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ON INTEGRATION IN BANACH SPACES, XI
(INTEGRATION WITH RESPECT TO POLYMEASURES)

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INTRODUCTION

In the case of integration with respect to an operator valued measure \( m: \mathcal{P} \to L(X, Y) \) countably additive in the strong operator topology, \( \mathcal{P} \)-measurable functions \( f: T \to X \) with continuous \( L_1 \)-pseudonorm \( \hat{m}(f, \cdot): \sigma(\mathcal{P}) \to [0, +\infty] \) form a complete pseudonormed linear space \( L^1(m) \), which shares many important properties of the classical \( L^1(\mu) \) spaces, see Parts II – VII. In particular, the Lebesgue Dominated Convergence Theorem (LDCT) holds in \( L^1(m) \), see Theorem 11.17.

Concerning integration with respect to a \( d \)-polymeasure \( \Gamma: \mathcal{P}_d \to L^d(X^d; Y) \) separately countably additive in the strong operator topology, in Theorem IX.7 we extended the LDCT to the class \( \mathcal{L}_1(\Gamma) \) of integrable \( d \)-tuples of functions \( (f_i) = (f_1, \ldots, f_d) \) whose multiple \( L_1 \)-gauge \( \hat{\Gamma}(f_i, (\cdot)): \sigma(\mathcal{P}_d) \to [0, +\infty] \) is separately continuous, see Definition 2 below. If \( \kappa_0 \subseteq Y \), then \( f \in L^1(m) \) if and only if \( f \) is \( \mathcal{P} \)-measurable and \( \hat{m}(f, T) < +\infty \). For \( d > 1 \) the analog is not true for the class \( \mathcal{L}_1(\Gamma) \). Nonetheless, it is true for the greater class \( \mathcal{L}_1(\Gamma) \) introduced by Definition 3. Namely, in Theorem 5 we prove the ,,if'' part, and, postponing the case of dimensions \( d > 2 \) to the forthcoming Part XIII, in Theorem 8 we prove the implication \( (f_i) \in \mathcal{L}_1(\Gamma) \Rightarrow \hat{\Gamma}((f_i), (T_i)) < +\infty \) for \( d = 2 \).

That \( \mathcal{L}_1(\Gamma) \) is the ,,right'' class is confirmed by Theorem 6 (Fubini theorem in \( \mathcal{L}_1(\Gamma) \)) and Theorem 10 (LDCT in \( \mathcal{L}_1(\Gamma) \)). As a byproduct we explain why a third of the definition of strict MT integrability in \([1]\) is enough, see the paragraph after Corollary 3 of Theorem 6.

We shall use freely the notation and concepts of the previous parts, treated as chapters, particularly the abbreviated notation from Part VIII.

THE CLASSES \( \mathcal{L}_1(\Gamma) \) AND \( \mathcal{L}_1(\Gamma) \)

Throughout this paper, if not specified otherwise, we assume that \( \Gamma: \mathcal{P}_d \to L^d(X^d; Y) \) is a given operator valued \( d \)-polymeasure separately countably additive in the strong operator topology with locally \( \sigma \)-finite semivariation \( \hat{\Gamma} \) on \( X\sigma(\mathcal{P}_d) \), see the beginning of Part IX = \([13]\).
Let us first introduce a useful notion.

**Definition 1.** For each \( i = 1, \ldots, d \) let \( f_i, f_{i,n} : T_i \to X_i, n = 1, 2, \ldots \) be \( \mathcal{P}_r \)-measurable functions. We say that the sequence of \( d \)-tuples \((f_{i,n})\), \( n = 1, 2, \ldots \) converges \( \Gamma \)-almost everywhere, shortly \( \Gamma \)-a.e., to the \( d \)-tuple \((f_i)\) if there are sets \( N_i \in \sigma(\mathcal{P}_i) \) \( i = 1, \ldots, d \) such that \( \Gamma(N_1, T_2, \ldots, T_d) = \ldots = \Gamma(T_1, \ldots, T_{d-1}, N_d) = 0 \), and \( f_{i,n}(t_i) \to f_i(t_i) \) for each \( t_i \in T_i - N_i, i = 1, \ldots, d \).

Obviously, in our previous theorems we may replace convergence everywhere of \( d \)-tuples of measurable or integrable functions by convergence \( \Gamma \)-a.e.

The title of this section indicates that there are two worthwhile generalizations of the space \( \mathcal{L}_1(m) \). Having in mind the notions from Parts I and II let us recall that a function \( g : T \to X \) belongs to \( \mathcal{L}_1(m) \) if \( g \) is \( \mathcal{P} \)-measurable and its \( L_1 \)-pseudo-norm \( \hat{m}(g, \cdot) : \sigma(\mathcal{P}) \to [0, +\infty] \) is continuous (equivalently, exhaustive). By Corollary of Theorem II.5 then \( \hat{m}(g, T) < +\infty \). (Here is an elementary proof of this fact: Put \( G = \{ t \in T, g(t) \neq 0 \} \), take \( G_k \in \mathcal{P}, k = 1, 2, \ldots \) such that \( G_k \uparrow G \) and \( \hat{m}(G_k) < < +\infty \) for each \( k = 1, 2, \ldots \). Define \( G'_k = G_k \cap \{ t \in T, |g(t)| \leq k \}, k = 1, 2, \ldots \). Then \( G - G'_k \to \emptyset \), hence there is a \( k_1 \) such that \( \hat{m}(g, G - G'_k) < 1 \). But then \( \hat{m}(g, T) = \hat{m}(g, G) \leq \hat{m}(g, G'_k) + 1 \leq k_1 \hat{m}(G'_k) + 1 < +\infty \).) This suggests the following, as we shall see, ,,strong” generalization of \( \mathcal{L}_1(m) \).

