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ON $\alpha$-IDEALS AND GENERALIZED $\alpha$-IDEALS IN SEMIGROUPS

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Let $S$ be a semigroup. A non-empty subset $A$ of $S$ is called a generalized $(m, n)$-ideal of $S$ if the inclusion

$$A^m S A^n \subseteq A$$

holds, where $m, n$ are arbitrary non-negative integers. Here, as usual, $A^0 S A^n = S A^n$, $A^m S A^0 = A^m S$ and $A^0 S A^0 = S$ (see [4]). A generalized $(m, n)$-ideal $A$ of $S$ is said to be an $(m, n)$-ideal of $S$ if $A$ is a subsemigroup of $S$. It is easy to see that one-side (left or right) ideals are particular cases of $(m, n)$-ideals. S. Lajos, in [3], [5], [6] and [7], characterized certain classes of semigroups through the generalized (1, 1)-ideals.

In this paper we shall generalize some results on $(m, n)$-ideals and generalized $(m, n)$-ideals in semigroups. In section 1 we introduce the $\alpha$-ideals and the generalized $\alpha$-ideals in semigroups, where $\alpha$ is a finite sequence of zeros and units containing at least one zero.

In Section 2 we characterize the semigroups for which any generalized $\alpha$-ideal is an $\alpha$-ideal. Moreover, we prove among other things that if every generalized $(3, 3)$-ideal is a $(3, 3)$-ideal then every generalized $\alpha$-ideal is an $\alpha$-ideal.

In Section 3 we investigate the semigroup by all generalized $\alpha$-ideals of a semigroup $S$. In particular, we prove that there is an isomorphism between the semigroup which consists of all $101$-ideals of a regular semigroup $S$ and the semigroup which consists of all $101$-ideals of the semigroup $S/\mu$, where $\mu$ is the maximal idempotent separating congruence on $S$. Moreover, we answer a question by S. Lajos on the semigroup which consists of all left ideals of a semigroup $S$.

The reader is referred to [2] for basic notions and terminology of algebraic semigroups theory.

1.

By $X^*$ we denote the free monoid over an alphabet $X$. Let $S$ be a semigroup. By $\mathcal{P}(S)$ we denote the semigroup of all subsets of $S$ under set product with the unity $\emptyset$. For $\alpha \in \{0, 1\}^*$ we shall define $f^S_\alpha: \mathcal{P}(S) \to \mathcal{P}(S)$ as follows: $f^S_\emptyset(A) = \emptyset$ if $\alpha$ is the empty word and

$$f^S_\alpha(A) = A_1 A_2 \ldots A_n$$

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if \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_n, \alpha_i \in \{0, 1\} \), where
\[
A_i = \begin{cases} 
A & \text{for } \alpha_i = 1, \\
S & \text{for } \alpha_i = 0.
\end{cases}
\]

1.1. Lemma. \( f_2^S(A) \subseteq f_2^S(B) \), whenever \( A \subseteq B \subseteq S \).

1.2. Lemma. \( f_2^S(A) \cap f_2^S(A) = f_2^S(A) f_2^S(A) \) for \( A \subseteq S \).

Let us put \( A = \{0, 1\}^* \setminus \{1\}^* \).

1.3. Lemma. If \( A \in \mathcal{P}(S) \) and \( \alpha \in A \), then \( A f_2^S(A) \subseteq f_2^S(A) \).

Proof. We can suppose that \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_n, \alpha_i \in \{0, 1\} \), and \( \alpha_j = 0 \) for some \( j \) with \( \alpha_i = 1 \) for \( i < j \). If \( j = 1 \), then \( A f_2^S(A) = A S A_2 \ldots A_n \subseteq S A_2 \ldots A_n = f_2^S(A) \).

If \( j > 1 \), then \( A f_2^S(A) = A^j S A_{j+1} \ldots A_n \subseteq A^{j-1} S A_{j+1} \ldots A_n = f_2^S(A) \).

1.4. Lemma. If \( A \in \mathcal{P}(S) \) and \( \alpha \in A \), then \( f_2^S(A) f_2^S(A) \subseteq f_2^S(A) \).

Proof. We have \( f_2^S(A) = A_1 A_2 \ldots A_n \) and \( A_j = S \) for some \( j \). Then \( f_2^S(A) f_2^S(A) = A_1 A_2 \ldots A_{j-1} (A_j \ldots A_n A_1 \ldots A_j) A_{j+1} \ldots A_n \subseteq A_1 A_2 \ldots A_{j-1} S A_{j+1} \ldots A_n =\)

\( = f_2^S(A) \).

