Czechoslovak Mathematical Journal

Jaromír Duda

Mal'cev conditions for directly decomposable compatible relations

Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 4, 674–680

Persistent URL: http://dml.cz/dmlcz/102343

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

MAL'CEV CONDITIONS FOR DIRECTLY DECOMPOSABLE COMPATIBLE RELATIONS

JAROMÍR DUDA, Brno

(Received November 11, 1987)

1. PRELIMINARIES

Mal'cev conditions for varieties having directly decomposable tolerances and directly decomposable reflexive compatible relations were given independently in [9] and [4]. Varieties having directly decomposable tolerance classes and directly decomposable relation classes were studied in [3]. The aim of this paper is to show that:

(i) the direct decomposability of tolerances (reflexive compatible relations) coincides with the direct decomposability of tolerance classes (relation classes, respectively) in varieties of algebras;

(ii) Mal'cev conditions from [9], [4] can be replaced by simpler ones;

(iii) all the above mentioned properties of tolerances and reflexive compatible relations in a variety V can be considered only on the square $F_V(2) \times F_V(2)$ of the V-free algebra $F_V(2)$ over two free generators.

To make this paper selfcontained we recall some definitions:

Definition 1. Let A, B be algebras of the same type. The kernels Π_A, Π_B of the canonical projections $pr_A: A \times B \to A$, $pr_B: A \times B \to B$, respectively, are called *factor congruences* on $A \times B$. A binary relation R on $A \times B$ is called a *subfactor relation* whenever $R \subseteq \Pi_A$ or $R \subseteq \Pi_B$.

Definition 2. Let R be a reflexive binary relation on a set A and let $a \in A$. Then the subset $[a] R = \{x \in A; \langle x, a\rangle \in R\}$ is called a *relation class* of R. In particular $[a] T$ is called a *tolerance class* provided T is a tolerance on A.

Definition 3. Let A, B be algebras of the same type. The product $A \times B$ has *directly decomposable relations* (*relation classes*) if every relation R (relation class C) on $A \times B$ is a product of its projections $\langle pr_A, pr_A\rangle R$ and $\langle pr_B, pr_B\rangle R$ ($pr_A C$ and $pr_B C$, respectively).

A variety of algebras V has some of the properties listed above whenever for every $A, B \in V$, $A \times B$ has the respective property.

In what follows, by a relation on an algebra A we mean a *compatible relation* on A, i.e. a subalgebra of $A \times A$. It is well known and frequently used that for any subset $M \subseteq A \times A$ the least tolerance $T(M)$ and the least reflexive relation $R(M)$ on A containing M exist. The functional descriptions of $T(M)$ and $R(M)$ are adopted from [1].

The symbol $\boldsymbol{c}^{\rightarrow}$ stands for the finite sequence $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$.

2. DIRECTLY DECOMPOSABLE TOLERANCES

Theorem 1. *For a variety V the following conditions are equivalent:*

(1) *V has directly decomposable subfactor tolerances;*

(2) *there exist binary terms $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n, \boldsymbol{d}_1, \ldots, \boldsymbol{d}_n$ and a $(2 + n)$-ary term $\boldsymbol{r}$ such that the identities*

$$x = \boldsymbol{r}(x, y, \boldsymbol{c}^{\rightarrow}(x, y)),$$
$$y = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, y)),$$
$$y = \boldsymbol{r}(y, x, \boldsymbol{c}^{\rightarrow}(x, y))$$

hold in V.

Proof. (1) $\Rightarrow$ (2): Consider the principal tolerance $T(\langle x, x\rangle, \langle y, x\rangle)$ on the square $F_V(2) \times F_V(2)$ of the V-free algebra $F_V(2)$ over free generators x and y. Since $\langle\langle x, x\rangle, \langle y, x\rangle\rangle \in T(\langle x, x\rangle, \langle y, x\rangle)$ the hypothesis yields $\langle\langle x, y\rangle, \langle y, y\rangle\rangle \in$ $\in T(\langle x, x\rangle, \langle y, x\rangle)$. Using the functional description of $T(\langle x, x\rangle, \langle y, x\rangle)$, see [1], the identites (2) readily follow.

