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ON THE THEORY OF B- AND BR-SPACES IN GENERAL TOPOLOGY

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1. B- and Br-spaces. A $T_2$ topological space $E$ is called a $B_r$-space ($B$-space) if every continuous, nearly open bijection (surjection) $f$ from $E$ onto an arbitrary $T_2$ space $F$ is open. Here $f: E \to F$ is called nearly open if for every $x \in E$ and every neighbourhood $U$ of $x$ the set $\text{cl}(f(U))$ is a neighbourhood of $f(x)$.

The notions of B- and Br-spaces in the above sense have first been used by T. Husain in the categories of locally convex vector spaces ([$\text{Hu}_1$]) and topological groups ([$\text{Hu}_2$]). They have been chosen in reminiscence of V. Pták's open mapping theorems ([$\text{P}$], [$\text{Ko}$]). We have adopted Husain's definition for the topological case. References concerning the classical theory of B- and Br-spaces and groups are [$\text{P}$], [$\text{Ko}$], [$\text{AEK}$], [$\text{Hu}$], [$\text{Ba}$], [$\text{Pe}$], [$\text{Gr}$], [$\text{Su}$], etc. In a purely topological context, Br-spaces have been considered in [We], [BP], although the term 'Br-space' has not been used there. Further references are [$\text{W}$], [$\text{St}$], [$\text{Ni}$].

Every $T_2$ locally compact space is a B-space and every B-space is a Br-space. In [We], Weston proved that every completely metrizable space is a Br-space. In [BP] this has been generalized to Čech complete spaces. In [$\text{Ni}$] we have further generalized this to obtain.

Proposition 1. Every $T_2$ semi-regular topological space $E$ containing a dense Čech complete subspace is a Br-space. In particular, this is true for monotonically Čech complete spaces.

In [$\text{Ni}$] we have given a direct proof. Proposition 1 may also be deduced from Byczkowski and Pols' result [BP] if we use the following

Lemma. Let $E$ be a $T_2$ semi-regular space and let $F$ be a $T_2$ space. Let $f: E \to F$ be a continuous, nearly open bijection and suppose there exists a dense subset $D$ of $E$ such that $f|D: D \to f(D)$ is open. Then $f$ is open.

Proof. Let $x \in E$ and a neighbourhood $U$ of $x$ be fixed. Choose a regular-open neighbourhood $V$ of $x$ contained in $U$. We prove $\text{int}(\text{cl}(f(V))) \subset f(U)$. Let $z \in \text{int}(\text{cl}(f(V)))$, $z = f(y)$. Let $W$ be a neighbourhood of $y$ with $f(W) \subset \text{int}(\text{cl}(f(V)))$. It is sufficient to prove $W \subset V$. So let $w \in W$ and let $O$ be a regular-open neighbourhood of $w$ contained in $W$. Proving that $O \cap V \neq \emptyset$ remains.
Since \( O, V \) are regular-open in \( E \), \( O \cap D, V \cap D \) are regular-open in \( D \), hence \( f(O \cap D), f(V \cap D) \) are regular-open in \( f(D) \). By note that int cl \( f(O) \cap f(D) \) and int cl \( f(V) \cap f(D) \) are as well regular-open in \( f(D) \) and this implies int cl \( f(O) \cap f(D) = f(V) \cap f(D) \). Since \( O \subset W \) implies int cl \( f(O) \subset \) int cl \( f(V) \) we obtain the desired result \( O \cap V \cap \emptyset. \)

In \([N_3]\) we have investigated an interesting class of \( B \)-spaces.

**Proposition 2.** Every Lindelöf \( P \)-space is a \( B \)-space.

Using the lemma above, one may obtain the following result. Here ‘locally Lindelöf’ means that every point has a base of neighbourhoods consisting of Lindelöf subspaces.

**Proposition 3.** Every \( T_2 \) semi-regular locally Lindelöf space \( E \) containing a dense set of \( P \)-points is a \( B_r \)-space.

