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COHERENCE IN VARIETIES OF ALGEBRAS

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The concept of coherent subalgebras comes from D. Geigher [5]. Coherent congruence blocks and coherent ideals are introduced in a similar way. The present paper deals with varieties of algebras having coherent congruence blocks and with varieties of algebras having coherent ideals. Using Mal'cev's characterizations of these two classes we deduce their relationships to permutable, n-permutable, modular, regular, and permutable at 0 varieties. For the sake of completeness we start with a brief review of D. Geigher's results [5].

1. A NOTE ON COHERENT SUBALGEBRAS

Definition 1. An algebra A has *coherent subalgebras* if for any subalgebra B of A the assumption $[b]\,\Theta \subseteq B$ for some $b \in B$, $\Theta \in \operatorname{Con} A$, implies $[x]\,\Theta \subseteq B$ for every $x \in B$.

A variety V has *coherent subalgebras* whenever each V-algebra has this property.

Theorem 1 (D. Geigher [5]). *For a variety V the following conditions are equivalent:*

(1) *V has coherent subalgebras;*

(2) *there exist an integer n, ternary terms $b_1, \ldots, b_n$, and a $(1 + n)$-ary term r such that the identities*

$$y = r(x, b_1(x, y, z), \ldots, b_n(x, y, z))\,,$$
$$z = b_i(x, x, z)\,, \quad 1 \leqq i \leqq n\,,$$

hold in V.

Corollary 1 (D. Geigher [5]). *Any variety with coherent subalgebras is permutable.*

Proof. Apparently, $p(x, y, z) = r(z, b_1(y, x, z), \ldots, b_n(y, x, z))$ is a Mal'cev ternary term entailing the permutability of V, see [9].

Corollary 2 (D. Geigher [5]). *Any variety with coherent subalgebras is regular.*

Proof. Since $x = \boldsymbol{r}(x, \boldsymbol{b}_1(x, x, z), \ldots, \boldsymbol{b}_n(x, x, z)) = \boldsymbol{r}(x, z, \ldots, z)$ we find that $(z = \boldsymbol{b}_i(x, y, z),\ 1 \leqq i \leqq n) \Leftrightarrow x = y$ which means that $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ are Csákány terms ensuring the regularity of V, see [2].

2. COHERENT CONGRUENCE BLOCKS

Replacing subalgebras by congruence blocks in the above Definition 1 the concept of coherent congruence blocks is obtained. Varieties of algebras having coherent congruence blocks were already studied by H. Werner and R. Wille [10]. In this section we first simplify the Mal'cev condition from [10].

Definition 2. An algebra A has *coherent congruence blocks* if for any congruence block C in A the assumption $[c]\ \Theta \subseteq C$ for some $c \in C$, $\Theta \in \operatorname{Con} A$, implies $[x]\ \Theta \subseteq C$ for every $x \in C$.

A variety V has *coherent congruence blocks* whenever each V-algebra has this property.

The symbol $\Theta(S)$ denotes the least congruence on A containing the subset $S \subseteq A \times A$.

Theorem 2. *For a variety V the following conditions are equivalent:*

(1) *V has coherent congruence blocks;*

(2) *there exist an integer n, ternary terms $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, and ternary terms $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$ such that the identities*

$$x = \boldsymbol{s}_1(x, y, \boldsymbol{b}_1(x, y, x)),$$
$$\boldsymbol{s}_i(x, y, x) = \boldsymbol{s}_{i+1}(x, y, \boldsymbol{b}_{i+1}(x, y, x)), \quad 1 \leqq i < n,$$
$$y = \boldsymbol{s}_n(x, y, x), \quad \textit{and}$$
$$z = \boldsymbol{b}_i(x, x, z), \quad 1 \leqq i \leqq n,$$

hold in V;

(3) *there exist an integer n and ternary terms $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ such that*

$$z = \boldsymbol{b}_i(x, x, z), \quad 1 \leqq i \leqq n, \quad \textit{and}$$
$$(x = \boldsymbol{b}_i(x, y, x), \quad 1 \leqq i \leqq n) \Rightarrow x = y$$

hold in V.

