B_iy_i - Ty_i\| < \varepsilon$ and $B_jy_j = 0$ for $j \neq i$, because then (6) is satisfied by $B = \sum_{i=1}^n B_i$. Thus for proving strong density of $\text{Alg}(R, S)$ in $B(X)$ we have to show that for given linearly independent $y_0, y_1, \ldots, y_n$ in $X$, positive $\varepsilon$ and an element $z$ in $X$ there is an operator $B$ in $\text{Alg}(R, S)$ such that
\[(7) \quad \|By_0 - z\| < \varepsilon \quad \text{and} \quad By_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

Let $(x_i)$ and $(f_i)$ be a sequence of elements and functionals satisfying conditions (1)–(5) and put
\[(8) \quad R = \sum_{i=1}^{\infty} 2^{-i}(f_i \otimes x_{i+1}) \quad \text{and} \quad S = \sum_{i=1}^{\infty} 2^{-i}(f_{i+1} \otimes x_i),
\]

where $f \otimes x$ is the one-dimensional operator on $X$ given by $u \mapsto f(u)x$. By (5) we have $\|R\|, \|S\| \leq M$ and so $R, S \in B(X)$.

First we show that all operators $f_m \otimes x_n$, $1 \leq m, n < \infty$, are in $\text{Alg}(R, S)$. To this end we prove the formula
\[(9) \quad f_m \otimes x_n = 2^p R^{n-1} (4SR - RS) S^{m-1}, \quad p = \left(\begin{array}{c} m \\ 2 \end{array}\right) + \left(\begin{array}{c} n \\ 2 \end{array}\right),
\]

where $\left(\begin{array}{c} k \\ 2 \end{array}\right)$ is 0 if $k = 1$. Using (1) we obtain immediately from (8)
\[(10) \quad Rx_n = 2^{-n} x_{n+1} \quad \text{and} \quad Sx_n = 2^{-n+1} x_{n-1}, \quad n = 1, 2, \ldots,
\]

where $x_0 = 0$. This implies
\[(11) \quad (4SR - RS) x_1 = x_1 \quad \text{and} \quad (4SR - RS) x_m = 0 \quad \text{for} \quad m \neq 1.
\]

Denote by $A_{m,n}$ the right-hand side operator in formula (9). We see by (10) and (11)
that $A_{m,n}x_k = 0$ for $k \neq m$ and

$$A_{m,n}x_m = 2^p R^{n-1}(4SR - RS) \times 2^{-m+1} \ldots 2^{-1}x_k = 2^q R^{n-1}x_1 = x_n,$$

$$p = \binom{m}{2} + \binom{n}{2}, \quad q = \binom{n}{2}.$$ 

We infer that $A_{m,n} = f_m \otimes x_n$ since both operators agree on linearly dense sequence $(x_i)$ and formula (9) holds true.

Let $y_0, y_1, \ldots, y_n$ be linearly independent elements in $X$, let $z$ be in $X$ and let $\varepsilon$ be a positive number. We shall be done if we find an operator $B$ in $\text{Alg}(R, S)$ so that relations (7) are satisfied. To this end observe that the sequences \( \{f_i(y_0)\} \) \( i=1, \ldots, n \) are linearly independent elements in $l^\infty$, otherwise there would exist non-zero coefficients $\lambda_0, \ldots, \lambda_n$ such that

$$0 = \sum_{j=0}^{n} \lambda_j f_i(y_j) = f_i(\sum_{j=0}^{n} \lambda_j y_j)$$

for all $i$ and by (3) we would have $\sum_{j=0}^{n} \lambda_j y_j = 0$ which is a contradiction. The linear independence of these sequences implies that there are indices $i_0, i_1, \ldots, i_n$ such that the finite sequences

$$\{f_{i_0}(y_0)\}^{k=0}, \{f_{i_1}(y_1)\}^{k=0}, \ldots, \{f_{i_n}(y_n)\}^{k=0}$$

are linearly independent. This means that there are coefficients $c_0, c_1, \ldots, c_n$ such that

$$\sum_{k=0}^{n} c_k f_{i_k}(y_0) = 1 \quad \text{and} \quad \sum_{k=0}^{n} c_k f_{i_k}(y_j) = 0, \quad 1 \leq j \leq n.$$ 

Define now $A = \sum_{k=0}^{n} c_k (f_{i_k} \otimes x_1)$.

By (9) we have $A \in \text{Alg}(R, S)$ and by (12) we conclude

$$Ay_0 = \sum_{k=0}^{n} c_k f_{i_k}(y_0) x_1 = x_1 \quad \text{and} \quad Ay_i = \sum_{k=0}^{n} c_k f_{i_k}(y_i) x_1 = 0,$$

$$1 \leq i \leq n.$$ 

For a given positive $\varepsilon$ and $z$ in $X$ we find by (2) a finite linear combination

$$z_0 = \sum_{j=1}^{m} b_j x_j$$

such that

$$\|z - z_0\| < \varepsilon.$$ 

Define $B = \sum_{j=1}^{m} b_j (f_1 \otimes x_j) A$. This is again an operator in $\text{Alg}(R, S)$. By (13) and (14) we have

$$By_0 = \sum_{j=1}^{m} b_j f_1 (Ay_0) x_j = \sum_{j=1}^{m} b_j x_j = z_0.$$ 

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and

$$By_i = \sum_{j=1}^{m} b_j f_i(Ay_j) x_j = 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$  

Thus by (15) we obtain (7) and the conclusion follows.

Remarks. 1° Without assuming that $X$ is a separable space our conclusion fails to be true. This follows from the fact that for any operators $R, S$ in $B(X)$ and for any element $x$ in $X$ the orbit $0(x) = \{Tx: T \in \text{Alg}(R, S)\}$ is separable. Thus for non-separable $X$ there is an element $z$ in $X$ such that $\text{dist}(z, 0(x)) = 1$, and there is no operator $T$ in $\text{Alg}(R, S)$ satisfying $\|Tx - (f \otimes z)x\| < 1$ if $f(x) = 1$.

2° It is still an open problem whether $B(X)$ can be separable for an infinite-dimensional Banach space (for this information the authors are indebted to Tadeusz Figiel), and for most familiar Banach spaces the algebra $B(X)$ is known to be non-separable. Thus for many cases (perhaps for all cases) our theorem cannot be improved by replacing strong generation by generation. In particular the algebra $B(H)$ cannot be generated by two operators if $H$ is an infinite-dimensional Hilbert space.

3° In our theorem we can replace strong generation by weak generation (generation in the weak operator topology). This follows from the fact that for linear subspaces of $B(X)$ their closures in strong and in weak operator topologies coincide.

References

[8] W. Żelazko: $B(X)$ is generated in strong operator topology by two subalgebras with square zero, submitted.

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