**Definition 2.** Let \( g_i : T_i \to X_i, i = 1, \ldots, d \). We say that the \( d \)-tuple \((g_i)\) belongs to \( \mathcal{L}_1(\Gamma) \) if \( g_i \) is \( \mathcal{P}_r \)-measurable for each \( i = 1, \ldots, d \), and the \( L_1 \)-gauge \( \hat{m}([g_i], (\cdot)) : \mathcal{X}(\mathcal{P}_i) \to [0, +\infty] \) is separately continuous (equivalently, separately exhaustive). By Theorem VIII.6 then \( \hat{m}([g_i], (T_i)) < +\infty \).

The last fact may be again proved in an elementary way. The following lemma is also immediate.

**Lemma 1.** Let \((g_i) \in \mathcal{L}_1(\Gamma)\). Then:

1) If \((f_1, g_2, \ldots, g_d) \in \mathcal{L}_1(\Gamma)\), then \((f_1 + g_1, g_2, \ldots, g_d) \in \mathcal{L}_1(\Gamma)\). The analogs hold for the coordinates \( i = 2, \ldots, d \).

2) If \( f_i : T_i \to X_i, i = 1, \ldots, d \), are \( \mathcal{P}_r \)-measurable and \( |f_i(t_i)| \leq |g_i(t_i)| \) for \( \Gamma \)-almost every \( t_i \in T_i \), then \((f_i) \in \mathcal{L}_1(\Gamma)\).

3) If \( \varphi_i : T_i \to K_i, i = 1, \ldots, d \), are bounded scalar valued \( \mathcal{P}_r \)-measurable functions, then \((\varphi_i g_i) \in \mathcal{L}_1(\Gamma)\). Particularly \((a_i g_i) \in \mathcal{L}_1(\Gamma)\) for any scalars \( a_i, i = 1, \ldots, d \).

4) If \( U : Y \to Z \) is a bounded linear operator, then \((g_i) \in \mathcal{L}_1(\Gamma U)\).

It is easy to see that Theorem IX.7, with the convergence \( \Gamma \)-almost everywhere, is a generalization to \( \mathcal{L}_1(\Gamma) \) of the Lebesgue Dominated Convergence Theorem in \( \mathcal{L}_1(m) \), i.e., of Theorem II.17. By this theorem \( \mathcal{L}_1(\Gamma) \subset \mathcal{F}_1(\Gamma) \). The next theorem is a generalization of Theorems II.16 and V.1, i.e., of the Vitali Convergence Theorem in \( \mathcal{L}_1(m) \).

**Theorem 1.** For each \( i = 1, \ldots, d \) let \( f_i, f_{i,n} : T_i \to X_i \) be \( \mathcal{P}_r \)-measurable, let
(f_{i,n}) \in \mathcal{D}_1(\Gamma) \text{ for each } n = 1, 2, \ldots, \text{ and let } (f_{i,n}) \to (f_i) \text{ } \Gamma\text{-almost everywhere. Then the following conditions are equivalent:}

\begin{itemize}
  \item[(a)] \( (f_i) \in \mathcal{D}_1(\Gamma) \) and \( \hat{F}[[(f_{i,n}), (A_i)]] \to \hat{F}[[(f_i), (A_i)]] \) for each \( (A_i) \in X\sigma(\mathcal{P}_i) \);
  \item[(b)] the \( L_1 \)-gauges \( \hat{F}[[(f_{i,n}), (\cdot)]] : X\sigma(\mathcal{P}_i) \to [0, +\infty] \), \( n = 1, 2, \ldots \) are separately uniformly continuous (equivalently, separately uniformly exhaustive on \( X\mathcal{P}_i \)), and
  \item[(c)] \( \hat{F}[[(f_{i,n}), (A_i)]] \to \hat{F}[[(f_i), (A_i)]] \) uniformly with respect to \( (A_i) \in X\sigma(\mathcal{P}_i) \);
\end{itemize}

and if they hold, then

\[ \lim_{n \to \infty} \int_{(A_i)} (f_{i,n}) d\Gamma = \int_{(A_i)} (f_i) d\Gamma \]

uniformly with respect to \( (A_i) \in X\sigma(\mathcal{P}_i) \).

**Proof.** Clearly \( a) \Rightarrow b) \) by separate monotonicity and separate continuity of the \( L_1 \)-gauge \( \hat{F}[[(f), (\cdot)]] : X\sigma(\mathcal{P}_i) \to [0, +\infty] \). The equivalence in \( b) \) is a consequence of the Fatou property of the \( L_1 \)-gauge, see Theorem VIII.4 and also Theorem 11 in [22].

\( b) \Rightarrow c) \). For each \( i = 1, \ldots, d \) put \( F_i = \bigcup_{n=0}^{\infty} \{ t_i \in T_i, f_{i,n}(t_i) = 0^i \} \), where \( f_{i,0} = f_i \).

