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DIRECT PRODUCT DECOMPOSITIONS OF DIRECTED GROUPS

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In this paper we investigate the relations between congruences and direct product decompositions of directed groups.

1. PRELIMINARIES

It is well-known that for each algebra $\mathscr{A} = (A; F)$ there is a one-to-one correspondence between the direct product decompositions of $\mathscr{A}$ and the families $(\theta_i \mid i \in I)$ of congruence relations of $\mathscr{A}$ which satisfy the following conditions:

(1) $\bigwedge(\theta_i \mid i \in I) = \mathrm{id}_A$;

(2) $\bigvee(\theta_i \mid i \in I) = A \times A$;

(3) given a family $(x_i \mid i \in I)$ of elements of A, there exists an element $x \in A$ such that $x \, \theta_i \, x_i$ for each $i \in I$.

(Cf., e.g., [4].)

The correspondence under consideration is as follows. If φ is an isomorphism of $\mathscr{A}$ onto $\Pi(\mathscr{A}_i \mid i \in I)$, $i \in I$ and $x, y \in A$, then we put $x \, \theta_i \, y$ if $\varphi(x)(i) = \varphi(y)(i)$. Then the family $(\theta_i \mid i \in I)$ of congruence relations satisfies the conditions (1), (2) and (3). Conversely, if $(\theta_i \mid i \in I)$ is a family of congruence relations of $\mathscr{A}$ satisying (1), (2) and (3), then the mapping $x \to ([x] \, \theta_i \mid i \in I)$ is an isomorphism of $\mathscr{A}$ onto $\Pi(\mathscr{A}/\theta_i \mid i \in I)$.

We shall investigate the question whether (or to what extent) an analogous result is valid for directed groups.

Let us recall that an algebraic system $\mathscr{G} = (G; +, \leqq)$ is a partially ordered group if

(i) $(G; +)$ is a group;

(ii) $(G; \leqq)$ is a partially ordered set;

(iii) whenever $a, b, x, y \in G$, the implication

$$x = y \Rightarrow a + x + b = a + y + b$$

is valid.

If, moreover, $(G; \leqq)$ is a directed set, then $\mathscr{G}$ is said to be a directed group.

The direct product of partially ordered groups is defined in a natural way (cf., e.g., Fuchs [1]). Let θ be an equivalence relation on G. For $x \in G$ we put, as usual,

$[x]\,\theta = \{y \in G \mid y\,\theta\,x\}$. If θ is a congruence of the group $(G; +)$ and if for each $x \in G$, the set $[x]\,\theta$ is a convex subset of $(G; \leqq)$, then θ is said to be a *congruence relation of* $\mathcal{G}$. This definition is in accordance with the notion of homomorphism for partially ordered groups (cf. [1]).

Let θ be a congruence relation on a partially ordered group $\mathcal{G}$. The corresponding factor structure will be denoted by $\mathcal{G}/\theta$. Hence $\mathcal{G}/\theta = (G_1; +, \leqq)$, where $(G_1; +)$ is the group $(G; +)/[0]\,\theta$, and for $[x]\,\theta, [y]\,\theta \in G_1$ we have $[x]\,\theta \leqq [y]\,\theta$ if and only if there exist $x_1 \in [x]\,\theta$ and $y_1 \in [y]\,\theta$ such that $x_1 \leqq y_1$.

Let φ be an isomorphism of $\mathcal{G}$ onto the direct product $\Pi(\mathcal{G}_i \mid i \in I)$. For $i \in I$, $x \in G$ and $y \in G$ we put $x\,\theta_i\,y$ if $\varphi(x)(i) = \varphi(y)(i)$. It is easy to verify that θ_i is a congruence relation of $\mathcal{G}$. A congruence relation constructed in this way will be called a d-congruence relation. Also, it is obvious that the family $(\theta_i \mid i \in I)$ satisfies the conditions (1), (2) and (3).

A congruence relation θ of G will be said to be a *k-congruence relation* if it satisfies the following condition:

(k) If $a, b, c, d \in G$, $a \leqq c \leqq d$, $b \leqq d$ $(a \geqq c \geqq d,\ b \geqq d)$ and $a\,\theta\,b$, then there is $e \in G$ such that $c \leqq e \leqq d$, $b \leqq e$ $(c \geqq e \geqq d,\ b \geqq e)$ and $c\,\theta\,e$.

Equivalence relations on directed sets satisfying this condition were investigated by M. Kolibiar [4] and the author [3].