(2) $\Rightarrow$ (1): Let T be a subfactor tolerance on $A \times B \in V$, say $T \subseteq \Pi_A$. Assuming the identities (2) we find

$$x' = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, x')),$$
$$y = \boldsymbol{r}(y, z, \boldsymbol{c}^{\rightarrow}(y, z)),$$
$$x' = \boldsymbol{r}(x, x, \boldsymbol{d}^{\rightarrow}(x, x')),$$
$$z = \boldsymbol{r}(z, y, \boldsymbol{c}^{\rightarrow}(y, z)),$$

i.e. $\langle\langle x, y\rangle, \langle x, z\rangle\rangle \in T$ implies $\langle\langle x', y\rangle, \langle x', z\rangle\rangle \in T$ for any $x, x' \in A$, $y, z \in B$. The proof is complete.

Remark 1. The Mal'cev condition for varieties having directly decomposable subfactor congruences was given by J. Hagemann [8].

Lemma 1. *Let A, B be algebras of the same type. For any tolerance class $[\langle z_1, z_2\rangle]\, T$ on $A \times B$ the following conditions are equivalent:*

(1) *$[\langle z_1, z_2\rangle]\, T$ is directly decomposable;*

(2) (i) *$\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$ implies $\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$;*

(ii) *$\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\, T$ imply $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\, T$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): Let $\langle a, b\rangle, \langle a', b'\rangle \in [\langle z_1, z_2\rangle]\, T$. Then $\langle a, z_2\rangle, \langle z_1, b'\rangle \in [\langle z_1, z_2\rangle]\, T$,

by (2) (i). So $\langle a, b' \rangle \in [\langle z_1, z_2 \rangle] T$, by (2) (ii). The last argument establishes the direct decomposability of the tolerance class $[\langle z_1, z_2 \rangle] T$.

Lemma 2. *Let A, B be algebras of the same type. The following conditions are equivalent:*

(1) *$A \times B$ has directly decomposable tolerances;*

(2) *$A \times B$ has directly decomposable subfactor tolerances and directly decomposable tolerance classes.*

Proof. Only the implication (2) $\Rightarrow$ (1) is nontrivial: Let T be a tolerance on $A \times B$ and let $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$, $\langle\langle a', b'\rangle, \langle c', d'\rangle\rangle \in T$. Since $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$, $\langle\langle c, d\rangle, \langle c, d\rangle\rangle \in T$, the direct decomposability of tolerance classes yields $\langle\langle a, d\rangle, \langle c, d\rangle\rangle \in T$. Then also $\langle\langle a, d'\rangle, \langle c, d'\rangle\rangle \in T$ by the direct decomposability of subfactor tolerances. Analogously from $\langle\langle c', d'\rangle, \langle c', d'\rangle\rangle$, $\langle\langle a', b'\rangle, \langle c', d'\rangle\rangle \in T$ we find $\langle\langle c', b'\rangle, \langle c', d'\rangle\rangle \in T$ and, further, $\langle\langle c, b'\rangle, \langle c, d'\rangle\rangle \in T$. Altogether $\langle\langle a, d'\rangle, \langle c, d'\rangle\rangle$, $\langle\langle c, b'\rangle, \langle c, d'\rangle\rangle \in T$ which implies $\langle\langle a, b'\rangle, \langle c, d'\rangle\rangle \in T$. This proves the direct decomposability of T, see [2; Thm 1, p. 227].

Theorem 2. *For a variety V the following conditions are equivalent:*

(1) *V has directly decomposable tolerances;*

(2) *V has directly decomposable tolerance classes;*

(3) *there exist ternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ and a $(4 + n)$-ary term s such that the identities*

$$\begin{aligned} x &= s(x, y, z, z, \mathbf{p}^{\rightarrow}(x, y, z))\,, \\ y &= s(x, y, z, z, \mathbf{q}^{\rightarrow}(x, y, z))\,, \\ z &= s(z, z, x, y, \mathbf{p}^{\rightarrow}(x, y, z))\,, \\ z &= s(z, z, x, y, \mathbf{q}^{\rightarrow}(x, y, z)) \end{aligned}$$

hold in V;