Proof. (1) ⇒ (2): Let $A = F_V(x, y, z)$ be the V-free algebra with free generators x, y, and z. Take the congruence $\Theta(x, y)$ and denote by C the congruence block $[x]\ \Theta(\{x\} \times [z]\ \Theta(x, y))$. Then $[z]\ \Theta(x, y) \subseteq C$, by construction. Hence $[x]\ \Theta(x, y) \subseteq C$, by hypothesis. In this way we find that $y \in C$, or equivalently, $\langle x, y\rangle \in \Theta(\{x\} \times [z]\ \Theta(x, y))$. Since $\Theta(-)$ is evidently an algebraic closure operator on $A \times A$ we can write $\langle x, y\rangle \in \Theta(\langle x, b_1\rangle, \ldots, \langle x, b_n\rangle)$ for some $b_i \in [z]\ \Theta(x, y)$, $1 \leqq i \leqq n$. Thus $b_i = \boldsymbol{b}_i(x, y, z)$ and $z = \boldsymbol{b}_i(x, x, z)$, $1 \leqq i \leqq n$. Further, applying the functional description of the congruence $\Theta(\langle x, b_1\rangle, \ldots, \langle x, b_n\rangle)$ from [4] we get

$$x = \boldsymbol{q}_1(x, y, z, x, \boldsymbol{b}_1(x, y, z)),$$
$$\boldsymbol{q}_i(x, y, z, \boldsymbol{b}_i(x, y, z), x) = \boldsymbol{q}_{i+1}(x, y, z, x, \boldsymbol{b}_{i+1}(x, y, z)), \quad 1 \leqq i < n,$$
$$y = \boldsymbol{q}_n(x, y, z, \boldsymbol{b}_n(x, y, z), x)$$

for some quinary terms $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

Defining $\boldsymbol{p}_i(x, y, z) = \boldsymbol{q}_i(x, z, x, \boldsymbol{b}_i(y, z, x), \boldsymbol{b}_i(x, y, x))$, $1 \leqq i \leqq n$, we obtain the identities

$$\begin{aligned} \boldsymbol{p}_1(x, z, z) &= \boldsymbol{q}_1(x, z, x, \boldsymbol{b}_1(z, z, x), \boldsymbol{b}_1(x, z, x)) = \\ &= \boldsymbol{q}_1(x, z, x, x, \boldsymbol{b}_1(x, z, x)) = x, \\ \boldsymbol{p}_i(x, x, z) &= \boldsymbol{q}_i(x, z, x, \boldsymbol{b}_i(x, z, x), \boldsymbol{b}_i(x, x, x)) = \boldsymbol{q}_i(x, z, x, \boldsymbol{b}_i(x, z, x), x) = \\ &= \boldsymbol{q}_i(x, z, x, x, \boldsymbol{b}_{i+1}(x, z, x)) = \boldsymbol{p}_{i+1}(x, z, z), \quad 1 \leqq i < n, \\ \boldsymbol{p}_n(x, x, z) &= \boldsymbol{q}_n(x, z, x, \boldsymbol{b}_n(x, z, x), \boldsymbol{b}_n(x, x, x)) = \\ &= \boldsymbol{q}_n(x, z, x, \boldsymbol{b}_n(x, z, x), x) = z, \end{aligned}$$

which prove the $(n + 1)$-permutability of $\boldsymbol{V}$, see [8]. Thanks to this fact we can suppose that the terms $\boldsymbol{q}_i$, $1 \leqq i \leqq n$, do not depend on the fourth variable, see [4] again. Thus

$$x = \boldsymbol{q}_1(x, y, z, \boldsymbol{b}_1(x, y, z)),$$
$$\boldsymbol{q}_i(x, y, z, x) = \boldsymbol{q}_{i+1}(x, y, z, \boldsymbol{b}_{i+1}(x, y, z)), \quad 1 \leqq i < n,$$
$$y = \boldsymbol{q}_n(x, y, z, x).$$

Setting $x = z$ we conclude that

$$x = \boldsymbol{s}_1(x, y, \boldsymbol{b}_1(x, y, x)),$$
$$\boldsymbol{s}_i(x, y, x) = \boldsymbol{s}_{i+1}(x, y, \boldsymbol{b}_{i+1}(x, y, x)), \quad 1 \leqq i < n,$$
$$y = \boldsymbol{s}_n(x, y, x)$$

for some ternary terms $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n$, as required.