1.5. Lemma. If \( A \in \mathcal{P}(S) \) and \( \alpha \in A \), then \( f_2^S(A \cup f_2^S(A)) \subseteq f_2^S(A) \).

Proof. Let \( A \in \mathcal{P}(S) \). First we note that for every positive integer \( n \) we have
\[
(1) \quad f_2^S(A \cup f_2^S(A)) = S^n = f_2^S(A),
\]
where \( \alpha = 0^n \).

Let \( \alpha \in A \). We prove our statement by induction on the length \( n \) of \( \alpha \). It follows from (1) that the result is true for \( n = 1 \). Assume now that \( n \geq 2 \) and the result holds for \( n - 1 \).

Case 1: \( \alpha = 1\beta \). Then \( \beta \in A \) and \( f_2^S(A \cup f_2^S(A)) \subseteq f_2^S(A) \). Using Lemmas 1.1, 1.2, 1.3 and 1.4 we obtain
\[
\begin{align*}
f_2^S(A \cup f_2^S(A)) &= (A \cup A f_2^S(A)) f_2^S(A \cup f_2^S(A)) \subseteq \\
&= (A \cup A f_2^S(A)) f_2^S(A \cup f_2^S(A)) \subseteq (A \cup A f_2^S(A)) f_2^S(A) \subseteq \\
&= A f_2^S(A) \cup A f_2^S(A) f_2^S(A) = A f_2^S(A) = f_2^S(A).
\end{align*}
\]

Case 2: \( \alpha = \beta \). This is dual to Case 1.

Case 3: \( \alpha = 0\beta 0 \), where \( \beta \in \{0, 1\}^* \). According to (1) we can suppose that \( \beta \notin \{0\}^* \).
Then \( n \geq 3, \beta = \alpha_2 \ldots \alpha_n \) and \( I \neq 0 \), where \( i \in I \) if and only if \( i \in \{2, \ldots, n-1\} \)
and \( \alpha_i = 1 \). Therefore \( f_2^S(A \cup f_2^S(A)) = S A_2 \ldots A_{n-1} S \), where \( A_i = A \cup f_2^S(A) \) for \( i \in I \) and \( A_i = S \) for \( i \in \bar{I} = \{2, \ldots, n-1\} \setminus I \).

By Z we denote the set of all words of \( \{0, 1\}^* \) having the length \( n - 2 \). Let us put
\[
\begin{align*}
B_{i_0} &= A, B_{i_1} = f_2^S(A) \text{ if } i \in I, \quad B_{i_0} = S = B_{i_1} \text{ if } i \in \bar{I} \quad \text{and} \quad B_i = S B_{2 \gamma} \ldots B_{n-1, \gamma_{n-1}} S \\
&\quad \text{if } \gamma = \gamma_2 \ldots \gamma_{n-1} \in Z. \quad \text{It is easy to show that } B_i = f_2^S(A) \quad \text{if } \gamma = 0^{n-2} \text{ and } B_i \subseteq \\
&\quad S f_2^S(A) \quad \text{if } \gamma = 0^{n-2}. \quad \text{Hence we have } \quad f_2^S(A \cup f_2^S(A)) = \bigcup_{\gamma \in Z} B_i \subseteq f_2^S(A).
\end{align*}
\]

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1.6. Definition. Let \( \alpha \in \Lambda \). A non-empty subset \( M \) of a semigroup \( S \) is called a generalized \( \alpha \)-ideal of \( S \) if \( f^S_\alpha(M) \subseteq M \). A generalized \( \alpha \)-ideal \( M \) of \( S \) is said to be an \( \alpha \)-ideal of \( S \) if \( M \) is a subsemigroup of \( S \).

1.7. Theorem. Let \( A \) be a non-empty subset of a semigroup \( S \). Then \( A \cup f^S_\alpha(A) \) is a generalized \( \alpha \)-ideal of \( S \) for every \( \alpha \in \Lambda \).

The proof follows from Lemma 1.5.

2.

S. Lajos, in [3], gave an example of a semigroup for which certain generalized \((m, n)\)-ideal are not \((m, n)\)-ideals. F. Catino, in [1], characterized the semigroups for which any generalized \((1, 1)\)-ideal is a \((1, 1)\)-ideal.

2.1. Theorem. Let \( S \) be a semigroup and \( \alpha \in \Lambda \). Then every generalized \( \alpha \)-ideal of \( S \) is an \( \alpha \)-ideal of \( S \) if and only if \( ab \in f^S_\alpha(\{a, b\}) \) for all \( a, b \in S \).

