Jiří Rachůnek
Polars in autometrized algebras


Persistent URL: [http://dml.cz/dmlcz/102344](http://dml.cz/dmlcz/102344)

**Terms of use:**

© Institute of Mathematics AS CR, 1989

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
POLARS IN AUTOMETRIZED ALGEBRAS

Jiří RACHŮNEK, Olomouc

(Received January 20, 1987)

In the paper [3], K. L. N. Swamy introduced the notion of the autometrizied algebra which is a generalization, for example, of l-groups and Brouwerian algebras. Ideals in autometrizied algebras are studied by K. L. N. Swamy and N. P. Rao in [4], where the polars of ideals are also introduced. Prime ideals in autometrizied algebras are studied by the author in [2]. In this paper properties of polars in connections with ideals and prime ideals are discussed.

A system \( A = (A, +, \leq, *) \) is called an autometrizied algebra if

1. \((A, +)\) is a commutative semigroup with zero element 0;
2. \((A, \leq)\) is an ordered set and
   \[ \forall a, b, c \in A ; \quad a \leq b \Rightarrow a + c \leq b + c ; \]
3. \(* : A \times A \rightarrow A\) is a mapping such that
   \[ \forall a, b \in A ; \quad a \ast b \geq 0 \quad \text{and} \quad a \ast b = 0 \iff a = b , \]
   \[ \forall a, b \in A ; \quad a \ast b = b \ast a , \]
   \[ \forall a, b, c \in A ; \quad a \ast c \leq (a \ast b) + (b \ast c) . \]

If \((A, \leq)\) is a lattice and

\[ \forall a, b, c \in A ; \quad a + (b \lor c) = (a + b) \lor (a + c) , \]
\[ a + (b \land c) = (a + b) \land (a + c) , \]

then \( A \) is called a lattice algebra (an l-algebra).

If

\[ \forall a \in A ; \quad a \leq a \ast 0 , \]
\[ \forall a, b, c, d \in A ; \quad (a + c) \ast (b + d) \leq (a \ast b) + (c \ast d) , \]
\[ \forall a, b, c, d \in A ; \quad (a \ast c) \ast (b \ast d) \leq (a \ast b) + (c \ast d) , \]
\[ \forall a, b \in A ; \quad a \leq b \Rightarrow \exists x \geq 0 ; \quad a + x = b , \]

then we say that \( A \) is a normal algebra.

If

\[ \forall a \in A ; \quad a \geq 0 \Rightarrow a \ast 0 = a , \]

then \( A \) is called a semiregular algebra.
An ordered semigroup $A$ with zero element $0$ is said to be an \textit{interpolation semigroup} if
\[
\forall a, b, c \in A ; \quad [0 \leq a, b, c, a \leq b + c \Rightarrow \\
\exists 0 \leq b_1 \leq b, 0 \leq c_1 \leq c ; \quad a = b_1 + c_1].
\]

For instance, every commutative $l$-group and every Brouwerian algebra is an interpolation semigroup. (For $l$-groups see e.g. [1, p. 21], for Brouwerian algebras it follows from the distributivity.)

If $A = (A, +, \leq, *)$ is an autometrized algebra, $0 \neq I \subseteq A$, then $I$ is said to be an \textit{ideal} in $A$ if
\[
\forall a, b \in I ; \quad a + b \in I,
\]
\[
\forall a \in I , \quad x \in A ; \quad x \ast 0 \leq a \ast 0 \Rightarrow x \in I.
\]

Let us suppose that $A = (A, +, \leq, *)$ is an autometrized $l$-algebra, $a, b \in A$. We say that $a$ and $b$ are \textit{orthogonal} (notation $a \perp b$) whenever
\[
(a \ast 0) \land (b \ast 0) = 0.
\]

If $B \subseteq A$, then
\[
B^\perp = \{ x \in A ; x \perp b \text{ for all } b \in B \}
\]
is called the \textit{polar} of the set $B$.

We say that $C \subseteq A$ is a \textit{polar in} $A$ if there exists $B \subseteq A$ such that $C = B^\perp$. The set of all polars in an algebra $A$ will be denoted by $\mathcal{P}(A)$.

\textbf{Theorem 1.} Any polar in a normal interpolation autometrized $l$-algebra $A$ is an ideal in $A$.