(4) *there exist binary terms $f_1, \ldots, f_{n+2}, g_1, \ldots g_{n+2}, h_1, \ldots, h_n, k_1, \ldots, k_n$ and $(4 + n)$-ary terms s_1, s_2 such that the identities*

$$\left.\begin{aligned} x &= s_1(x, y, \mathbf{f}^{\rightarrow}(x, y)) \\ x &= s_1(y, x, \mathbf{g}^{\rightarrow}(x, y)) \\ y &= s_1(y, x, \mathbf{f}^{\rightarrow}(x, y)) \\ x &= s_1(x, y, \mathbf{g}^{\rightarrow}(x, y)) \end{aligned}\right\} (\Sigma_1)$$

$$\left.\begin{aligned} x &= s_2(x, y, y, y, \mathbf{h}^{\rightarrow}(x, y)) \\ y &= s_2(x, x, y, x, \mathbf{k}^{\rightarrow}(x, y)) \\ y &= s_2(y, x, y, y, \mathbf{h}^{\rightarrow}(x, y)) \\ x &= s_2(x, x, x, y, \mathbf{k}^{\rightarrow}(x, y)) \end{aligned}\right\} (\Sigma_2)$$

hold in V.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) was already shown in [3; Thm 4, p. 400].

(3) ⇒ (4): Setting $z = y$ in the identities (3) we find that

$$\begin{aligned} x &= \boldsymbol{s}(x, y, y, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ y &= \boldsymbol{s}(x, y, y, y, \boldsymbol{q}^{\rightarrow}(x, y, y)), \\ y &= \boldsymbol{s}(y, y, x, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ y &= \boldsymbol{s}(y, y, x, y, \boldsymbol{q}^{\rightarrow}(x, y, y)). \end{aligned}$$

Interchange the variables x and y in the second and the fourth identities. Then

$$\begin{aligned} x &= \boldsymbol{s}(x, y, y, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ x &= \boldsymbol{s}(y, x, x, x, \boldsymbol{q}^{\rightarrow}(y, x, x)), \\ y &= \boldsymbol{s}(y, y, x, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ x &= \boldsymbol{s}(x, x, y, x, \boldsymbol{q}^{\rightarrow}(y, x, x)). \end{aligned}$$

Defining

$$\begin{aligned} \boldsymbol{s}_1(a, b, w^{\rightarrow}) &= \boldsymbol{s}(a, w_{n+1}, b, w_{n+2}, w_1, \ldots, w_n), \\ \boldsymbol{f}^{\rightarrow}(x, y) &= \boldsymbol{p}_1(x, y, y), \ldots, \boldsymbol{p}_n(x, y, y), y, y, \quad \text{and} \\ \boldsymbol{g}^{\rightarrow}(x, y) &= \boldsymbol{q}_1(y, x, x), \ldots, \boldsymbol{q}_n(y, x, x), x, x \end{aligned}$$

we get the identities (Σ_1).

Setting $z = y$ in the first and the third identities (3) and $z = x$ in the remaining ones we obtain

$$\begin{aligned} x &= \boldsymbol{s}(x, y, y, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ y &= \boldsymbol{s}(x, y, x, x, \boldsymbol{q}^{\rightarrow}(x, y, x)), \\ y &= \boldsymbol{s}(y, y, x, y, \boldsymbol{p}^{\rightarrow}(x, y, y)), \\ x &= \boldsymbol{s}(x, x, x, y, \boldsymbol{q}^{\rightarrow}(x, y, x)). \end{aligned}$$

Now the identities (Σ_2) follow for

$$\begin{aligned} \boldsymbol{s}_2(a, b, c, d, w^{\rightarrow}) &= \boldsymbol{s}(a, c, b, d, w^{\rightarrow}), \\ \boldsymbol{h}^{\rightarrow}(x, y) &= \boldsymbol{p}^{\rightarrow}(x, y, y), \quad \text{and} \\ \boldsymbol{k}^{\rightarrow}(x, y) &= \boldsymbol{q}^{\rightarrow}(x, y, x). \end{aligned}$$

