

Ján Jakubík

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ON A RADICAL CLASS OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

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The radical class under consideration in the present note is the class \mathcal{S}_{pec} (for the definitions, cf. below). In [2] the question was proposed whether there exists a torsion class T of lattice ordered groups such that

$$\mathcal{S}_{pec} = T^C,$$

where C is the completion closure operation. It will be shown that the answer to this question is "No".

1. PRELIMINARIES

We recall the following basic notions we shall need in the sequel.

A torsion class (cf. Martinez [5], and Darnel [2]) is a nonempty collection of lattice ordered groups closed with respect to convex l -subgroups, joins of convex l -subgroups, and homomorphic images.

Let \mathcal{G} be the class of all lattice ordered groups and let T be a torsion class. For every $G \in \mathcal{G}$ we denote by $T(G)$ the join of all convex l -subgroups of G that belong to T . Then $T(G) \in T$ and $T(G)$ is an l -ideal of G .

For each convex l -subgroup H of G we denote by \bar{H}_G the order closure of H in G ; i.e., \bar{H}_G is the intersection of all closed l -subgroups H_i of G with $H \subseteq H_i$. Next, for each torsion class T we put

$$(*) \quad T^C = \{\bar{T(G)}_G : G \in \mathcal{G}\}.$$

An element $0 < x \in G$ is said to be *special* if it has exactly one value.

For the definition of lex-subgroup of a lattice ordered group cf. [1], 2.27.

Let $x \in G$; we denote by $[x]$ the convex l -subgroup of G generated by x .

1.1. Lemma. (Cf. [1], Theorem 2.14.) *Let $G \in \mathcal{G}$, $0 < x \in G$. Then the following conditions are equivalent:*

- (i) x is special.
- (ii) $[x]$ is a proper lexico extension.

A lattice ordered group G is said to be *special valued* if every positive element

of G is a join of special elements (cf. [2]); let \mathcal{S}_{pec} be the class of all special valued lattice ordered groups.

A nonempty subclass of \mathcal{G} is said to be a *radical class* (cf. [4]) if it is closed with respect to convex l -subgroups, joins of convex l -subgroups and isomorphic images.

Let \mathcal{R} be the collection of all radical classes of lattice ordered groups. The collection \mathcal{R} is partially ordered by inclusion. Then \mathcal{R} is a complete lattice [4]. For $T \in \mathcal{R}$ let T^c be as in (*). If $G \in \mathcal{G}$, then $T(G)$ is defined similarly as in the case when T is a torsion class. Again, $T(G) \in T$ and $T(G)$ is an l -ideal of G .

1.2. Lemma. (Cf. [2].) $\mathcal{S}_{\text{pec}} \in \mathcal{R}$.

1.3. Lemma. (Cf. [2].) *The mapping $T \rightarrow T^c$ is a closure operator on \mathcal{R} . In particular, $T^c \in \mathcal{R}$ and $T \subseteq T^c$ for each $T \in \mathcal{R}$.*

The following assertion is obvious.

1.4. Lemma. *There exists a (unique up to isomorphism) root P such that*

- (i) P has a greatest element p_0 ;
- (ii) each bounded chain in P is finite;
- (iii) if $p \in P$, then the system $L(p)$ of all elements covered by p has the power \aleph_0 .

2. AN EXAMPLE

Let P be as in Section 1. For each $p \in P$ let $G_p = Z$ (the additive group of all integers with the natural linear order). Let G be the lexicographic product

$$\Gamma_{p \in P} G_p$$

(cf. [3]). Then Ω is a lattice ordered group (cf., e.g., [1], Chap. IV). If $f \in G$, then we denote by $f(p)$ the p -th component of f .

Let $p \in P$ be fixed. Denote

$$A(p) = \{f \in G : f(q) = 0 \text{ whenever } q \not\leq p\},$$

$$B(p) = \{f \in G : f(q) = 0 \text{ whenever } q \prec p\}.$$

Then we evidently have

2.1. Lemma. $A(p)$ is a proper lex extension of the lattice ordered group $B(p)$. For each $x \in A(p) \setminus B(p)$ with $x > 0$ the relation

$$[x] = A(p)$$

is valid.

Then in view of 2.1 and 1.1 we obtain

2.2. Lemma. *Let $p \in P$ and $0 < x \in A(p) \setminus B(p)$. Then x is special.*

Let $0 < y \in G$. Let $P(y)$ be the system of all $p \in P$ such that

- (i) $y(p) \neq 0$,
- (ii) if $p \in P(y)$, $q \in P$, $q > p$, then $y(q) = 0$.

We must have $P(y) \neq \emptyset$. For each $p \in P(y)$ there exists a uniquely determined element $x^{(p)} \in G$ such that $x^{(p)}(q) = y(q)$ whenever $q = p$, and $x^{(p)}(q) = 0$ otherwise.

2.3. Lemma. *Let $0 < y \in G$. Then $0 < x^{(p)} \in A(p) \setminus B(p)$, and $y = \bigvee_{p \in P(y)} x^{(p)}$. If p, q are distinct elements of $P(y)$, then $x^{(p)} \wedge y^{(q)} = 0$.*

The proof is immediate.

In view of 2.1, 2.2 and 2.3 we infer:

2.4. Corollary. *Let y and $x^{(p)}$ be as in 2.3. Then each $x^{(p)}$ is special, hence $G \in \mathcal{S}pec$.*

Again, let p be a fixed element of P . Let $D(p)$ be the set of all elements $x \in B(p)$ such that the set

$$\{p \in L(p): x(p) \neq 0\}$$

is finite. Then $D(p)$ is a convex l -subgroup of $B(p)$. Because G is abelian, $D(p)$ is an l -ideal of G and clearly $D(p) \subset B(p)$. Put

$$B'(p) = B(p)/D(p).$$

$B'(p)$ is an archimedean lattice ordered group and if $0 < z \in B'(p)$, then the interval $[0, z]$ of $B'(p)$ fails to be linearly ordered. Therefore $B'(p) \neq \{0\}$ and $B'(p)$ has no special element. Thus

$$(1) B'(p) \notin \mathcal{S}pec.$$

2.5. Proposition. *Let T be a torsion class of lattice ordered groups. Then $\mathcal{S}pec \neq T^c$.*

Proof. By way of contradiction, assume that the relation

$$(2) \mathcal{S}pec = T^c$$

is valid. Let G be as above. In view of 2.4 and (2) we have $G \in T^c$. Hence

$$G = T^c(G) = \overline{T(G)}_G.$$

Thus $T(G) \neq \{0\}$. Hence there exists $0 < y \in T(G)$. Since $T(G) \in T$ and $[y]$ is a convex l -subgroup of $T(G)$, we obtain $[y] \in T$.

There exists $p \in P(y)$. Then $B(p)$ is a convex l -subgroup of $[y]$, which yields that $B(p) \in T$. Since T is closed with respect to homomorphisms, $B'(p) \in T$. In view of (2) and 1.3 we have $T \subseteq T^c = \mathcal{S}pec$, whence $B'(p) \in \mathcal{S}pec$. This contradicts (1).

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Author's address: 040 01 Košice, Ždanovova 6, Czechoslovakia (Matematický ústav SAV, dislokované pracovisko).