\textbf{Proof.} Let $B \subseteq A$, $x, y \in B^\perp$, $b \in B$. Since $A$ is normal, we have
\[
[(x + y) \ast 0] \land (b \ast 0) \leq [(x \ast 0) + (y \ast 0)] \land (b \ast 0).
\]

But $A$ is also an interpolation algebra, hence we obtain
\[
0 \leq [(x + y) \ast 0] \land (b \ast 0) \leq [(x \ast 0) \land (b \ast 0)] + [(y \ast 0) \land (b \ast 0)] = 0,
\]
therefore $x + y \in B^\perp$.

If $x \in B^\perp$, $a \in A$, $a \ast 0 \leq x \ast 0$, then evidently $a \in B^\perp$.

For an autometrized algebra $A$ the set of all its ideals will be denoted by $\mathcal{I}(A)$. If $A$ is normal, then $\mathcal{I}(A)$ ordered by set inclusion is a complete algebraic lattice in which the infimum of any system of ideals is formed by the intersection of that system ([4, Theorem 1]). If $B \subseteq A$, then we denote the smallest ideal in $A$ containing $B$ by $I(B)$. For $a \in A$ we shall write $I(a)$ instead of $I(\{a\})$.

We have
\[
I(B) = \{ x \in A ; x \ast 0 \leq (b_1 \ast 0) + \ldots + (b_n \ast 0), b_1, \ldots, b_n \in B \},
\]
\[
I(a) = \{ x \in A ; x \ast 0 \leq m(a \ast 0), \text{ for some positive integer } m \}.
\]

\textbf{Theorem 2.} If $A$ is a semiregular normal interpolation autometrized $l$-algebra,
\( B \subseteq A, \text{ then} \)

\[
B^\perp = \{x \in A; I(x) \cap I(B) = \{0\}\}.
\]

**Proof.** a) Let \( x \in B^\perp, \ z \in I(x) \cap I(B) \). Then there exist an integer \( m \geq 0 \) and elements \( b_1, \ldots, b_n \in B \) such that

\[
z \ast 0 \leq m(x \ast 0),
\]

\[
z \ast 0 \leq (b_1 \ast 0) + \ldots + (b_n \ast 0).
\]

Since \( A \) is an interpolation algebra, it follows that

\[
0 \leq z \ast 0 \leq m(x \ast 0) \wedge [(b_1 \ast 0) + \ldots + (b_n \ast 0)] \leq
\]

\[
\leq [m(x \ast 0) \wedge (b_1 \ast 0)] + \ldots + [m(x \ast 0) \wedge (b_n \ast 0)] \leq
\]

\[
\leq m[(x \ast 0) \wedge (b_1 \ast 0)] + \ldots + m[(x \ast 0) \wedge (b_n \ast 0)] = 0 + \ldots + 0 = 0,
\]

hence \( z \ast 0 = 0 \), and this means that \( z = 0 \). Therefore \( I(x) \cap I(B) = \{0\} \).

b) Let us suppose that \( x \in A \) is such that \( I(x) \cap I(B) = \{0\} \). Let \( b \in B \). Let us denote \( c = (x \ast 0) \wedge (b \ast 0) \). Then the semiregularity of the algebra \( A \) implies

\[
c \ast 0 = c \leq x \ast 0,
\]

hence \( c \in I(x) \). Similarly

\[
c \ast 0 = c \leq b \ast 0,
\]

and thus \( c \in I(B) \).

But then \( c = 0 \) by the assumption, therefore \( x \in B^\perp \).

**Corollary.** Any polar in a semiregular normal interpolation autometritized l-algebra \( A \) is the polar of an ideal in \( A \).

**Proof.** If \( B^\perp \) is a polar in \( A \), then Theorem 2 implies \( B^\perp = I(B^\perp) \).

An ideal \( I \) in an autometritized algebra \( A \) is called a prime ideal in \( A \) if

\[
\forall J, K \in \mathcal{I}(A); \ J \cap K = I \Rightarrow J = I \text{ or } K = I.
\]

In addition, if \( A \) is a semiregular normal autometritized l-algebra, \( I \) a prime ideal in \( A \), then

\[
\forall a, b \in A; \ 0 = a \wedge b \Rightarrow a \in I \text{ or } b \in I.
\]

([2, Theorem 4].)

We denote the set of all prime ideals in \( A \) by \( \mathcal{I}_p(A) \).