Proof. Suppose that \( M^2 \subseteq M \) for every generalized \( \alpha \)-ideal \( M \) of \( S \). Let \( a, b \in S \) and put \( A = \{a, b\} \) and \( M = A \cup f^S_\alpha(A) \). According to Theorem 1.7, \( M \) is a generalized \( \alpha \)-ideal of \( S \) and so \( ab \in M^2 \subseteq M \). If \( ab \in A \), then \( ab = a \) or \( ab = b \). In both cases we have \( ab \in f^S_\alpha(A) \).

Assume that \( ab \in f^S_\alpha(\{a, b\}) \) for all \( a, b \in S \). Let \( M \) be a generalized \( \alpha \)-ideal of \( S \). If \( x \in M^2 \), then \( x = ab \), where \( a, b \in M \) and so, by Lemma 1.1, we have \( ab \in f^S_\alpha(\{a, b\}) \subseteq f^S_\alpha(M) \subseteq M \). Therefore \( M^2 \subseteq M \) and the proof is complete.

Let us put \( W(a, b) = \{a^2, b^2, ba^2, ab^2, aba\} \). Recall that an element \( a \) of a semigroup \( S \) is said to be left regular if \( a \in a^2S \). Dually, a right regular element of \( S \).

2.2. Theorem. Let \( S \) be a semigroup and \( \beta \in \{0, 1\}^* \). Then the following statements are equivalent:

1. For any \( \alpha \in \{0, 1\}^* \), every generalized \( \alpha 0 \beta \)-ideal of \( S \) is an \( \alpha \beta \)-ideal of \( S \).
2. Every generalized \( \alpha 0 \beta \)-ideal of \( S \) is a \( \alpha 0 \beta \)-ideal of \( S \).
3. For all \( a, b \in S \) we have \( ab \in W(a, b) \ S f^S_\beta(\{a, b\}) \) and moreover \( a^2 \) is a left regular element of \( S \).

Proof. 1 \( \Rightarrow \) 2. It is clear.

2 \( \Rightarrow \) 3. Let \( a, b \in S \). Put \( A = \{a, b\} \). By Theorem 2.1 and Lemma 1.2 we have \( ab \in f^S_{\alpha \beta}(A) = A^3 f^S_\beta(A) \). This implies \( ab \in W(a, b) S f^S_\beta(A) \) or \( ab \in bab S f^S_\beta(A) \). In the second case we obtain \( ab \in b^2 a b S f^S_\beta(A) \subseteq b^2 S f^S_\beta(A) \subseteq W(a, b) S f^S_\beta(A) \).

Moreover, \( a = b \) or we have \( a^2 \in a^3S \) and so \( a^2 \in a^4S \).

3 \( \Rightarrow \) 1. Let \( a, b \in S \). Then \( ab \in W(a, b) S f^S_\beta(A) \), where \( A = \{a, b\} \), and \( a^2 \in a^mS, b^2 \in b^mS \) for all integers \( m \geq 2 \). It is easy to show that \( ab \in A^mS f^S_\beta(A) \) for all integers \( m \geq 2 \), whenever \( ab \in \{a^2, b^2, ba^2, ab^2\} S f^S_\beta(A) \). Suppose that \( ab \in aba S f^S_\beta(A) \). We shall distinguish two cases.

Case 1: \( ba \in \{b^2, a^2, ab^2, ba^2\} S f^S_\beta(A) \). Then \( ba \in A^mS f^S_\beta(A) \) for all integers \( m \geq 2 \) and so \( ab \in A^mS f^S_\beta(A) \) for all integers \( m \geq 2 \).
Case 2: $ba \in babSf_0^S(A)$. Then $ab \in (ab)^2 Sf_0^S(A)$ and so $ab \in (ab)^m Sf_0^S(A) \subseteq A^n Sf_0^S(A)$ for all integers $m \geq 2$.

Therefore we have $ab \in A\alpha Sf_0^S(A)$ for all integers $m \geq 2$ and so $ab \in f_0^S(A)$. $Sf_0^S(A) = f_{\alpha\beta}(A)$ for all $\alpha \in \{0, 1\}^\ast$. It follows from Theorem 2.1 that every generalized $\alpha\beta$-ideal of $S$ is an $\alpha\beta$-ideal of $S$ and the proof is complete.

We recall that an element $a$ of a semigroup $S$ is called completely regular if there exists an element $x$ of $S$ such that $a = axa$, $ax = xa$. It is well known that an element of $S$ is completely regular if it is left regular and right regular.

