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CHARACTERIZATIONS OF DUAL SEMIGROUPS

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The study of dual semigroups was firstly initiated by Št. Schwarz in 1960 [3]. The structure of dual semigroups with zero radicals was investigated by him in 1971 [4]. He proved that a dual semigroup is a 0-direct union of completely 0-simple subsemigroups if and only if the intersection of all maximal ideals of S is zero. In 1977, O. Steinfeld [7] noticed that the following conditions are equivalent:

(i) S is a dual semigroup.

(ii) S is a regular semigroup in which the product of any two different idempotents of S is zero.

(iii) S is an inverse semigroup in which every non-zero idempotent is primitive.

At the first glimpse, these two results seem to be independent and unrelated. It is the purpose of this article to unify these results into one. As a consequence, a list of results obtained by G. B. Preston, P. S. Venkatesen, L. Márki and O. Steinfeld, (see [8], p. 89) in the literature concerning inverse semigroups and regular semigroups are linked together with dual semigroups. Characterization theorems for dual semigroups with zero radicals, 0-simple dual semigroups and reduced dual semigroups are also obtained. We refer to Št. Schwarz [3] and O. Steinfeld [8] for all terminology and definitions not given here. Throughout the paper, S is a semigroup with zero.

1. PRELIMINARIES

Let X be a subset of a semigroup S. Let $r(X) = \{z \in S \mid xa = 0 \text{ for all } x \in X\}$ and let $\ell(X) = \{b \in S \mid bx \in 0 \text{ for all } x \in X\}$. A semigroup S is called dual if $r\ell(R) = R$ and $\ell r(L) = L$ hold simultaneously for every right nad left ideal R and L of S respectively.

The radical $W(S)$ of a semigroup S is defined to be the set-theoretic union of all nilpotent right ideals of S. As in ring theory, $W(S)$ coincides with the set-theoretic union of all nilpotent left ideals of S or the set-theoretic union of all nilpotent two-sided ideals of S. Thus a semigroup with zero radical means that the semigroup contains no non-trivial nilpotent right (left, two-sided) ideals.

The Frattini radical M^* of S is the intersection of all maximal ideals of S. The prime radical P^* of S is the intersection of all prime ideals of S.

For the sake of convenience, we summarize here some results of Schwarz which will be frequently used and quoted.

Theorem 1.1 (Schwarz [3] and [4]).
Let S be a dual semigroup. Then S has the following properties:

(i) $E \neq \emptyset$ *and* $M^* \cap E = \emptyset$, *where E is the set of all idempotents of S.*

(ii) $S^2 = S$ *and* $ef = 0$ *for distinct idempotents e and f of S.*

(iii) *Every non-zero right (left, two-sided) ideal of S contains a 0-minimal right (left, two-sided) ideal of S.*

(iv) *Let I be a two-sided ideal of S which has non-zero intersection with the radical of S. Then every left (right, two-sided) ideal of I is also a left (right, two-sided) ideal of S.*

(v) **(First decomposition theorem).** $S = \bigcup_{e \in E} eS$ $(= \bigcup_{e \in E} Se)$, *with* $e_iS \cap e_jS = \{0\}$. $(Se_i \cap Se_j = \{0\})$ *for every* $e_i \neq e_j \in E$. *Moreover, each of the summand contains an unique idempotent and an unique 0-minimal right (left) ideal of S.*

(vi) **(Second decomposition theorem).** *If* $W(S) = \{0\}$, *then* $S = \bigcup_{i \in \Lambda} M_i$, *where M_i's are 0-simple dual subsemigroups (ideals) of S and* $M_iM_j = M_i \cap M_j = \{0\}$ *for all* $i \neq j \in \Lambda$.

Theorem 1.2. *Let S be a 0-simple dual semigroup. Then $\{eS \mid e \in E\}$ is the set of all 0-minimal right ideals of S. Likewise, $\{Se \mid e \in E\}$ is the set of all 0-minimal left ideals of S.*