**Proof.** Let \( f : E \to F \) be a continuous, nearly open bijection onto the \( T_2 \) space \( F \). We may assume that \( F \) is semi-regular. Let \( D \) denote the set of \( P \)-points in \( E \). We prove that \( f \mid D : D \to f(D) \) is open. First note that every point of \( f(D) \) is a \( P \)-point in \( F \). Indeed, let \( G_n, n = 1, 2, \ldots \) be open sets containing \( y = f(x), x \in D \). Choose open sets \( V_n, n = 1, 2, \ldots \) in \( E \) having \( x \in V_n \), int cl \( f(V_n) \subset G_n \). Then \( V = \bigcap_n V_n \) is a neighbourhood of \( x \) having int cl \( f(V) \subset G_n \), \( x = 1, 2, \ldots \).

Let \( x \in D \) and a Lindelöf neighbourhood \( U \) of \( x \) be fixed. We claim that \( \text{cl} (f(U)) \cap \bigcap f(D) = f(U) \cap f(D) \). Assume the contrary and let \( z \in \text{cl} (f(U)) \cap f(D), \ y \in D \). Let \( \Phi \) denote the filter of neighbourhoods of \( z \), then \( \{ f(U) \cap O : O \in \Phi \} \) is an open cover of \( f(U) \), hence there exist \( O_n \in \Phi, n = 1, 2, \ldots \) having \( f(U) = \bigcup_n f(U) \cap O_n \), a contradiction since we have \( \bigcap_n O_n \in \Phi \).

It follows from our lemma that every \( T_2 \) semi-regular space \( E \) containing a dense \( B_r \)-subspace is itself a \( B_r \)-space. The corresponding result for \( B \)-spaces is not valid. In § 7 we shall present an example of a completely regular space \( E \) containing a dense Lindelöf \( P \)-subspace which is not a \( B \)-space.

In \([N_3]\) we have investigated another interesting class of \( B_r \)-spaces. Let \( S \) be a cofinal subset of \( \omega_1 \). Let \( S^* \) denote the set of \( f \in \omega_1^{\omega} \) having \( f^* = \sup \{ f(n) : n < \omega_1 \} \), \( \in S \). Give \( \omega_1 \) the discrete topology and let \( \omega_1^\omega \) and \( S^* \) have the product topology. Recall that \( S \) is called stationary if it intersects every closed cofinal subset of \( \omega_1 \). We have the following

**Proposition 4.** ([\( N_2 \), [FK] for (1) \( \Leftrightarrow \) (2)]. Let \( S \subset \omega_1 \) be cofinal. Then the following statements are equivalent:

1. \( S \) is stationary;
2. \( S^* \) is a Baire space;
3. \( S^* \) is a \( B_r \)-space.

This provides examples of metrizable \( B_r \)-spaces which do not contain any dense completely metrizable subspace, since clearly \( S^* \) contains a dense completely metrizable subspace if and only if \( S \) contains a closed cofinal subset.
2. Order interpretation. We introduce an order relation $\leq$ on the set of all $T_2$ topologies on a fixed set $E$ by postulating that $\tau_1 \leq \tau_2$ is satisfied if and only if $\text{id}: (E, \tau_2) \to (E, \tau_1)$ is continuous and nearly open. Then $(E, \tau)$ is a $B_r$-space if and only if $\tau$ is minimal among $T_2$ topologies on $E$. Dually one may consider the $\leq$ maximal topologies. It turns out that these can be internally characterized as follows.

**Proposition 5.** $\tau$ is maximal with respect to $\leq$ if and only if every dense subset of $(E, \tau)$ is open. □

**Open problem.** Obtain an internal characterization of $\leq$ minimal (i.e. $B_r$) topologies.

Using the Kuratowski/Zorn lemma one easily proves that given any $T_2$ topology $\tau$ on $E$, there exists a $\leq$ maximal topology $\tau_0$ having $\tau \subset \tau_0$.