Using the same method of proof as in Theorem 2.2, we obtain:

2.3. Theorem. Let $S$ be a semigroup. Then the following statements are equivalent:

1. For all $\alpha, \beta \in \{0, 1\}^\ast$, every generalized $\alpha\beta$-ideal of $S$ is an $\alpha\beta$-ideal of $S$.
2. Every generalized $1^301^3$-ideal of $S$ is a $1^301^3$-ideal of $S$.
3. For all $a, b \in S$ we have $ab \in W(a, b) SW(b, a)$ and moreover $a^2$ is a completely regular element of $S$.

3.

Let $\varrho$ be a congruence on a semigroup $S$. Put $T = S/\varrho$ and define $\psi: \mathcal{P}(T) \to \mathcal{P}(S)$ as follows:

$$\psi(M) = \bigcup_{z \in M} z$$

for any $M \subseteq T$.

3.1. Lemma. Let $P, Q \in \mathcal{P}(T)$. Then $\psi(P) \subseteq \psi(Q)$ if and only if $P \subseteq Q$.

3.2. Lemma. Let $P, Q \in \mathcal{P}(T)$. Then $\psi(P) \psi(Q) \subseteq \psi(PQ)$.

3.3. Lemma. Let $M \in \mathcal{P}(T)$ and $P_i \in \mathcal{P}(T)$ for $i = 1, 2, \ldots, n$. Then $\psi(P_1) \psi(P_2) \ldots \psi(P_n) \subseteq \psi(M)$ if and only if $P_1P_2 \ldots P_n \subseteq M$.

Proof. Suppose that $\psi(P_1) \psi(P_2) \ldots \psi(P_n) \subseteq \psi(M)$. Then for every $y \in M$ there exist $i_1, i_2, \ldots, i_n$ such that $z_{i_1}z_{i_2} \ldots z_{i_n} \in \psi(M)$. Conversely, assume now that $P_1P_2 \ldots P_n \subseteq M$. Using Lemma 3.1 and Lemma 3.2 we obtain $\psi(P_1) \psi(P_2) \ldots \psi(P_n) \subseteq \psi(P_1P_2 \ldots P_n) \subseteq \psi(M)$.

3.4. Lemma. Let $M \in \mathcal{P}(T)$. Then $f_\alpha^S(\psi(M)) \subseteq \psi(M)$ if and only if $f_\alpha^T(M) \subseteq M$.

3.5. Lemma. Let $M$ be a non-empty subset of $T$. Then $M$ is an $\alpha$-ideal [a generalized $\alpha$-ideal] of $T$ if and only if $\psi(M)$ is an $\alpha$-ideal [a generalized $\alpha$-ideal] of $S$.

3.6. Lemma. Let $P_i \in \mathcal{P}(T)$ for $i = 1, 2, \ldots, n$ and $\psi(P_1) \psi(P_2) \ldots \psi(P_n) \subseteq \psi(T)$. Then $\psi(P_1P_2 \ldots P_n) = \psi(P_1) \psi(P_2) \ldots \psi(P_n)$.

Proof. Suppose that $\psi(P_1) \psi(P_2) \ldots \psi(P_n) = \psi(M)$ for some $M \in \mathcal{P}(T)$. It follows from Lemma 3.3 that $P_1P_2 \ldots P_n \subseteq M$. According to Theorem 2.2 we have $\psi(M) \subseteq
Lemma 3.1 implies $M \subseteq P_1P_2 \ldots P_n$ and this completes the proof.

Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(T)$. By $[\mathcal{A}]$, $[\psi(\mathcal{A})]$, respectively, we denote the subsemigroup of $\mathcal{P}(T)$ generated by $\mathcal{A}$, the subsemigroup of $\mathcal{P}(S)$ generated by $\psi(\mathcal{A})$.

3.7. Lemma. Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(T)$ such that $[\psi(\mathcal{A})] \subseteq \psi(\mathcal{P}(T))$. Then $\psi([\mathcal{A}])$ is an isomorphism of $[\mathcal{A}]$ onto $[\psi(\mathcal{A})]$.

Proof. Assume that $M \in [\mathcal{A}]$, then $M = P_1P_2 \ldots P_n$, where $P_i \in \mathcal{A}$ $(i = 1, 2, \ldots, n)$. Hence we have $\psi(P_1) \psi(P_2) \ldots \psi(P_n) \in [\psi(\mathcal{A})] \subseteq \psi(\mathcal{P}(T))$. According to Lemma 3.6, we obtain $\psi(M) \in [\psi(\mathcal{A})]$. Thus $\psi([\mathcal{A}]) \subseteq [\psi(\mathcal{A})]$.