By local \( \sigma \)-finiteness of the semivariation \( \hat{F} \) on \( X\sigma(\mathcal{P}_i) \), see the beginning of Part IX, there is a sequence of \( d \)-tuples of sets \( (F_{i,k}) \in \mathcal{P}_i, k = 1, 2, \ldots \) such that \( F_{i,k} \nearrow F_i \)

for each \( i = 1, \ldots, d \), and \( \hat{F}(F_{i,k}) \leq +\infty \) for each \( k = 1, 2, \ldots \).

Owing to the Fatou property of the multiple \( L_1 \)-gauge, see Theorem VIII.4, we have

\[ \hat{F}[(f_{i,n}), (A_i)] = \hat{F}[(f_i), (A_i)] \leq \inf_{n} \hat{F}[(f_{i,n}), (A_i)] \]

for each \( (A_i) \in X\sigma(\mathcal{P}_i) \). Hence \( f_i \in \mathcal{D}_1(\Gamma) \) by \( b) \). Thus \( \hat{F}[(f_{i,n}), (T_i)] \leq +\infty \) for each \( n = 0, 1, 2, \ldots \), \( (f_{i,0}) = (f_i) \).

For \( A_i \in \sigma(\mathcal{P}_i) \) put \( \mu_i(A_i) = \sup \{ \hat{F}[(f_{1,n}, f_{2,n}, \ldots, f_{d,n}), (A_1, T_2, \ldots, T_d)] : n \in \{0, 1, \ldots, \} \} \). Similarly we define \( \mu_i : \sigma(\mathcal{P}_i) \to [0, +\infty] \) for \( i = 2, \ldots, d \). \( b) \) implies that each \( \mu_i, i = 1, \ldots, d \), is a subadditive semimeasure in the sense of Definition 1 in [22]. Since the Egoroff-Lusin theorem, see Section 1.4 in Part I, still holds if \( \mu \) is a semimeasure, for each \( i = 1, \ldots, d \) there are sets \( N_i \in \sigma(\mathcal{P}_i) \) and \( F_i \in \mathcal{P}_i \), \( k = 1, 2, \ldots \) such that \( \mu_i(N_i) = 0, F_i' \nearrow F_i - N_i \), and on each \( F_i' \), \( k = 1, 2, \ldots \) the sequence \( f_{i,n}, n = 1, 2, \ldots \) converges uniformly to the function \( f_i \).

Finally, put \( F_{i,k} = F_{i,k} \cap F_i' \cap \{ t_i \in T_i, |f_i(t_i)| \leq k \} \), \( i = 1, \ldots, d \), and \( k = 1, 2, \ldots \). Without loss of generality we may suppose that \( |f_{i,n}(t_i)| \leq 2k \) for each \( t_i \in F_{i,k}, i = 1, \ldots, d \), and \( k = 1, 2, \ldots \).

Let \( \varepsilon > 0 \). Since \( F_{i,k} \nearrow F_i - N_i \) for each \( i = 1, \ldots, d \), by \( b) \) there is a positive integer \( k_{\varepsilon} \) such that

\[ \hat{F}[(f_{i}, \ldots, f_{j-1}, F_j - N_j - F_{j,k_{\varepsilon}}, T_{j+1}, \ldots)] < \varepsilon/2d \]

for each \( j = 1, \ldots, d \). Hence

\[ \hat{F}[(f_{i}, f_{i,k_{\varepsilon}})] \leq \hat{F}[(f_{i}, (T_i)] = \hat{F}[(f_{i}, (F_i - N_i)] \leq \hat{F}[(f_{i}, (F_{i,k_{\varepsilon}})] + \varepsilon/2 \].
Since the uniformly bounded sequence \( f_{i,n} \cdot 1_{E_{i,k_n}}, \quad n = 1, 2, \ldots \) converges uniformly to the function \( f_i \cdot 1_{E_{i,k_n}} \) for each \( i = 1, \ldots, d \), and since \( \tilde{f}(F_{i,k_n}^*) < +\infty \), there is a positive integer \( n_0 \) such that
\[
|\tilde{f}((f_{i,n}), (F_{i,k_n}^*)) - \tilde{f}((f_i), (F_{i,k_n}^*))| < \frac{\varepsilon}{2}
\]
for \( n \geq n_0 \). Hence \( b) \Rightarrow c) \).

Trivially \( c) \Rightarrow a) \).

The last assertion of the theorem is a consequence of Theorem X.11. The theorem is proved.

The following theorem is a generalization of Theorem II.5.

**Theorem 2.** Let \((g_i) \in \mathcal{L}_2(\Gamma)\). Then there are countably additive measures \( \lambda_i: \sigma(\mathcal{P}_i) \to [0, 1], \quad i = 1, \ldots, d \), such that \( \tilde{f}((g_i), (\ldots, T_{j-1}, \cdot, T_{j+1}, \ldots)): \sigma(\mathcal{P}_j) \to [0, +\infty) \) is \((\delta - \varepsilon)\) (equivalently \((0 - 0)\)) absolutely \( \lambda_j \)-continuous for each \( j = 1, \ldots, d \).