(4) ⇒ (1): The identities (Σ_2) ensure the direct decomposability of subfactor tolerances. Defining

$$\begin{aligned} \boldsymbol{r}(a, b, w^{\rightarrow}) &= \boldsymbol{s}_2(a, b, w_{n+1}, w_{n+2}, w_1, \ldots, w_n), \\ \boldsymbol{c}^{\rightarrow}(x, y) &= \boldsymbol{h}_1(x, y), \ldots, \boldsymbol{h}_n(x, y), y, y, \quad \text{and} \\ \boldsymbol{d}^{\rightarrow}(x, y) &= \boldsymbol{k}_1(x, y), \ldots, \boldsymbol{k}_n(x, y), y, x \end{aligned}$$

we get the identities from Theorem 1 (2).

Further, the identities (Σ_1) yield

$$\begin{aligned} x &= \boldsymbol{s}_1(x, z_1, \boldsymbol{f}^{\rightarrow}(x, z_1)), \\ z_2 &= \boldsymbol{s}_1(y, z_2, \boldsymbol{g}^{\rightarrow}(z_2, y)), \\ z_1 &= \boldsymbol{s}_1(z_1, x, \boldsymbol{f}^{\rightarrow}(x, z_1)). \\ z_2 &= \boldsymbol{s}_1(z_2, y, \boldsymbol{g}^{\rightarrow}(z_2, y)), \end{aligned}$$

which means that $\langle x, z_2\rangle \in [\langle z_1, z_2\rangle]\,T$ whenever $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\,T$. Similarly

$$\begin{aligned} y &= s_1(y, z_2, f^{\to}(y, z_2)), \\ z_1 &= s_1(x, z_1, g^{\to}(z_1, x)), \\ z_2 &= s_1(z_2, y, f^{\to}(y, z_2)), \\ z_1 &= s_1(z_1, x, g^{\to}(z_1, x)) \end{aligned}$$

follow from the identities (Σ_1). This establishes that $\langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\,T$ whenever $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\,T$.

Finally, we use again the identities (Σ_2). One easily checks that

$$\begin{aligned} x &= s_2(x, z_1, z_1, z_1, h^{\to}(x, z_1)), \\ y &= s_2(z_2, z_2, y, z_2, k^{\to}(z_2, y)), \\ z_1 &= s_2(z_1, x, z_1, z_1, h^{\to}(x, z_1)), \\ z_2 &= s_2(z_2, z_2, z_2, y, k^{\to}(z_2, y)), \end{aligned}$$

which proves that $\langle x, y\rangle \in [\langle z_1, z_2\rangle]\,T$ is a consequence of $\langle x, z_2\rangle, \langle z_1, y\rangle \in [\langle z_1, z_2\rangle]\,T$. Lemma 1 and Lemma 2 complete the proof.

Corollary 1. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable tolerances*;

(2) *$F_V(\mathbf{2}) \times F_V(\mathbf{2})$ has directly decomposable tolerances.*

3. DIRECTLY DECOMPOSABLE REFLEXIVE RELATIONS

In this section we generalize the above results to reflexive relations. Since the proofs of Theorems 3, 4 are very similar to those of Theorems 1, 2 we omit the details.

Theorem 3. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable subfactor reflexive relations*;

(2) *there exist binary terms $c_1, \ldots, c_n, d_1, \ldots, d_n$ and a $(1 + n)$-ary term u such that the identities*

$$\begin{aligned} x &= u(x, c^{\to}(x, y)), \\ y &= u(x, d^{\to}(x, y)), \\ y &= u(y, c^{\to}(x, y)) \end{aligned}$$

hold in V

Proof. Apply [1] and the proof of Theorem 1.