(2) ⇒ (3): Immediate.

(3) ⇒ (1): Let C denote the congruence block $[x]\,\Psi$ in an algebra $A \in \boldsymbol{V}$. Suppose further that $[c]\,\Theta \subseteq C$ for some $c \in C$, $\Theta \in \operatorname{Con} A$. It suffices to verify the inclusion $[x]\,\Theta \subseteq C$. To do this take an arbitrary element $y \in [x]\,\Theta$. Then $\Theta(x, y) \subseteq \Theta$ and so $[c]\,\Theta(x, y) \subseteq [c]\,\Theta \subseteq C$ holds. Consequently $\boldsymbol{b}_i(x, y, c) \in [c]\,\Theta(x, y) \subseteq C = [x]\,\Psi$, $1 \leqq i \leqq n$. Since $\langle x, c\rangle \in \Psi$ we get that $\langle \boldsymbol{b}_i(x, y, x), x\rangle \in \Psi$, $1 \leqq i \leqq n$. Now denote by $\bar{A}$ the factor algebra $A/\Psi \in \boldsymbol{V}$ consisting of elements $\bar{a} = [a]\,\Psi$, $a \in A$. The equalities $\boldsymbol{b}_i(\bar{x}, \bar{y}, \bar{x}) = \bar{x}$, $1 \leqq i \leqq n$, imply $\bar{x} = \bar{y}$, by hypothesis. In other words we have $\langle x, y\rangle \in \Psi$, i.e. $y \in [x]\,\Psi = C$, as required. The proof is complete.

We have already proved

Corollary 3. *Any variety with coherent congruence blocks is n-permutable for some $n > 1$.*

Further, we state

Corollary 4. *Any variety with coherent congruence blocks is congruence modular.*

Proof. Assume the identities (2) of Theorem 2. Introduce quaternary terms $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_{2n+1}$ via

$$\boldsymbol{d}_0(x, y, z, w) = x ,$$
$$\boldsymbol{d}_{2i-1}(x, y, z, w) = \boldsymbol{s}_i(x, w, \boldsymbol{b}_i(y, z, x)) , \quad 1 \leqq i \leqq n ,$$
$$\boldsymbol{d}_{2i}(x, y, z, w) = \boldsymbol{s}_i(x, w, x) , \quad 1 \leqq i \leqq n , \quad \text{and}$$
$$\boldsymbol{d}_{2n+1}(x, y, z, w) = w .$$

Then

$$\boldsymbol{d}_0(x, x, w, w) = x = \boldsymbol{s}_1(x, w, \boldsymbol{b}_1(x, w, x)) = \boldsymbol{d}_1(x, x, w, w) ,$$
$$\boldsymbol{d}_{2i-1}(x, y, y, w) = \boldsymbol{s}_i(x, w, \boldsymbol{b}_i(y, y, x)) = \boldsymbol{s}_i(x, w, x) = \boldsymbol{d}_{2i}(x, y, y, w) , \ 1 \leqq i \leqq n ,$$
$$\boldsymbol{d}_{2i}(x, x, w, w) = \boldsymbol{s}_i(x, w, x) = \boldsymbol{s}_{i+1}(x, w, \boldsymbol{b}_{i+1}(x, w, x)) = \boldsymbol{d}_{2i+1}(x, x, w, w) ,$$
$$1 \leqq i \leqq n ,$$
$$\boldsymbol{d}_{2n}(x, x, w, w) = \boldsymbol{s}_n(x, w, x) = w = \boldsymbol{d}_{2n+1}(x, y, z, w) ,$$