**Proof.** Let \( \mathcal{P}\{(g_i)\} = \{(f_i) \in \mathcal{F}\{(X_i, T_i), f_i \leq |g_i| \} \text{ for each } i = 1, \ldots, d\} \) and let \( \mathcal{M}\{(g_i)\} = \{\alpha_{f,i,j}, \alpha_{f,j,i}(A_i) = \int_{(A_i)} (f_i) dT_i, \quad (f_i) \in \mathcal{P}\{(g_i)\} \}. \) Since \( \tilde{f}((f_i), (A_i)) \leq \tilde{f}((g_i), (A_i)) \leq \tilde{f}((g_i), (A_i)) \) for each \( (f_i) \in \mathcal{P}\{(g_i)\} \) and each \( (A_i) \in \mathcal{X} \sigma(\mathcal{P}_i) \), the family \( \mathcal{M}\{(g_i)\} \) of vector \( d \)-polymeasures on \( \mathcal{X} \sigma(\mathcal{P}_i) \) is separately uniformly countably additive. Now the assertion of the theorem immediately follows from the well known result of Bartle, Dunford and Schwartz, see Theorem I.2.4 in [3] \( \tilde{f}((g_i), (T_i)) < +\infty \). The theorem is proved.

The next corollary is a generalization of the second part of the \(*\)-Theorem from Part I. Its validity is obvious.

**Corollary.** Let the semivariation \( \tilde{f}: \mathcal{X}\mathcal{P}_i \to [0, +\infty] \) be separately continuous. Then for each \( (A_i) \in \mathcal{X}\mathcal{P}_i \) there are countably additive measures \( \lambda_j, (A_i): A_j \cap \mathcal{P}_j \to [0, 1], \quad j = 1, \ldots, d \), such that \( \tilde{f}((\ldots, A_{j-1}, \cdot, A_{j+1}, \ldots)): A_j \cap \mathcal{P}_j \to [0, +\infty) \) is \((\delta - \varepsilon)\) absolutely \( \lambda_j, (A_i) \)-continuous for each \( j = 1, \ldots, d \). The analog holds if each \( \mathcal{P}_i, \quad i = 1, \ldots, d \) is replaced by \( \sigma(\mathcal{P}_i) \), and in both cases the semivariation \( \tilde{f} \) has locally control \( d \)-polymeasure on \( \mathcal{X} \sigma(\mathcal{P}_i) \).

We are now ready to prove the following generalization of Theorem V.4:

**Theorem 3.** Let \((g_i) \in \mathcal{L}_2(\Gamma)\). Then for each \( \varepsilon > 0 \) there is a positive integer \( N_\varepsilon \) such that whenever \( i \in \{1, \ldots, d\}, \quad f_{i,j}: T_i \to X_i, \quad j = 1, \ldots, N_\varepsilon \) are \( \mathcal{P}_i \)-measurable and
\[
\sum_{j=1}^{N_\varepsilon} |f_{i,j}| \leq |g_i|,
\]
then \( \tilde{f}((\ldots, g_{i-1}, f_{i,j}, g_{i+1}, \ldots), (T_i)) < \varepsilon \) for at least one \( j \in \{1, \ldots, N_\varepsilon\} \).

**Proof.** Let \( \varepsilon > 0 \). Using theorem 2 and its Corollary, similarly as in the proof of Theorem V.4 we obtain positive integers \( N_{i,\varepsilon}, \quad i = 1, \ldots, d \) with the corresponding properties. Clearly \( N_\varepsilon = \max \{N_{i,\varepsilon}, \quad i \in \{1, \ldots, d\}\} \) has the required property. The theorem is proved.
Let us again have the setting of Parts I and II. If $Y$ does not contain a subspace isomorphic to the space $c_0$, shortly if $c_0 \not\subseteq Y$, then by Theorem II.5 each $\mathcal{P}$-measurable function $g: T \to X$ with finite $L_1$-pseudonorm $\mathfrak{m}(g, T) < +\infty$ belongs to $\mathcal{L}_1(m)$. Since, due to the nice example of Hans Weber, see Part VIII, there are Hilbert space valued bimeasures defined on the Cartesian product of two $\sigma$-rings which are not uniform bimeasures, the analog of Theorem II.5 for $\mathcal{L}_1(\Gamma)$ for $d > 1$ does not hold. The idea how to define the "right" $\mathcal{L}_1(\Gamma)$ came from the following simple characterization of elements of $\mathcal{L}_1(m)$. This theorem may be proved similarly as Theorem 1 in [15].

**Theorem 4.** Let $g: T \to X$. Then $g \in \mathcal{L}_1(m)$ if and only if $g$ is $\mathcal{P}$-measurable and each $\mathcal{P}_g$-measurable function $f: T \to X$ with finite $L_1$-pseudonorm $\mathfrak{m}(f, T) < +\infty$ belongs to $\mathcal{L}_1(m)$.

Since, due to the nice example of Hans Weber, see Part VIII, there are Hilbert space valued bimeasures defined on the Cartesian product of two $\sigma$-rings which are not uniform bimeasures, the analog of Theorem II.5 for $\mathcal{L}_1(\Gamma)$ for $d > 1$ does not hold. The idea how to define the "right" $\mathcal{L}_1(\Gamma)$ came from the following simple characterization of elements of $\mathcal{L}_1(m)$. This theorem may be proved similarly as Theorem 1 in [15].