Proof. It is known that if I is a 0-minimal two-sided ideal of a semigroup S containing a 0-minimal right (left) ideal of S then I is the union of all 0-minimal right (left) ideals of S contained in I (see Clifford-Preston [Theorem 2.33 [1]). Thus, by theorem 1.1 (iv), $S = \bigcup_{i \in \Lambda} R_i (= \bigcup_{j \in \Gamma} L_j)$, where R_i's (L_i's) are 0-minimal right (left) ideals of S. By theorem 1.1 (v), $S = \bigcup_{e \in E} eS = \bigcup_{e \in E} R_i (*)$. And by theorem 1.1 (iii), $R_i \subset eS$ for some $i \in \Lambda$ and $e \in E$. If $R_i \subsetneqq eS$, then there exists $j \neq i$ such that $R_j \cap eS \neq \{0\}$ by $(*)$. Because $R_j \cap eS$ is a non-zero right ideal of S contained in R_j, therefore $R_j = R_j \cap eS \subset eS$ since R_j is 0-minimal. This implies that eS contains two distinct 0-minimal right ideals, namely, R_i and R_j, which contracts theorem 1.1 (v). Thus, $R_i = eS$. Now, let R be an arbitrary 0-minimal right ideal of S. Then by $(*)$ again, there exists $e \in E$ such that $R \cap es \neq \{0\}$. As R is an 0-minimal right ideal so $R = eS$. Hence, $\{eS \mid e \in E\}$ is the set of all 0-minimal right ideal of S. Similarly, the set of all 0-minimal left ideals of S is identified by $\{Se \mid e \in E\}$.

2. DUAL SEMIGROUPS WITH ZERO RADICALS

In this section, dual semigroups with zero radicals will be studied.

Lemma 2.1. *Any dual semigroup with zero radicals is a* 0-*direct union of all its* 0-*minimal right (left) ideals* $\{R_i\}_{i\in A}(\{L_j\}_{j\in\Gamma})$, *that is,* $S = \bigcup_{i\in A} R_i (= \bigcup_{j\in\Gamma} L_j)$, *with* $R_i \cap R_j = \{0\}$ $(L_i \cap L_j = \{0\})$ *for all* $i \neq j$.

Proof. By theorem 1.1 (vi) and theorem 1.2, M_i is expressed by $M_i = \bigcup\{R_{ij} | \, R_{ij}$'s are 0-minimal right ideals of $M_i\}$ (in fact, M_i can be regarded as an ideal of S). According to theorem 1.1 (iv), each R_{ij} is also an 0-minimal right ideal of S. Hence, $S = \bigcup M_i = \bigcup_{i\in A} \{R_i | \, R_i$ is a 0-minimal right ideals of $S\}$. Clearly $R_i \cap R_j = \{0\}$ for all $i \neq j$. The proof is completed.

Theorem 2.2. *Let S be a dual semigroup. Then the following conditions are equivalent:*

(i) *The radical* $W(S)$ *of* S *is zero.*

(ii) $\{eS| \, e \in E\}$ *is the set of all* 0-*minimal right ideals of* S.

(iii) $\{Se| \, e \in E\}$ *is the set of all* 0-*minimal left ideals of* S.

(iv) *Every* 0-*minimal two-sided ideal of* S *can be written as* SeS *for some* $e \in E$.

(v) *The Frattini radical* M^* *of* S *is zero.*

(vi) S *is a* 0-*direct union of completely* 0-*simple subsemigroups.*

(vii) *The prime radical* P^* *of* S *is zero.*

Proof. (i) ⇒ (ii) Mimic to the proof of theorem 1.2.

(ii) ⇒ (iv) Let I be an arbitrary 0-minimal two-sided ideal of S. Then $I \cap eS \neq \{0\}$ for some $e \in E$. By (ii), eS is 0-minimal, so $eS = eS \cap I \subseteq I$ and hence $\{0\} \neq SeS \subseteq \subseteq SI \subseteq I$. Consequently, $SeS = I$ for I is 0-minimal.

(iv) ⇒ (v) Suppose that $M^* \neq \{0\}$. Then, by the definition of M^*, there exists a 0-minimal two-sided ideal of S contained in M^*. Applying condition (iv), we obtain $SeS \subseteq M^*$ for some $e \in E$. This implies that $e \in M^*$ which violates theorem 1.1 (i). Thus $M^* = \{0\}$.

(v) ⇒ (vii) By theorem 1.1 (ii), we have $S^2 = S$. This condition implies that every maximal ideal of S is prime. Therefore $P^* \subset M^* = \{0\}$.

(vii) ⇒ (i). Suppose $W(S) \neq \{0\}$. Then there exists a non-zero ideal I of S such that $I^n = \{0\}$ for some integer $n > 1$. Since $I^n = \{0\} \subset P_i$ for any prime ideal P_i of S. So $I \subset \bigcap P_i = P^* = \{0\}$, a contradiction. Thus, $W(S) = \{0\}$.