and, finally,

$$\boldsymbol{d}_j(x, y, y, x) = x , \quad 0 \leqq j \leqq 2n , \quad \text{since}$$
$$\boldsymbol{d}_0(x, y, y, x) = x , \quad \text{by definition} ,$$
$$\boldsymbol{d}_1(x, y, y, x) = \boldsymbol{s}_1(x, x, \boldsymbol{b}_1(y, y, x)) = \boldsymbol{s}_1(x, x, x) = x ,$$
$$\boldsymbol{d}_2(x, y, y, x) = \boldsymbol{s}_1(x, x, x) = x ,$$
$$\boldsymbol{d}_3(x, y, y, x) = \boldsymbol{s}_2(x, x, \boldsymbol{b}_2(y, y, x)) = \boldsymbol{s}_2(x, x, x) = x ,$$
$$\ldots$$
$$\boldsymbol{d}_{2n}(x, y, y, x) = \boldsymbol{s}_n(x, x, x) = x ,$$
$$\boldsymbol{d}_{2n+1}(x, y, y, x) = x , \quad \text{by definition} .$$

In this way we have proved that $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_{2n+1}$ are Day terms entailing the modularity of $\boldsymbol{V}$, see [3].

Corollary 5 (H. Werner, R. Wille [10]). *Any permutable variety has coherent congruence blocks.*

Proof. Let $\boldsymbol{p}$ be a Mal'cev ternary term, see [9]. Apply Theorem 2 (3) with $n = 1$, $\boldsymbol{b}_1(x, y, z) = \boldsymbol{p}(y, x, z)$.

Proposition 1 (H. Werner, R. Wille [10]). *Any regular algebra has coherent congruence blocks.*

Proof. Immediate.

3. COHERENT IDEALS

The concept of an ideal in an algebra was introduced in [1, 7] as follows:

Definition 3. Let A an algebra with nullary operation 0. An $(m + n)$-ary term $\boldsymbol{t}$ of A is called an *ideal term* whenever $\boldsymbol{t}(a_1, \ldots, a_m, 0, \ldots, 0) = 0$ holds for any $a_1, \ldots, a_m \in A$.

A subset $\emptyset \in I$ of A is called an *ideal* in A whenever $\boldsymbol{t}(a_1, \ldots, a_m, i_1, \ldots, i_n) \in I$ for any $a_1, \ldots, a_m \in A$, $i_1, \ldots, i_n \in I$, and any ideal term $\boldsymbol{t}$ of A.

The symbol $I(S)$ denotes the least ideal in an algebra A containing the subset $S \subseteq A$.

Definition 4. A variety $\boldsymbol{V}$ with nullary operation 0 is called *permutable at* 0 whenever the identities

$$\boldsymbol{s}(x, x) = 0\,, \quad \boldsymbol{s}(x, 0) = x$$

hold in $\boldsymbol{V}$ for some binary term $\boldsymbol{s}$.

Lemma 1 (H.-P. Gumm, A. Ursini [7]). *Let A be an algebra from a variety permutable at* 0. *Then any ideal in A is a congruence block in A.*

Proof. [7; p. 49].

Definition 5. An algebra A with nullary operation 0 has *coherent ideals* whenever for any ideal I in A the assumption $[i]\,\Theta \subseteq I$ for some $i \in I$, $\Theta \in \operatorname{Con} A$, implies $[x]\,\Theta \subseteq I$ for every $x \in I$.

A variety $\boldsymbol{V}$ with nullary operation 0 has *coherent ideals* whenever each $\boldsymbol{V}$-algebra has this property.

Theorem 3. *Let $\boldsymbol{V}$ be a variety with nullary operation* 0. *The following conditions are equivalent*:

(1) *$\boldsymbol{V}$ has coherent ideals*;

(2) *there exist an integer n, ternary terms $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$, and a $(1+n)$-ary term $\boldsymbol{v}$ such that the identities*

$$y = \boldsymbol{v}(y, \boldsymbol{b}_1(0, y, 0), \ldots, \boldsymbol{b}_n(0, y, 0))\,,$$
$$0 = \boldsymbol{v}(y, 0, \ldots, 0)\,, \quad and \quad z = \boldsymbol{b}_i(x, x, z)\,, \quad 1 \leqq i \leqq n\,,$$

hold in $\boldsymbol{V}$.