**Definition 3.** Let $g_i: T_i \to X_i$, $i = 1, \ldots, d$. We say that $(g_i)$ belongs to $\mathcal{L}_1(\Gamma)$ if $g_i$ is $\mathcal{P}_g$-measurable for each $i = 1, \ldots, d$, and for any $\mathcal{P}_g$-measurable functions $f_i: T_i \to X_i$, $i = 1, \ldots, d$, the inequalities $|f_i| \leq |g_i|$ for each $i = 1, \ldots, d$ imply that $(f_i)$ is an integrable $d$-tuple, i.e., $(f_i) \in \mathcal{F}(\Gamma)$.

Obviously $\mathcal{L}_1(\Gamma) \subseteq \mathcal{L}_1(\Gamma)$. Further, we immediately obtain

**Lemma 2.** The assertions of Lemma 1 still hold if $\mathcal{L}_1(\Gamma)$ is replaced by $\mathcal{L}_1(\Gamma)$.

For a $\mathcal{P}_g$-measurable function $g_i: T_i \to X_i$ and $k = 1, 2, \ldots$ put

$$\mathcal{P}_{g_i,k} = \{t_i \in T_i; g_i(t_i) \geq 1/k\} \cap \mathcal{P}_i$$

and let $\mathcal{P}_{g_i} = \bigcup_{k=1}^{\infty} \mathcal{P}_{g_i,k}$. Then $\mathcal{P}_{g_i}$ is evidently a $\sigma$-ring and $g_i$ is $\mathcal{P}_{g_i}$-measurable. We shall use this notation as well as the following fact.

**Theorem 5.** Let $c_0 \not\subseteq Y$, let $g_i: T_i \to X_i$ be $\mathcal{P}_g$-measurable, $i = 1, \ldots, d$, and let $\mathcal{F}[(g_i), (T_i)] < +\infty$. Then $(g_i) \in \mathcal{L}_1(\Gamma)$.

**Proof.** For every $i = 1, \ldots, d$ take a sequence $g_{i,n} \in S(\mathcal{P}_i, X_i)$ such that $g_{i,n}(t_i) \to g_i(t_i)$ and $|g_{i,n}(t_i)| \leq |g_i(t_i)|$ for each $t_i \in T_i$. Clearly $g_{i,n} \in S(\mathcal{P}_i, X_i)$ for each $n = 1, 2, \ldots$. Let $\Gamma' = \Gamma: X \times \prod_{i=1}^{d} T_i \to L^a(\Gamma)$, whose semivariation on a set $A_t \in \mathcal{P}_{g_t,k}$ is bounded by $k. \mathcal{F}[(g_i), (T_i)]$ for all $n_2, \ldots, n_d = 1, 2, \ldots$, and all $(A_2, \ldots, A_d) \in \sigma(\mathcal{P}_2 \times \cdots \times \sigma(\mathcal{P}_d))$. By symmetry in coordinates the analogs hold for the coordinates $i = 2, \ldots, d$.

Since obviously $\mathfrak{m}_{1,\ldots,1}(g_i, T_i) \leq \mathcal{F}[(g_i), (T_i)] < +\infty$, and $c_0 \not\subseteq Y$ by assumption, $g_i \in \mathcal{L}_1(\mathcal{F}(m_1,\ldots,1))$ by Theorem II.5. Thus according to Definition I.2 the function $g_i$ is integrable with respect to the measure $m_{1,\ldots,1}(\cdot)$, and $\int A_t g_i \, dm_{1,\ldots,1} = \lim_{n_1 \to \infty} \int A_t g_{i,n_1} \, dm_{1,\ldots,1} \in Y$ exists for each $A_t \in \sigma(\mathcal{P}_t)$, hence also for each $A_t \in \sigma(\mathcal{P}_t)$.
i = 2, ..., d, and since clearly \( \int g_1 \, dm_{1,?,?} = \int (g_{i,ni}) \, d\Gamma \), we immediately obtain that \( (g_1, g_2, ..., g_{d,n_d}) \in \mathcal{F}_1(\Gamma) \) and

\[
\{ (g_{i,ni}) \} \, d\Gamma = \lim_{n_1 \to \infty} \int (g_{i,ni}) \, d\Gamma
\]

for each \( n_2, ..., n_d = 1, 2, ... \) and each \( (A_i) \in X\sigma(\mathcal{P}_i) \).

Similarly \( (g_1, x_2 \cdot \mathcal{A}_2, g_3, n_3, ..., g_{d,n_d}) \in \mathcal{F}_1(\Gamma) \) and

\[
\{ (g_{1,n_1}, x_2 \cdot \mathcal{A}_2, g_3, n_3, ..., g_{d,n_d}) \} \, d\Gamma = \lim_{n_1 \to \infty} \int (g_{1,n_1}) \, d\Gamma
\]

for each \( x_2 \in X_2 \) and each \( A_2 \in \sigma(\mathcal{P}_2) \), for any given \( n_3, ..., n_d \) and \( A_1, A_3, ..., A_d \). This equality implies that for any given \( n_3, ..., n_d \) and \( A_1, A_3, ..., A_d \) the mapping \( (A_2, x_2) \to \{ (g_{1,n_1}, x_2 \cdot \mathcal{A}_2, g_3, n_3, ..., g_{d,n_d}) \} \, d\Gamma \), \( A_2 \in \mathcal{P}_2, x_2 \in X_2 \), defines a measure \( m_{2,(?,?),?} : \mathcal{P}_2 \to L(X_2, Y) \) countably additive in the strong operator topology, whose semivariation on a set \( A_2 \in \mathcal{P}_2 \) is bounded by \( k. \hat{F}([g_1], (T_i)] < +\infty \).