**Theorem 3.** If \( A \) is an autometritized algebra, \( I \in \mathcal{I}(A), a \in A, a \notin I \), then there exists a prime ideal in \( A \) containing \( I \) and not containing \( a \).

**Proof.** Let \( a \in A, I \in \mathcal{I}(A), a \notin I \). Let us denote

\[
Z = \{J \in \mathcal{I}(A); I \subseteq J, a \notin J\}.
\]

Let us consider an arbitrary linearly ordered system \( (J_\alpha; \alpha \in \Gamma) \) of elements in \( Z \) and let

\[
K = \bigcup_{\alpha \in \Gamma} J_\alpha.
\]

683
If \(a, b \in K\), then there exist \(\alpha_1, \alpha_2 \in \Gamma\) such that \(a \in J_{\alpha_1}\), \(b \in J_{\alpha_2}\), and, for example, \(J_{\alpha_1} \supseteq J_{\alpha_2}\). Hence \(a, b \in J_{\alpha_1}\), and so \(a + b \in J_{\alpha_1} \subseteq K\).

It is obvious that if \(x \in A\), \(a \in K\), \(x \cdot 0 \leq a \cdot 0\), then \(x \in K\).

Thus \(K\) is an ideal in \(A\) and \(a \notin K\), therefore \(K \in Z\). This means that \(Z\) is an inductive set, therefore \(Z\) contains a maximal element.

Let us consider any maximal element \(L\) in \(Z\). Let \(M, N \in \mathcal{J}(A)\), \(M \cap N = L\), and let \(M \supseteq L\), \(N \supseteq L\). Then \(a \in M\), \(a \in N\), hence \(a \in M \cap N = L\), a contradiction.

Therefore \(L \in \mathcal{J}(A)\).

**Theorem 4.** For any element \(a \neq 0\) in an autometrized algebra \(A\) there exists a prime ideal in \(A\) not containing the element \(a\).

**Proof.** Since \([0] \in \mathcal{J}(A)\), the assertion is an immediate consequence of Theorem 3.

**Theorem 5.** If \(A\) is a semiregular autometrized \(l\)-algebra, \(B \subseteq A\), then \(B^\perp\) is equal to the intersection of all prime ideals in \(A\) not containing \(B\).

**Proof.** Let \(C\) be the intersection of all prime ideals in \(A\) not containing \(B\).

Let \(x \in B^\perp\), \(I \in \mathcal{J}_p(A)\), \(B \subseteq I\), \(b \in B \setminus I\). Then \((x \cdot 0) \wedge (b \cdot 0) = 0\) and consequently \(b \cdot 0 \notin I\). (If \(b \cdot 0 \in I\), then also \(b \in I\), because in the case of a semiregular algebra we have \(b \cdot 0 = (b \cdot 0) \cdot 0\).) Since \(I \in \mathcal{J}_p(A)\), it follows \(x \cdot 0 \in I\), and so also \(x \in I\). Therefore \(B^\perp \subseteq C\).

Conversely, let \(x \notin B^\perp\), i.e., let there exist \(b \in B\) such that \((x \cdot 0) \wedge (b \cdot 0) > 0\).

Let us consider \(I \in \mathcal{J}_p(A)\) such that \((x \cdot 0) \wedge (b \cdot 0) \notin I\). Then \(x \cdot 0 \notin I\), \(b \cdot 0 \notin I\).

The semiregularity of \(A\) yields \(x \notin I\), \(b \notin I\). Thus \(x \notin I\), \(B \notin I\), hence \(x \notin C\). But this means \(C \subseteq B^\perp\).

**Corollary.** Any polar in a semiregular normal autometrized \(l\)-algebra \(A\) is an ideal in \(A\).

Now let \(A\) be a semiregular normal autometrized \(l\)-algebra. Then \(\mathcal{J}(A)\) is a complete algebraic Brouwerian lattice and for \(I \in \mathcal{J}(A)\) we have that \(I^\perp\) is the pseudo-complement of \(I\) in \(\mathcal{J}(A)\). Further, the mapping that to any \(I \in \mathcal{J}(A)\) assigns \(I^{\perp \perp}\) is a closure operator on \(\mathcal{J}(A)\). ([4, Theorem 6, Lemma 7, Theorem 7].)