**Open problem.** Does a corresponding result hold for $\leq$ minimality?

3. Category. Since $T_2$ minimal (= $H$ minimal) topological spaces are clearly $B_r$-spaces, it follows from a result of Herrlich ([He]) that a $B_r$-space need not be a Baire space in general. One may ask, however, for a first category $B_r$-space which is completely regular. In [N$_3$] we have provided an example of this type constructing a first category Lindelöf $P$-space. On the other hand, all metrizable $B_r$-spaces known up to now are Baire spaces. In [N$_3$] we have obtained the following

**Theorem 1.** Every strongly zero-dimensional metrizable $B_r$-space is Baire. □

**Open problem.** Is it true that every metrizable $B_r$-space is a Baire space?

Note that theorem 1 may be used to prove that every suborderable metrizable $B_r$-space is a Baire space. Another partial positive answer is obtained for metrizable topological groups in view of the following

**Proposition 6.** ([N$_2$]) Every topological group which is a $B_r$-space (in the topological sense) is complete with respect to its two-sided uniformity. □

4. Products. The situation in the classical categories (see [Kö], [Gr]) suggests that the product of even two $B_r$-spaces need not be a $B_r$-space. In [N$_2$] we have obtained the expected counterexamples.

**Proposition 7.** Let $S, T \subset \omega_1$ be stationary sets. Then the following are equivalent:

1. $S \cap T$ is stationary;
2. $S^* \times T^*$ is a $B_r$-space. □

Clearly this provides the desired counterexamples for we may choose disjoint stationary subsets $S, T$ of $\omega_1$, then $S^*, T^*$ are $B_r$-spaces, but $S^* \times T^*$ is not.

One may ask for a $B_r$-space $E$ whose square $E \times E$ is no longer a $B_r$-space. Such an example can be obtained from the following construction.

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**Proposition 8.** Let $F$ be a strongly zero-dimensional metrizable Baire space such that for some $n \geq 2 F^n$ is no longer a Baire space. Suppose that $F$ is a $B_r$-space. Then there exists $r, 1 \leq r \leq n - 1$ such that $E = F^r$ is a $B_r$-space but $E \times E$ is not.

**Proof.** The construction is based on theorem 1 and the fact that finite products of strongly zero-dimensional metrizable spaces are strongly zero-dimensional and metrizable. Regard $F \times F$. If this is not a $B_r$-space, then $E = F$. Otherwise $F^2$ is a Baire space by theorem 1. Then regard $F^2 \times F^2$. If this is not $B_r$, then $E = F^2$. Otherwise $F^4$ is a Baire space. etc. □

In $[N_3]$ we have obtained a space $F$ as above using an example from $[FK]$.

Though no general positive results concerning products of $B_r$-spaces are to be expected, there are positive results in special situations. Namely the classes of $T_2$ minimal spaces, Čech complete spaces, Lindelöf $P$-spaces are examples of productive, countably productive, finitely productive classes of $B_r$-spaces.

**Open problem.** Given a $B_r$-space $E$ and a compact $T_2$ space $K$, must $E \times K$ be a $B_r$ space?

5. **Closed subspaces.** From the situation in the classical categories (concerning the open mapping theory) one would expect that closed subspaces of $B_r$-spaces are again $B_r$. In fact, the corresponding statements are known to be valid in the categories of locally convex vector spaces ([$Kō$]), linear topological spaces ([$AEK$]) and Abelian topological groups. In the case of topological groups the answer is not known (see $[Ba_2]$, $[Gr]$) although there are some positive partial results. In the topological case, the situation seems to be of a completely different nature for we have the

**Proposition 9.** Every $T_2$ semi-regular topological space $E$ is the closed subspace of some $B_r$-space $F$.

**Proof.** Let $F = E \times \{1\} \cup E \times \{2\}$ and define a topology on $F$ by imposing that $\{(x, 1)\}$ is a neighbourhood of $(x, 1)$ for each $x \in E$ and $U(x)$ is a neighbourhood of $(x, 2)$, whenever $x \in E$ and $U$ is a neighbourhood of $x$ in $E$, where $U(x)$ denotes the set $\{(y, i); y \in U \setminus \{x\}, i = 1, 2\} \cup \{(x, 2)\}$. Then $E \times \{2\}$ is a closed subspace of $F$ homeomorphic with $E$ and $E \times \{1\}$ is an open dense and discrete subspace of $F$. Since $F$ is semi-regular by construction, it is a $B_r$-space by proposition 1. □

6. **Sums of $B_r$-spaces.** The class of $B_r$-spaces behaves very strange with respect to summation. First note that the sum of even two $B_r$-spaces need not be a $B_r$-space. Indeed, let $S, T$ be disjoint stationary subsets of $\omega_1$, then $S^*, T^*$ are $B_r$-spaces but $S^* + T^*$ is not $B_r$ in view of the fact that $S^*, T^*$ are disjoint dense subspace of $\omega_1^\alpha$ and hence the natural mapping $f: S^* + T^* \to \omega_1^\alpha$ is a continuous nearly open bijection onto $f(S^* + T^*)$ which is not open.