Continuing as above we obtain that \( (g_1, g_2, g_3, ..., g_{d,n_d}) \in \mathcal{F}_2(\Gamma) \), and

\[
\{ (g_{1,n_1}, g_2, g_3, ..., g_{d,n_d}) \} \, d\Gamma = \lim_{n_2 \to \infty} \lim_{n_1 \to \infty} \int (g_{1,n_1}) \, d\Gamma
\]

for each \( n_3, ..., n_d = 1, 2, ... \) and each \( (A_i) \in X\sigma(\mathcal{P}_i) \).

Continuing in this manner we finally obtain that \( (g_i) \in \mathcal{F}_d(\Gamma) \), and

\[
\{ (g_{i,n_i}) \} \, d\Gamma = \lim_{n_d \to \infty} \lim_{n_{d-1} \to \infty} \lim_{n_{d-2} \to \infty} \cdots \lim_{n_1 \to \infty} \int (g_{i,n_i}) \, d\Gamma
\]

for each \( (A_i) \in X\sigma(\mathcal{P}_i) \).

Let us note that by symmetry in coordinates the analogs are valid for any permutation of \( \{1, ..., d\} \). Note finally that in Theorem 10 below we show that \( \mathcal{L}_1(\Gamma) \subset \mathcal{F}_1(\Gamma) \) in general. The theorem is proved.

Suppose now that each \( X_i, i = 1, ..., d \), is finite dimensional. Then according to Corollary of Theorem X.5 we have \( \mathcal{F}(\Gamma) = \mathcal{F}_1(\Gamma) \). Further \( \hat{F}([g_1], (T_i)] < +\infty \) for each \( (g_i) \in \mathcal{F}(\Gamma) \) by Theorems VIII.2 and VIII.3. Hence we have obtained the following

**Corollary.** Let each \( X_i, i = 1, 2, ... \) be finite dimensional and let \( c_0 \in Y \). Then \( \mathcal{F}(\Gamma) \subset \mathcal{F}_1(\Gamma) \subset \mathcal{L}_1(\Gamma) \).

One of our most important results is the following

**Theorem 6.** (The Fubini Theorem in \( \mathcal{L}_d(\Gamma) \).) Let \( (g_{i}) \in \mathcal{L}_1(\Gamma) \), let \( d > 1 \) and let \( d_1 \) be a positive integer such that \( 1 \leq d_1 < d \). Then:

1. \( (g_{i}) \in \mathcal{F}_d(\Gamma) \);
2. \( (g_1, ..., g_{d_1}, x_{d_1+1} \cdot \mathcal{A}_{d_1+1}, ..., x_d \cdot \mathcal{A}_d) \in \mathcal{F}(\Gamma) \) for each \( x_i \in X_i \) and \( A_i \in \mathcal{P}_{d_i}, i = d_1 + 1, ..., d \).
3. Let \( (A_1, ..., A_{d_1}) \in \sigma(\mathcal{P}_1) \times \cdots \times \sigma(\mathcal{P}_d) \) be given. For each \( x_i \in X_i \) and
At age $g_i = d_i + 1, \ldots, d$, we put

$$\Gamma_{d_i, (A_1, \ldots, A_{d_i})} (A_{d_i+1}, \ldots, A_d) (x_{d_i+1}, \ldots, x_d) =$$

$$= (f_{(A_1, \ldots, A_{d_i})} (g_1, \ldots, g_d, \ldots) d\Gamma) (A_{d_i+1}, \ldots, A_d) (x_{d_i+1}, \ldots, x_d) =$$

$$= \int_{(A_i)} (g_1, \ldots, g_d, x_{d_i+1} \cdot x_{A_{d_i+1}}, \ldots, x_d \cdot x_{A_d}) d\Gamma.$$

Then $\Gamma_{d_i, (\cdot)}: \mathcal{P}_{d_i+1} \times \ldots \times \mathcal{P}_d \to L^{d-d_i}(X_{d_i+1}, \ldots, X_d; \mathcal{Y})$, $\Gamma_{d_i, (\cdot)}$ is separately countably additive in the strong operator topology, and its semivariation $\hat{\rho}_{d_i, (\cdot)}$ is finite valued on $\mathcal{P}_{d_i+1} \times \ldots \times \mathcal{P}_d$ for each $k = 1, 2, \ldots$. Hence $\hat{\rho}_{d_i, (\cdot)}$ is locally $\sigma$-finite on $\mathcal{P}_{d_i+1} \times \ldots \times \mathcal{P}_d$ and also on $\sigma(\mathcal{P}_{d_i+1}) \times \ldots \times \sigma(\mathcal{P}_d)$.