As the equivalence of (v) and (vi) had been proved by Št. Schwarz mentioned in the introduction and (i) ⇒ (ii) ⇒ (iv) can be proved analogously as (i) ⇒ (ii) ⇒ (iv). The cycle of proof is completed.

Remark. $W(S)$, M^* and P^* of a semigroup S are, in general, different. In particular, Št. Schwarz [5] has already pointed out that if $S^2 = S$ then $P^* \subset M^*$ and $W(S) \subset M^*$. However, in a dual semigroup, we have shown that $W(S) = \{0\}$, $M^* = \{0\}$ and $P^* = \{0\}$ are all equivalent. Using this result, we are now able to sharpen Schwarz's characterization theorem for dual semigroups as mentioned in the introduction as follows:

Theorem 2.3. *A dual semigroup S is a union of completely* 0-*simple subsemigroups* $\Leftrightarrow P^* = \{0\} \Leftrightarrow M^* = \{0\} \Leftrightarrow W(S) = \{0\}$.

We now unify the mentioned results of Št. Schwarz and O. Steinfeld in one setting. It can be seen that dual semigroups with zero radicals have a lot of interesting properties.

Theorem 2.4 (Main Theorem). *Let S be a semigroup. Then the following statements are equivalent*:

(i) *S is a dual semigroup satisfying any one of the conditions* (i) *to* (iii) *as listed in theorem* 2.2.
(ii) *S is a regular semigroup in which the product of any two different idempotents is zero.*
(iii) *S is an inverse semigroups in which every non-zero idempotent is primitive.*
(iv) *S is an inverse semigroup and is the union of its* 0-*minimal left ideals.*
(v) *S is a inverse semigroup and is the union of its* 0-*minimal quasi-ideals.*
(vi) *S is a union of some of its quasi-ideals and these quasi-ideals form a special complete system.*
(vii) *S is a* 0-*direct union of ideals which are completely* 0-*simple inverse subsemigroups of S.*
(viii) *S is a* 0-*direct union of ideals which are* 0-*simple dual subsemigroups of S.*
(ix) *S is a* 0-*direct union of ideals which are Brandt subsemigroups of S.*

Proof. The equivalence of (i), (ii), (iii) can be proved by using theorem 2.2 and the mentioned characterization theorem of O. Steinfeld [7]. The equivalence of (vii), (viii) and (ix) were shown by Steinfeld in [8]. The equivalence of (ii) to (iv) were the results obtained by G. B. Prestion, P. S. Venkatesen, O. Steinfeld and L. Márki. (see Steinfeld [98], p. 88 – 89).

Št. Schwarz [6] has shown that every dual semigroup with zero radical is a 0-direct union of 0-simple dual subsemigroups of *S*. (refered theorem 1.1 (vi)). By using theorem 2.4, the following characterization theorem for dual semigroups with zero radicals is now obtained.

Theorem 2.5. *A semigroup S is a dual semigroup with zero radical if and only if S is a* 0-*direct union of ideals which are* 0-*simple dual subsemigroups of S.*

3. 0-SIMPLE DUAL SEMIGROUPS

It is easily seen that 0-simple semigroups are semigroups with zero radicals. The converse statement is obviously not true. The following theorem is a chatacterization theorem for 0-simple dual semigroups with zero radicals.

Theorem 3.1. *Let S be a dual semigroup with zero radical. Then the following conditions are equivalent.*

(i) *S is 0-simple.*
(ii) $SeS = Sf$ *for any* $e, f \in E$.
(iii) $eSfS = eS$ *for any* $e, f \in E$.
(iv) $SeSf = Sf$ *for any* $e, f \in E$.
(v) *Every non-zero right ideal and left ideal of S has non-zero intersection.*

Proof. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (iii). We first notice that $\{0\} \neq eSfS \subseteq eS$. Then by the 0-minimality of eS; we have $eS = eSfS$.

(iii) $\Rightarrow$ (ii). As $S = \bigcup_{f \in E} fS$, so $SeS = (\bigcup_{f \in E} fS)\, eS = \bigcup_{f \in E} fSeS = \bigcup_{f \in E} fS = S$.

(iv) $\Rightarrow$ (v). Because every non-zero right ideal R and left ideal L of S containing 0-minimal right ideal eS and 0-minimal left ideal fS, thus $\{0\} \neq eSf = eS \cap Sf \subset \subset R \cap L$ by (iv).

(v) $\Rightarrow$ (iv). As $eSf = eS \cap Sf$, so $eSf \neq \{0\}$ by (v).

