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ON A RADICAL CLASS OF LATTICE ORDERED GROUPS

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The radical class under consideration in the present note is the class $\mathcal{S}pec$ (for the definitions, cf. below). In [2] the question was proposed whether there exists a torsion class T of lattice ordered groups such that

$$\mathcal{S}pec = T^C ,$$

where C is the completion closure operation. It will be shown that the answer to this question is "No".

1. PRELIMINARIES

We recall the following basic notions we shall need in the sequel.

A torsion class (cf. Martinez [5], and Darnel [2]) is a nonempty collection of lattice ordered groups closed with respect to convex l-subgroups, joins of convex l-subgroups, and homomorphic images.

Let $\mathcal{G}$ be the class of all lattice ordered groups and let T be a torsion class. For every $G \in \mathcal{G}$ we denote by $T(G)$ the join of all convex l-subgroups of G that belong to T. Then $T(G) \in T$ and $T(G)$ is an l-ideal of G.

For each convex l-subgroup H of G we denote by $\bar{H}_G$ the order closure of H in G; i.e., $\bar{H}_G$ is the intersection of all closed l-subgroups H_i of G with $H \subseteq H_i$. Next, for each torsion class T we put

$$(*) \qquad T^C = \{\overline{T(G)}_G \colon G \in \mathcal{G}\} .$$

An element $0 < x \in G$ is said to be *special* if it has exactly one value.

For the definition of lex-subgroup of a lattice ordered group cf. [1], 2.27.

Let $x \in G$; we denote by $[x]$ the convex l-subgroup of G generated by x.

1.1. Lemma. (Cf. [1], Theorem 2.14.) *Let $G \in \mathcal{G}$, $0 < x \in G$. Then the following conditions are equivalent*:

(i) *x is special.*

(ii) *$[x]$ is a proper lexico extension.*

A lattice ordered group G is said to be *special valued* if every positive element of G is a join of special elements (cf. [2]); let $\mathcal{S}pec$ be the class of all special valued lattice ordered groups.

A nonempty subclass of $\mathcal{G}$ is said to be a *radical class* (cf. [4]) if it is closed with respect to convex l-subgroups, joins of convex l-subgroups and isomorphic images.

Let $\mathcal{R}$ be the collection of all radical classes of lattice ordered groups. The collection $\mathcal{R}$ is partially ordered by inclusion. Then $\mathcal{R}$ is a complete lattice [4]. For $T \in \mathcal{R}$ let T^C be as in $(*)$. If $G \in \mathcal{G}$, then $T(G)$ is defined similarly as in the case when T is a torsion class. Again, $T(G) \in T$ and $T(G)$ is an l-ideal of G.

1.2. Lemma. (Cf. [2].) $\mathcal{S}pec \in \mathcal{R}$.

1.3. Lemma. (Cf. [2].) *The mapping $T \to T^C$ is a closure operator on $\mathcal{R}$.*
In particular, $T^C \in \mathcal{R}$ and $T \subseteq T^C$ for each $T \in \mathcal{R}$.
The following assertion is obvious.

1.4. Lemma. *There exists a (unique up to isomorphism) root P such that*

(i) *P has a greatest element p_0;*
(ii) *each bounded chain in P is finite;*
(iii) *if $p \in P$, then the system $L(p)$ of all elements covered by p has the power $\aleph_0$.*

2. AN EXAMPLE

Let P be as in Section 1. For each $p \in P$ let $G_p = Z$ (the additive group of all integers with the natural linear order). Let G be the lexicographic product

$$\Gamma_{p\in P}\, G_p$$

(cf. [3]). Then Ω is a lattice ordered group (cf., e.g., [1], Chap. IV). If $f \in G$, then we denote by $f(p)$ the p-th component of f.

Let $p \in P$ be fixed. Denote
$A(p) = \{f \in G: f(q) = 0 \text{ whenever } q \nleq p\}$,
$B(p) = \{f \in G: f(q) = 0 \text{ whenever } q \nless p\}$.

Then we evidently have

2.1. Lemma. *$A(p)$ is a proper lex extension of the lattice ordered group $B(p)$. For each $x \in A(p) \setminus B(p)$ with $x > 0$ the relation*

$$[x] = A(p)$$

is valid.

Then in view of 2.1 and 1.1 we obtain

2.2. Lemma. *Let $p \in P$ and $0 < x \in A(p) \setminus B(p)$. Then x is special.*

Let $0 < y \in G$. Let $P(y)$ be the system of all $p \in P$ such that

(i) $y(p) \neq 0$,
(ii) if $p \in P(y)$, $q \in P$, $q > p$, *then* $y(q) = 0$.

We must have $P(y) \neq \emptyset$. For each $p \in P(y)$ there exists a uniquely determined element $x^{(p)} \in G$ such that $x^{(p)}(q) = y(q)$ whenever $q = p$, and $x^{(p)}(q) = 0$ otherwise.

2.3. Lemma. *Let* $0 < y \in G$. *Then* $0 < x^{(p)} \in A(p) \setminus B(p)$, *and* $y = \bigvee_{p \in P(y)} x^{(p)}$. *If* p, q *are distinct elements of* $P(y)$, *then* $x^{(p)} \wedge y^{(q)} = 0$.

The proof is immediate.

In view of 2.1, 2.2 and 2.3 we infer:

2.4. Corollary. *Let* y *and* $x^{(p)}$ *be as in* 2.3. *Then each* $x^{(p)}$ *is special, hence* $G \in \mathcal{S}pec$.

Again, let p be a fixed element of P. Let $D(p)$ be the set of all elements $x \in B(p)$ such that the set

$$\{p \in L(p): x(p) \neq 0\}$$

is finite. Then $D(p)$ is a convex l-subgroup of $B(p)$. Because G is abelian, $D(p)$ is an l-ideal of G and clearly $D(p) \subset B(p)$. Put

$$B'(p) = B(p)/D(p)\,.$$

$B'(p)$ is an archimedean lattice ordered group and if $0 < z \in B'(p)$, then the interval $[0, z]$ of $B'(p)$ fails to be linearly ordered. Therefore $B'(p) \neq \{0\}$ and $B'(p)$ has no special element. Thus

(1) $B'(p) \notin \mathcal{S}pec$.

2.5. Proposition. *Let* T *be a torsion class of lattice ordered groups. Then* $\mathcal{S}pec \neq T^C$.

Proof. By way of contradiction, assume that the relation

(2) $\mathcal{S}pec = T^C$

is valid. Let G be as above. In view of 2.4 and (2) we have $G \in T^C$. Hence

$$G = T^C(G) = \overline{T(G)}_G\,.$$

Thus $T(G) \neq \{0\}$. Hence there exists $0 < y \in T(G)$. Since $T(G) \in T$ and $[y]$ is a convex l-subgroup of $T(G)$, we obtain $[y] \in T$.

There exists $p \in P(y)$. Then $B(p)$ is a convex l-subgroup of $[y]$, which yields that $B(p) \in T$. Since T is closed with respect to homomorphisms, $B'(p) \in T$. In view of (2) and 1.3 we have $T \subseteq T^C = \mathcal{S}pec$, whence $B'(p) \in \mathcal{S}pec$. This contradicts (1).

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