3) $(g_{d_i+1}, \ldots, g_d) \in \mathcal{L}_1(\Gamma_{d_i, (\cdot)})$ for each $(\cdot) = (A_1, \ldots, A_{d_i}) \in \sigma(\mathcal{P}_{d_i}) \times \ldots \times \sigma(\mathcal{P}_d)$,

$$\hat{\rho}_{d_i, (\cdot)}[(g_{d_i+1}, \ldots, g_d), (A_{d_i+1}, \ldots, A_d)] \leq \hat{\rho}[(g_i), (A_i)]$$

and

$$\int_{(A_{d_i+1}, \ldots, A_d)} (g_{d_i+1}, \ldots, g_d) d\Gamma_{d_i, (\cdot)} = \int_{(A_i)} (g_i) d\Gamma$$

for each $(A_i) \in \chi \sigma(\mathcal{P}_1)$, and

4) $\int_{(A_{d_i+1}, \ldots, A_d)} (g_{d_i+1}, \ldots, g_d) d(\int_{(A_1, \ldots, A_{d_i+1}, \ldots, A_d)} (g_1, \ldots, g_d, \ldots) d\Gamma) = \int_{(A_i)} (g_i) d\Gamma =$

$$= \int_{(A_1, \ldots, A_{d_i})} (g_1, \ldots, g_d) d \int_{(A_{d_i+1}, \ldots, A_d)} (\ldots, g_{d_i+1}, \ldots, g_d) d\Gamma$$

for each $(A_i) \in \chi \sigma(\mathcal{P}_1)$, where the $d_i$-polymeasure

$$\int_{(\ldots, A_{d_i+1}, \ldots, A_d)} (\ldots, g_{d_i+1}, \ldots, g_d) d\Gamma: \mathcal{P}_{d_i} \times \ldots \times \mathcal{P}_{d_i} \to L^{d_i}(X_1, \ldots, X_{d_i}; \mathcal{Y})$$

is defined similarly as the $(d - d_i)$-polymeasure in 2), and has similar properties as the latter.

Proof. We now prove the theorem under the additional assumption that $\hat{\rho}[(g_i), (T_i)] < +\infty$. In Theorem 8 below we prove that for $d = 2$, $(g_i) \in \mathcal{L}_1(\Gamma) \Rightarrow \hat{\rho}[(g_i), (T_i)] < +\infty$. In the forthcoming Part XIII, Theorem 12 we will prove this implication for an arbitrary dimension $d$.

Having this additional assumption we first show that the proof of Theorem 5 remains valid in this new situation. To this end we must show that $g_1 \in \mathcal{L}_1(m_{1, (\cdot), (\cdot)})$ in the notation of this proof. According to Theorem 4 we must prove that $f_1: T_1 \to X_1$ is integrable with respect to the measure $m_{1, (\cdot), (\cdot)}$ provided $f_1$ is $\mathcal{P}_{d_i}$ measurable and $|f_1| \leq |g_1|$. Let $f_1$ be such a function.

Since $(f_1, g_2, m_2, \ldots, g_{d_m}, m_{d_m}) \in \mathcal{F}(\Gamma)$ by the definition of $\mathcal{L}_1(\Gamma)$, the set function $v_0$, $v_0(A_1) = \int_{(A_1, \ldots, A_{d_i})} (f_1, g_2, m_2, \ldots, g_{d_m}, m_{d_m}) d\Gamma$, $A_1 \in \sigma(\mathcal{P}_1)$, is a countably additive vector measure by the separate countable additivity of the indefinite integral with respect to $\Gamma$, see Theorems IX.3 and IX.4.

Let $f_{1, n}: T_1 \to X_1$, $n = 1, 2, \ldots$ be a sequence of $\mathcal{P}_{d_i}$ measurable functions such that $f_{1, n} \to f_1$ and $|f_{1, n}| \leq |f_1|$. Since the semivariation $m_{1, (\cdot), (\cdot)}$ is $\sigma$-finite on $\mathcal{P}_{d_i}$, each $f_{1, n}$, $n = 1, 2, \ldots$ is integrable with respect to $m_{1, (\cdot), (\cdot)}$.

For $A_1 \in \sigma(\mathcal{P}_{d_i})$ put $v_n(A_1) = \int_{A_1} f_{1, n} dm_{1, (\cdot), (\cdot)} =$
Then \( \mu: \sigma(\mathcal{P}_g) \to [0, 2] \) is a subadditive submeasure in the sense of Definition 1 in [21] \( \tilde{v}_n(A_1) = \sup \{ |v_n(B_1)|, B_1 \in \sigma(\mathcal{P}_g), B_1 \subseteq A_1 \} \), i.e. \( \tilde{v}_n \) is the supremation of \( v_n \), see Definition VIII.2.

Put \( F = \{ t \in T_1, f_1(t_1) \neq 0 \} \in \sigma(\mathcal{P}_g) \). According to Egoroff-Lusin theorem, see Section 1.4 in [5], which remains valid for the subadditive submeasure \( \mu \), there is a set \( N \in \sigma(\mathcal{P}_g) \) with \( \mu(N) = 0 \) and a sequence of sets \( F_k \in \sigma(\mathcal{P}_g), k = 1, 2, \ldots \) such that \( F_k \not\subseteq F - N \), and on each \( F_k \), \( k = 1, 2, \ldots \) the sequence \( f_{1,n}, n = 1, 2, \ldots \) converges uniformly to the function \( f_1 \). Since the semivariation \( \bar{m}_{1,1}(\cdot)(F) \) is \( \sigma \)-finite, without loss of generality we may and will suppose that \( \bar{m}_{1,1}(A \cap F_k) < +\infty \) for each \( k = 1, 2, \ldots \). But then, clearly, the functions \( f_1 \cdot \chi_{F_k \cup N}, k = 1, 2, \ldots \) are integrable with respect to the measure \( \bar{m}_{1,1}(\cdot)(\cdot) \) and