Summarize theorem 2.4, theorem 3.1 and some known results of O. Steinfeld ([8], p. 70, 88, 89 and 90), we are able to formulate the following theorem which characterizes 0-simple dual semigroups.

Theorem 3.2. *The following conditions on a semigroup are equivalent:*

(i) *S is a 0-simple dual semigroup.*
(ii) *S is a dual semigroup satisfying any one of the conditions* (i) *to* (ix) *as listed in theorem* 2.4 *and* $eSf \neq \{0\}$ *for any* $e, f \in E$.
(iii) $S = \bigcup_{e \in E} Se$, *where Se's are pairwise left similar 0-minimal left ideals of S and* $ef = 0$ *for* $e \neq f \in E$.
(iv) *S is a union of some of its quasi-ideals and these quasi-ideals form a special homogeneous complete system.*
(v) *S is an inverse semigroup and is the union of its 0-minimal left ideals which are pairwise left similar.*
(vi) *S is a completely 0-simple inverse semigroup.*

Proof. (i) $\Leftrightarrow$ (ii). These have been established by theorem 2.2 and theorem 3.1.

(ii) $\Rightarrow$ (iii). Clearly, $S = \bigcup_{e \in E} Se$ and $ef = 0$ for every $e \neq f \in E$. Also, it can be easily shown that every non-zero right ideal and left ideal of a dual semigroup has non-zero intersection, that is, $eSf \neq \{0\}$. Apply Proposition 6.12 (b) of Steinfeld ([8], p. 42), all Se are left similar 0-minimal left ideals of S.

(iii) $\Rightarrow$ (ii). By theorem 10.1 of Steinfeld ([8], p. 79), S is a regular semigroup. Thus S is dual since $ef = 0$ for $e \neq f \in E$ [7]. By Proposition 6.12 (b) of [8], we have $eSf \neq \{0\}$. The equivalence of (iv), (v) and (vi) were shown in the Corollary 10.11 of Steinfeld ([8], p. 90) and the equivalence of (i) and (iv) were given in [7]. Thus the proof is completed.

4. REDUCED DUAL SEMIGROUPS

A semigroup S is said to be *reduced if* S does not contain any non-zero nilpotent elements. It is trivial to see that reduced semigroups are semigroups with zero radicals, but the converse is generally not true. The following is an example.

Example 4.1. Let $S = \{0, a, b, c, d\}$ be the semigroup defined below:

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	c	0
b	0	0	b	0	d
c	0	0	c	0	a
d	0	d	0	b	0

Clearly, the radical of S is $\{0\}$ but S is not reduced.

The following is a characterization theorem for reduced dual semigroups with zero radicals.

Theorem 4.2. *Let S be a dual semigroup with zero radical. Then the following conditions are equivalent:*

(i) *S is reduced.*

(ii) *$eSf \neq \{0\}$ for ary $e \neq f \in E$.*

(iii) *Every 0-minimal ideal of S contains an unique non-zero idempotent of S.*

(iv) *S is a 0-disjoint union of groups with 0.*

Proof. (i) $\Rightarrow$ (i). Since S is dual, by theorem 1.1 (ii), we have $ef = fe = 0$ for $e \neq f \in E$. Thus, $(eSf)^2 = eS(fe)\,Sf = \{0\}$. This implies that $eSf \neq \{0\}$ for S is reduced.

(ii) $\Rightarrow$ (iii). Let M be an 0-minimal ideal of S. Suppose M contains at least two distinct non-zero idempotents, say, e and f. Then by lemma 2.2 of Schwarz [3] and theorem 1.2, M is itself a 0-simple dual semigroup. If $eMf = \{0\}$, then $Mf = (MeM)f = M(eMf) = \{0\}$. This implies that $f^2 = f = 0$, a contradiction. Thus $eMf \neq \{0\}$, but this contradicts (ii). Hence M contains an unique non-zero idempotent.

(iii) $\Rightarrow$ (iv). A 0-simple semigroup containing a unique non-zero idempotent is known to be a group with 0. Since M is itself 0-simple, M must be a group with 0. Thus, by Theorem 1.1 (vi), S is a 0-disjoint union of groups with 0.

(iv) $\Rightarrow$ (v). Trivial.

In closing, we remark that a dual semigroup S is 0-simple and reduced if and only if S is a group with 0. This conclusion comes immediately from theorem 3.1 and theorem 4.2 as 0-simplicity and reducedness are incompatible conditions on a semigroup unless the semigroup itself contains an unique idempotent.

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