On the other hand there are certain positive results on sums of $B_r$-spaces.

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Proposition 10. ([N₂]) Given any \( B_r \)-space \( E \), the sum \( E + E \) is a \( B_r \)-space. □

In \([N₂]\) we have investigated summation with Čech complete summands and have obtained the following interesting

Theorem 2. Let \( E \) be a completely regular \( B_r \)-space. Then the following statements are equivalent:
(1) \( E \) is a Baire space;
(2) \( E + F \) is a \( B_r \)-space whenever \( F \) is Čech complete. □

As a consequence of theorem 1 and theorem 2 we deduce that \( E + F \) is a \( B_r \)-space if \( E \) is a strongly zero-dimensional metrizable \( B_r \)-space and \( F \) is Čech complete. On the other hand, if \( E \) is a Lindelöf \( P \)-space of the first category, theorem 2 provides a Čech complete space \( F \) such that \( E + F \) is no longer a \( B_r \)-space.

Another positive result on sums is the following

Proposition 11. Given a \( B_r \)-space \( E \) and a \( T_2 \) locally compact space \( L \), the sum \( E + L \) is a \( B_r \)-space.

Proof. Let \( f: E + L \to F \) be a continuous, nearly open bijection onto the \( T_2 \) space \( F \). Since \( f \mid E: E \to f(E) \), \( f \mid L: L \to f(L) \) are as well nearly open, we have \( E \cong f(E) \), \( L \cong f(L) \). It remains to prove that \( f(E) \) is closed in \( F \). But this follows from the fact that \( f(L) \) is open in its \( T_2 \) extension \( \text{int cl}(f(L)) \) and so is open in \( F \). □

7. \( B \)-spaces. It has been an open question for a long time whether there exist \( B_r \)-complete locally convex vector spaces which are not \( B \)-complete. Finally, an example of this type has been found by Valdivia ([V]). In the category of topological groups the corresponding counterexample was constructed in [Su]. Now in the purely topological case the situation is different. While the class of \( B_r \)-spaces is considerably large, \( B \)-spaces seem to be of a rather special type. In fact, even completely metrizable spaces need not be \( B \)-spaces. An example may be found in [BP].

Example. A \( T_2 \) minimal space need not be a \( B \)-space. Indeed, let \( E \) denote the \( T_2 \) minimal space constructed in [He], whose point set is \( R_0 \cup R_1 \cup R_2 \), where \( R_0 = \mathbb{R} \setminus \mathbb{Q} \cap \mathbb{I} \times \{0\} \), \( R_i = \mathbb{Q} \cap \mathbb{I} \times \{i\} \), \( i = 1, 2 \). Define \( f: E \to I \) by \( f(x, i) = x \), then \( f \) is a continuous, nearly open surjection which is not open.

Concerning sums of \( B \)-spaces we have the following

Proposition 12. ([N₃]). Let \( E \) be a completely regular \( B \)-space. Then the following statements are equivalent:
(1) \( E + L \) is a \( B \)-space whenever \( L \) is \( T_2 \) locally compact;
(2) \( E + K \) is a \( B \)-space whenever \( K \) is \( T_2 \) compact;
(3) \( E + \beta E \) is a \( B \)-space;
(4) \( E \) is locally compact. □

Let \( E \) be a non-discrete Lindelöf \( P \)-space. Then \( E \) is a \( B \)-space but \( E + \beta E \) is not
since $E$ is not locally compact. On the other hand, $E + E$ is clearly a $B$-space since it is Lindelöf $P$. This proves that the lemma from § 1 is not valid for surjective mappings $f$ resp. the class of $B$-spaces is not closed with respect to taking $T_2$ extensions.

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