2. THE SYSTEM OF ALL CONGRUENCE RELATIONS OF $\mathcal{G}$

In what follows we assume that $\mathcal{G}$ is a directed group. Let Con $\mathcal{G}$ be the system of all congruence relations of $\mathcal{G}$ and let S be a subset of Con $\mathcal{G}$. Let α be a cardinal. Consider the following condition:

$(c(\mathcal{G}, S, \alpha))$ Each d-congruence relation of G belongs to S; if $(\theta_i \mid i \in I)$ is a family of congruence relations belonging to S such that (1), (2) and (3) hold, and if card $I \leqq \alpha$, then the mapping $x \to ([x]\,\theta_i \mid i \in I)$ is an isomorphism of $\mathcal{G}$ onto $\Pi(\mathcal{G}/\theta_i \mid i \in I)$.

Now we are interested in the validity of the following assertions:

(A) For each directed group $\mathcal{G}$ the condition $c(\mathcal{G}, \text{Con}\,\mathcal{G}, 2)$ is satisfied.

(B) For each directed group $\mathcal{G}$ there exists a subset S of Con $\mathcal{G}$ such that for each positive integer α the condition $c(\mathcal{G}, S, \alpha)$ is satisfied.

(C) For each directed group $\mathcal{G}$ there exists a subset S of Con $\mathcal{G}$ such that for each cardinal α the condition $c(\mathcal{G}, S, \alpha)$ is satisfied.

We begin with showing that the assertion (A) does not hold.

2.1. Proposition. *There exists a directed group G such that the condition* $c(\mathcal{G}, \text{Con}\,\mathcal{G}, 2)$ *fails to be valid.*

Proof. Let G be the set of all pairs (x, y) of integers. The operation $+$ in G is defined component-wise. For (x, y) and (x_1, y_1) in G we put $(x, y) \leqq (x_1, y_1)$ if either (i) $(x, y) = (x_1, y_1)$, or (ii) $x < x_1$ and $y < y_1$. Then $\mathcal{G} = (G; +, \leqq)$ is a directed group.

We set $(x, y)\,\theta_1\,(x_1, y_1)$ if $x = x_1$. Similarly we put $(x, y)\,\theta_2\,(x_1, y_1)$ if $y = y_1$. Both θ_1 and θ_2 belong to Con $\mathcal{G}$ and the family $(\theta_i \mid i = 1, 2)$ satisfies the conditions (1), (2) and (3).

Let us consider the mapping $\varphi(z) = ([z]\,\theta_1, [z]\,\theta_2)$ of $\mathcal{G}$ onto $\Pi(\mathcal{G}/\theta_i \mid i = 1, 2)$. For $z = (1, 0)$ we have $z \ngeqq 0$, but $\varphi(z)$ is strictly positive in $\Pi(\mathcal{G}/\theta_i \mid i = 1, 2)$. Therefore the condition $c(\mathcal{G}$, Con $\mathcal{G}$, 2) fails to be valid.

3. THE SYSTEM OF ALL k-CONGRUENCE RELATIONS OF $\mathcal{G}$

In this section it will be shown that the assertion (B) holds and that (C) is not true. Let $\mathrm{Con}_k\,\mathcal{G}$ be the set of all k-congruence relations of $\mathcal{G}$.

3.1. Definition. (Cf. [4].) Let $\mathcal{P} = (P; \leqq)$ be a directed set. An equivalence relation θ on P will be called a *congruence relation of* $\mathcal{P}$ if the following conditions are satisfied:

(i) For each $a \in P$, $[a]\,\theta$ is a convex subset of $\mathcal{P}$.

(ii) The condition (k) is fulfilled.

If θ is a congruence relation of $\mathcal{P}$, then $\mathcal{P}/\theta$ has the usual meaning. The direct product of partially ordered sets $\mathcal{P}_i$ $(i \in I)$ will be denoted by $\Pi(\mathcal{P}_i \mid i \in I)$.