Proof. (1) $\Rightarrow$ (2): Consider the ideal $I(\{x\} \cup [z]\,\Theta(x, y))$ in the $\boldsymbol{V}$-free algebra $F_{\boldsymbol{V}}(x, y, z)$. Since $[z]\,\Theta(x, y) \subseteq I(\{x\} \cup [z]\,\Theta(x, y))$ we have $y \in [x]\,\Theta(x, y) \subseteq I(\{x\} \cup [z]\,\Theta(x, y))$, by hypothesis. This means that

$$y = \boldsymbol{u}(x, y, z, x, b_1(x, y, z), \ldots, b_n(x, y, z))\,,$$
$$0 = \boldsymbol{u}(x, y, z, 0, 0, \ldots, 0)$$

for some $(4 + n)$-ary term $\boldsymbol{u}$ and suitable $b_i \in [z]\,\Theta(x, y)$, $1 \leqq i \leqq n$. Then $b_i = \boldsymbol{b}_i(x, y, z)$ and $z = \boldsymbol{b}_i(x, x, z)$, $1 \leqq i \leqq n$. The remaining identities from (2) follow by setting $z = x = 0$.

(2) $\Rightarrow$ (1): One can easily verify that the binary term $s(x, y) = v(x, b_1(y, x, 0), \ldots$ $\ldots, b(y, x, 0))$ satisfies the identities $s(x, x) = 0$ and $s(x, 0) = x$. Thus an arbitrary ideal I in $A \in V$ is a congruence block $I = [0]\, \Psi$ for some $\Psi \in \text{Con}\, A$, see Lemma 1. Suppose further that $[i]\, \Theta \subseteq I$ for some $i \in I$, $\Theta \in \text{Con}\, A$. We want to prove that $[x]\, \Theta \subseteq I$ for any $x \in I$. To this end take an arbitrary element $y \in [x]\, \Theta$. Then $\langle x, y\rangle \in \Theta$ implies $\Theta(x, y) \subseteq \Theta$. Since $b_j(x, y, i) \in [i]\, \Theta(x, y)$, $1 \leqq j \leqq n$, we have also $b_j(x, y, i) \in [i]\, \Theta \subseteq I = [0]\, \Psi$, $1 \leqq j \leqq n$. Combining this with $\langle 0, x\rangle$, $\langle 0, i\rangle \in \Psi$ the conclusion $b_j(0, y, 0) \in [0]\, \Psi = I$, $1 \leqq j \leqq n$, follows. Consequently $y = v(y, b_1(0, y, 0), \ldots, b_n(0, y, 0)) \in I$, as required.

Corollary 6. *Let V be a variety with nullary operation* 0. *Then V is permutable at* 0 *whenever V has coherent ideals.*

Proof. See the proof of Theorem 3.

Corollary 7. *Let V be a permutable variety with nullary operation* 0. *Then V has coherent ideals.*

Proof. Let p be a Mal'cev ternary term entailing the permutability of V. Define $n = 3$, $b_1(x, y, z) = b_2(x, y, z) = b_3(x, y, z) = p(y, x, z)$, and $v(x, y, z, w) = p(y, z, w)$.

Then

$$v(y, b_1(0, y, 0), b_2(0, y, 0), b_3(0, y, 0)) = p(p(y, 0, 0), p(y, 0, 0), p(y, 0, 0)) = y\,,$$

$$v(y, 0, 0, 0) = p(0, 0, 0) = 0\,, \quad \text{and}$$

$$b_i(x, x, z) = p(x, x, z) = z\,, \quad 1 \leqq i \leqq 3\,.$$

Theorem 3 (2) completes the proof.

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