Theorem 4. *For a variety V the following conditions are equivalent*:

(1) *V has directly decomposable reflexive relations*;

(2) *V has directly decomposable relation classes*;

(3) *there exist ternary terms $p_1, \ldots, p_n, q_1, \ldots, q_n$ and a $(2 + n)$-ary term v such*

that the identities

$$x = \boldsymbol{v}(x, y, \boldsymbol{p}^{\rightarrow}(x, y, z)) ,$$
$$y = \boldsymbol{v}(x, y, \boldsymbol{q}^{\rightarrow}(x, y, z)) ,$$
$$z = \boldsymbol{v}(z, z, \boldsymbol{p}^{\rightarrow}(x, y, z)) ,$$
$$z = \boldsymbol{v}(z, z, \boldsymbol{q}^{\rightarrow}(x, y, z))$$

hold in V;

(4) *there exist binary terms* $f_1, \ldots, f_{n+1}, g_1, \ldots, g_{n+1}, h_1, \ldots, h_n, k_1, \ldots, k_n$ *and* $(2 + n)$*-ary terms* v_1, v_2 *such that the identities*

$$x = \boldsymbol{v}_1(x, \boldsymbol{f}^{\rightarrow}(x, y)) ,$$
$$x = \boldsymbol{v}_1(y, \boldsymbol{g}^{\rightarrow}(x, y)) ,$$
$$y = \boldsymbol{v}_1(y, \boldsymbol{f}^{\rightarrow}(x, y)) ,$$
$$x = \boldsymbol{v}_1(x, \boldsymbol{g}^{\rightarrow}(x, y)) ,$$
$$x = \boldsymbol{v}_2(x, y, \boldsymbol{h}^{\rightarrow}(x, y)) ,$$
$$y = \boldsymbol{v}_2(x, y, \boldsymbol{k}^{\rightarrow}(x, y)) ,$$
$$y = \boldsymbol{v}_2(y, y, \boldsymbol{h}^{\rightarrow}(x, y)) ,$$
$$x = \boldsymbol{v}_2(x, x, \boldsymbol{k}^{\rightarrow}(x, y))$$

hold in V.

Proof. (1) ⇒ (2) is trivial.

The implication (2) ⇒ (3) was already proved in [3; Thm 5, pp. 400–401].

The rest of the proof follows the same lines as in the proof of Theorem 2.

Corollary 2. *For a variety* V *the following conditions are equivalent*:

(1) V *has directly decomposable reflexive relations*;

(2) $F_V(\mathbf{2}) \times F_V(\mathbf{2})$ *has directly decomposable reflexive relations.*

Example 1. The variety $\boldsymbol{L}$ of all lattices satisfies all the above identities. This follows directly from the fact that $F_L(\mathbf{2}) \cong \mathbf{2} \times \mathbf{2}$.

4. CONCLUSION

The Mal'cev condition for varieties having *directly decomposable congruences* was given by G. A. Fraser and A. Horn in [6]. Using the method exhibited in Section 2 of this paper one easily checks that also the direct decomposability of congruences in varieties can be considered only on the square $F_V(\mathbf{2}) \times F_V(\mathbf{2})$. The simplification of the original Fraser-Horn identities is shown in [5].

References

[1] *Chajda, I.:* Lattices of compatible relations. Arch. Math. (Brno) *13* (1977), 89–96.

[2] *Duda, J.:* Directly decomposable compatible relations. Glasnik Matem. (Zagreb) *19* (1984), 225–229.

[3] *Duda, J.:* Varieties having directly decomposable congruence classes. Časopis pěstování matem. *111* (1986), 394—403.
[4] *Duda, J.:* Varieties with directly decomposable diagonal subalgebras form Mal'cev classes. Preprint.
[5] *Duda, J.:* Fraser-Horn identities can be writen in two variables. Algebra Univ. *26* (1989), 178—180.
[6] *Fraser, G. A.* and *Horn, A.:* Congruence relations in direct products. Proc. Amer. Math. Soc. *26* (1970), 390—394.
[7] *Grätzer, G.:* Universal Algebra. Second Expanded Edition. Springer-Verlag, Berlin–Heidelberg–New York, 1979.
[8] *Hagemann, J.:* Congruences on products and subdirect products of algebras. Preprint Nr. 219, TH-Darmstadt, 1975.
[9] *Niederle, J.:* Decomposability conditions for compatible relations. Czech. Math. Journal *33* (*108*) (1983), 522—524.

Author's address: 616 00 Brno 16, Kroftova 21, Czechoslovakia.