**Theorem 6.** a) If \(B_a \subseteq A\), \(\alpha \in \Gamma\), then

\[
\bigcap_{\alpha \in \Gamma} B_a^\perp = \left( \bigcup_{\alpha \in \Gamma} B_a \right)^\perp
\]

b) If \(B_a \in \mathcal{J}(A)\), \(\alpha \in \Gamma\), then

\[
\bigcap_{\alpha \in \Gamma} B_a^\perp = \left( \bigvee_{\alpha \in \Gamma} B_a \right)^\perp
\]

for the supremum in \(\mathcal{J}(A)\).

**Proof.** a) Let \(x \in \bigcap_{\alpha \in \Gamma} B_a^\perp\). Then \(x \perp b\) for each \(b \in \bigcup_{\alpha \in \Gamma} B_a\), hence \(\bigcap_{\alpha \in \Gamma} B_a^\perp \subseteq \left( \bigcup_{\alpha \in \Gamma} B_a \right)^\perp\).

Conversely, if \(y \in \left( \bigcup_{\alpha \in \Gamma} B_a \right)^\perp\), then \(y \perp b\) for all \(b \in \bigcup_{\alpha \in \Gamma} B_a\), hence \(x \in \bigcap_{\alpha \in \Gamma} B_a^\perp\), and so \(\left( \bigcup_{\alpha \in \Gamma} B_a \right)^\perp \subseteq \bigcap_{\alpha \in \Gamma} B_a^\perp\).

b) The assertion now follows immediately from Corollary of Theorem 2.

684
Corollary. a) If $B \subseteq A$, then
\[ B \subseteq B^{\perp\perp}, \quad B^{\perp} = B^{\perp\perp}. \]
b) $B \subseteq A$ is a polar in $A$ if and only if $B = B^{\perp\perp}$.

Now by [4, Theorem 7] we obtain that $\mathcal{P}(A)$ ordered by set inclusion is a Boolean algebra. If $B \in \mathcal{P}(A)$, then its complement is evidently formed by its polar $B^{\perp}$. In addition, by Theorem 6 $\mathcal{P}(A)$ is a Moore system, hence $\mathcal{P}(A)$ is a complete lattice.

**Theorem 7.** If $B_{\alpha}, \alpha \in \Gamma$, are any *polars* in $A$, then in the complete lattice $\mathcal{P}(A)$ we have
\[ \bigwedge_{\alpha \in \Gamma} B_{\alpha} = \bigcap_{\alpha \in \Gamma} B_{\alpha}, \quad \bigvee_{\alpha \in \Gamma} B_{\alpha} = \left( \bigcap_{\alpha \in \Gamma} B_{\alpha}^{\perp} \right)^{\perp}. \]

**Proof.** The first equality follows from Theorem 6.

Further, for any $C \in \mathcal{P}(A)$ we have $C \supseteq \bigcup_{\alpha \in \Gamma} B_{\alpha}$ if and only if $C^{\perp} \subseteq \bigcup_{\alpha \in \Gamma} B_{\alpha}^{\perp}$ and this is satisfied by Theorem 6 if and only if $C \supseteq \left( \bigcap_{\alpha \in \Gamma} B_{\alpha}^{\perp} \right)^{\perp}$. Hence the second equality follows.

**Theorem 8.** The mapping that to any $I \in \mathcal{I}(A)$ assigns $I^{\perp\perp}$ is a surjective lattice homomorphism of $\mathcal{I}(A)$ onto $\mathcal{P}(A)$.

**Proof.** The assertion follows immediately from [4, Theorem 7] and from Corollary of Theorem 6.

Let us denote $a^{\perp} = \{a\}^{\perp}$ for $a \in A$.

**Theorem 9.** If $A$ is an interpolation semiregular normal autometrized $l$-algebra, $a, b \in A$, then
\[ a^{\perp\perp} \cap b^{\perp\perp} = ((a \ast 0) \land (b \ast 0))^{\perp\perp}, \]
\[ a^{\perp\perp} \lor b^{\perp\perp} = ((a \ast 0) \lor (b \ast 0))^{\perp\perp}. \]

**Proof.** If $A$ is an interpolation algebra, then by [2, Proposition 2] we have
\[ I(a) \cap I(b) = I((a \ast 0) \land (b \ast 0)), \]
\[ I(a) \lor I(b) = I((a \ast 0) \lor (b \ast 0)). \]

Therefore the assertion is a consequence of Theorem 8.

**References**


*Author's address: 771 46 Olomouc, Leninova 26, Czechoslovakia (Přírodo-vědecká fakulta UP).*