Let $A \in [\psi(\mathcal{A})]$. Then $A = \psi(P_1) \psi(P_2) \ldots \psi(P_n) \in \psi(\mathcal{P}(T))$, where $P_i \in \mathcal{A}$ $(i = 1, 2, \ldots, n)$. It follows from Lemma 3.6 that $A = \psi(P_1P_2 \ldots P_n) \in [\psi(\mathcal{A})]$. Therefore $\psi([\mathcal{A}]) = [\psi(\mathcal{A})]$. By Lemma 3.6 and Lemma 3.1 we obtain that $\psi([\mathcal{A}])$ is an isomorphism of $[\mathcal{A}]$ onto $[\psi(\mathcal{A})]$. The proof is complete.

Let $\alpha \in A$. By $\mathcal{F}_S[\mathcal{F}_S]$ we denote the subsemigroup of $\mathcal{P}(S)$ generated by all $\alpha$-ideals $[\alpha]$-ideals of $S$. An equivalence relation $\sigma(\mathcal{F}_s) [\sigma(\mathcal{F}_S)]$ on $S$ is defined by the rule that

$$(a, b) \in \sigma(\mathcal{F}_S) \iff \forall H \in \mathcal{F}_S: \quad a \in H \Leftrightarrow b \in H$$

$$(a, b) \in \sigma(\mathcal{F}_S) \iff \forall H \in \mathcal{F}_S: \quad a \in H \Leftrightarrow b \in H$$

3.8. Theorem. Let $S$ be a semigroup and $\alpha \in A$. If $\varrho$ is a congruence on $S$ such that $\varrho \subseteq \sigma(\mathcal{F}_S) [\sigma(\mathcal{F}_S)]$ then the semigroups $\mathcal{F}_S$ and $\mathcal{F}_S[\mathcal{F}_S]$ and $\mathcal{F}_S[\mathcal{F}_S]$ are isomorphic.

Proof. It is easy to show that according to Lemma 3.5, $\varrho \subseteq \sigma(\mathcal{F}_S)$ implies $\mathcal{F}_S \subseteq \psi(\mathcal{P}(T))$, where $T = S/\varrho$. By $\mathcal{A}$ we denote the set of all $\alpha$-ideals of $T$. Lemma 3.5 implies that $\psi(\mathcal{A})$ is the set of all $\alpha$-ideals of $S$ and so $\mathcal{F}_S = [\psi(\mathcal{A})]$ and $\mathcal{F}_S = [\mathcal{A}]$. Analogously for generalized $\alpha$-ideals. The rest of the proof follows from Lemma 3.7.

If $S$ is a regular semigroup and $\alpha = 101$, by Proposition 4.1 of [11], $\sigma(\mathcal{F}_{101}) = \mathcal{H}$.

3.9. Corollary. Let $S$ be a regular semigroup and let $\varrho$ be a congruence relation on $S$ such that $\varrho \subseteq \mathcal{H}$. Then the semigroups $\mathcal{F}_{101}$ and $\mathcal{F}_{101}$ are isomorphic.

Theorem 2 in [9] is a consequence of the last corollary.

This Corollary gives more information on the semigroup $\mathcal{F}_{101}$: For instance, if $S$ is a $\omega$-regular bisimple semigroup it follows from Corollary 4 [10] that $\mathcal{F}_{101}$ is a rectangular band. Now, since $\mathcal{H}$ is a congruence relation and since $S/\mathcal{H}$ is isomorphic to the bicyclic semigroup $\mathcal{C}(p, q)$ (this latter has already been studied in [8]). Moreover from the description of $\mathcal{F}_{101}$ given in [8] one easily derives a description of $\mathcal{F}_{101}$.

The analogue of Corollary 3.9 does not hold if $\mathcal{H}$ is replaced by Green’s relations $\mathcal{L}$ and $\mathcal{R}$. Indeed, if $S$ is a left zero semigroup and $|S| > 1$, then $\mathcal{L} = S \times S$, $|\mathcal{F}_{101}| = 1$ and $|\mathcal{F}_{101}| = 1$, hence $\mathcal{F}_{101}$ and $\mathcal{F}_{101}$ are not isomorphic.

The following, however, is true, solving problem by S. Lajos.

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3.10. Corollary. Let $S$ be a semigroup. If $\varrho$ is a congruence contained in $\mathcal{L}$, then $\mathcal{F}_0^\varrho$ and $\mathcal{F}_0^{\mathcal{L}/\varrho}$ are isomorphic.

References


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