3.2. Theorem. (Cf. [4], Theorem 5.) *Let $(\theta_i \mid i \in I)$ be a family of congruence relations of a directed set $\mathcal{P} = (P; \leqq)$ satisfying the conditions* (1), (2) *and* (3). *Assume that I is finite. For each $x \in P$ put $\varphi(x) = ([x]\,\theta_i \mid i \in I)$. Then φ is an isomorphism of $\mathcal{P}$ onto $\Pi(\mathcal{P}/\theta_i \mid i \in I)$.*

3.3. Theorem. *Let $(\theta_i \mid i \in I)$ be a family of congruence relations of a directed group $\mathcal{G}$ satisfying the conditions* (1), (2) *and* (3). *Let I be finite. Then the mapping $x \to ([x]\,\theta_i \mid i \in I)$ is an isomorphism of $\mathcal{G}$ onto the direct product $\Pi(\mathcal{G}/\theta_i \mid i \in I)$.*

Proof. This is a consequence of 3.2 and of Theorem 3 of [2].

3.4. Corollary. *Let G be a directed group and let α be a positive integer. Then the condition $c(\mathcal{G}$, $\mathrm{Con}_k\,\mathcal{G}, \alpha)$ is valid.*

Thus we have verified that (B) holds; it suffices to put $S = \mathrm{Con}_k\,\mathcal{G}$. Now we shall show that the assertion of 3.4 cannot be extended to the case when α is infinite.

3.5. Example. Let R be the additive group of all reals with the natural linear order. Let N_0 be the set of all integers. Let I be an infinite set and let G be the set of all mappings g of I into R. For $g \in G$ and $i \in I$, $g(i)$ is said to be the *i-th component of g*. The operation $+$ in G is defined component-wise. For $x, y \in G$ we put $x \leqq y$ if some of the following conditions is valid:

(i) the set $\{i \in I \mid x(i) - y(i) \notin N_0\}$ is finite and $x(i) \leqq y(i)$ for each $i \in I$;

(ii) the set $\{i \in I \mid 0 < y(i) - x(i) < 1\}$ is finite and $x(i) \leqq y(i)$ for each $i \in I$.

Then $\mathcal{G} = (G; +, \leqq)$ is a directed group.

Let $x, y \in G$ and $i \in I$. We put $x\ \theta_i\ y$ if $x(i) = y(i)$. Next we set $x\ \theta'_i\ y$ if $x(j) = y(j)$ for each $j \in I \setminus \{i\}$. Then θ_i and θ'_i are congruence relations of $\mathcal{G}$ and we clearly have

3.6. Lemma. *Let $\mathcal{G}$ be as in* 3.5. *Then the mapping $\varphi(x) = ([x]\ \theta_i, [x]\ \theta'_i)$ is an isomorphism of $\mathcal{G}$ onto the direct product $\mathcal{G}/\theta_i \times \mathcal{G}/\theta'_i$.*

3.7. Corollary. *Let $\mathcal{G}$ be as in* 3.5. *For each $i \in I$, θ_i is a d-congruence relation of $\mathcal{G}$.*

3.8. Proposition. *Let $\mathcal{G}$ be as in* 3.5. *Let α be infinite. Then the condition $c(\mathcal{G}, \mathrm{Con}_k\ \mathcal{G}, \alpha)$ fails to hold.*

Proof. It is obvious that the family $(\theta_i \mid i \in I)$ satisfies the conditions (1), (2) and (3). By way of contradiction, assume that $c(\mathcal{G}, \mathrm{Con}_k\ \mathcal{G}, \alpha)$ is valid. Consider the mapping φ defined by $\varphi(x) = ([x]\ \theta_i \mid i \in I)$ for each $x \in G$. Let $y, z \in G$ be such that $y(i) = \frac{1}{2}$ and $z(i) = \frac{2}{3}$ for each $i \in I$. Then y and z are incomparable in G, but $\varphi(x) < \varphi(y)$. Hence φ fails to be an isomorphism, which is a contradiction to the assumption.

3.9. Proposition. *Let $\mathcal{G}$ be as in* 3.5 *and let α be an infinite cardinal. Then there exists no $S \subseteq \mathrm{Con}\ \mathcal{G}$ having the property that $c(\mathcal{G}, S, \alpha)$ is valid.*

Proof. By way of contradiction, assume that there exists $S \subseteq \mathrm{Con}\ \mathcal{G}$ such that the condition $c(\mathcal{G}, S, \alpha)$ holds. For each $i \in I$ let θ_i be as in 3.6. According to 3.7, all θ_i belong to S. But then by applying the same method as in the proof of 3.8 we arrive at a contradiction.

We have verified that the assertion (C) does not hold. This result shows that there is no possibility of sharpening the notion of congruence for directed groups in order to obtain "good" relations between congruences and direct product decompositions.

For a related result concerning directed sets cf. [3].

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