\[
\int_{(A_1, \ldots, A_d)} f_{1,n} \cdot \chi_{F_k \cup N} \, dm_{1,1}(\cdot)(\cdot) = \lim_{n \to \infty} \int_{(A_1, \ldots, A_d)} f_{1,n} \cdot \chi_{F_k \cup N} \, dm_{1,1}(\cdot)(\cdot) = \lim_{n \to \infty} \int_{(A_1, \ldots, A_d)} f_{1,n} \cdot \chi_{F_k \cup N} \, dm_{1,1}(\cdot)(\cdot) = \bar{m}_{1,1}(A_1 \cap (F_k \cup N))
\]

for each \( A_1 \in \sigma(\mathcal{P}_g) \) and each \( k = 1, 2, \ldots \). Since \( f_1 \cdot \chi_{F_k \cup N} \to f_1 \), and since the indefinite integrals \( \int f_1 \cdot \chi_{F_k \cup N} \, dm_{1,1}(\cdot)(\cdot) = \bar{m}_{1,1}(\cdot)(F_k \cup N), k = 1, 2, \ldots \) are uniformly countably additive by the countable additivity of the vector measure \( v_0: \sigma(\mathcal{P}_g) \to Y \), \( f_1 \) is integrable with respect to the measure \( m_{1,1}(\cdot)(\cdot) \) and \( \int f_{1,n} \, dm_{1,1}(\cdot)(\cdot) = v_0(A_1) \) for each \( A_1 \in \sigma(\mathcal{P}_g) \) by Theorem I.16. Hence \( g_1 \in \mathcal{L}_1(m_{1,1}(\cdot)(\cdot)) \), which we wanted to show.

Thus the rest of the proof of Theorem 5 remains valid under the given assumptions of this proof. Hence \( (g_i) \in \mathcal{F}_d(\Gamma) \) and

\[
(*) \quad \int_{(A_1, \ldots, A_d)} (g_i) \, d\Gamma = \lim_{n_1(\cdot) \to \infty} \lim_{n_2(\cdot) \to \infty} \lim_{n_3(\cdot) \to \infty} \cdots \lim_{n_d(\cdot) \to \infty} \int_{(A_1, \ldots, A_d)} (g_i, n) \, d\Gamma
\]

(for each \( (A_1) \in \mathcal{X} \sigma(\mathcal{P}_g) \) and each permutation \( p \) of \{1, \ldots, d\}.

Now 0) immediately follows from (*).

1) is clear from the definitions of \( \mathcal{P}_g \) and of \( \mathcal{L}_1(\Gamma) \).

2) follows from (*) by the uniform boundedness principle and by the Vitali-Hahn-Saks-Nikodym-(VHSN)-Theorem for polymeasures, see the beginning of Part VIII.

3) and 4) are direct consequences of (*) and of the corresponding definitions. The theorem is proved.

We immediately have the following

**Corollary 1.** Let \( (g_i) \in \mathcal{L}_1(\Gamma) \). Then the integrals on the right hand side below exist, the polymeasures obtained are separately countably additive in the strong
operator topologies and have $\sigma$-finite semivariations, and

$$
\int_{(A_d)} (g_d) \, d\Gamma = \int_{A_d} g_d \left( \int_{(A_d-1)} (g_{d-1}, \cdot) \, d(\int_{(A_{d-2}, \ldots, A_2, \cdot)} (g_2, \ldots) \, \ldots \right) \ldots \right)
$$

for each $(A_i) \in \mathcal{X} \sigma(\mathcal{P}_{g_i})$. The analogs hold for all permutations of $\{1, \ldots, d\}$ and all decompositions of $d$ as a sum of positive integers.

The next corollary requires the following

**Remark.** Let $m: \mathcal{P}_0 \to L(X, Y)$ be countably additive in the strong operator topology, let $g: T \to X$ be $\mathcal{P}_0$-measurable, and let its $L_1$-pseudonorm $\hat{m}(g, \cdot): \sigma(\mathcal{P}_0) \to [0, +\infty]$ be continuous. Then $\hat{m}(g, T) < +\infty$ by Corollary of Theorem II.5 (now the simple proof at the beginning of this section does not work since the semivariation $\hat{m}$ may take the value $+\infty$ on some sets of $\mathcal{P}_0$, see Section 1.1 in Part I). Hence $\mathcal{P}_{g, k} \subset \mathcal{P} \subset \mathcal{P}$ for each $k = 1, 2, \ldots$ by the Tschebyschef inequality, see Corollary of Theorem II.1. Thus $\mathcal{P}_g \subset \sigma(\mathcal{P})$, hence $g$ is $\mathcal{P}$-measurable. This implies that $g \in \mathcal{L}_1(m)$. In this way the requirement of $\sigma$-finiteness of the semivariation $\hat{m}$ is not needed for the definition of $\mathcal{L}_1(m)$. We use this fact in the following

**Corollary 2.** Let $g_i: T_i \to X_i$ be $\mathcal{P}_i$-measurable, $i = 1, \ldots, d$. Then the following two conditions are equivalent:

a) $(g_1) \in \mathcal{L}_1(\Gamma)$, and
b) the following $d$ conditions hold:

1) $g_1 \in \mathcal{L}_1((A_2, \ldots, A_d) (\cdot, x_2 \cdot x_{A_2}, \ldots, x_d \cdot x_{A_d}) \, d\Gamma)$ for each $(A_2, \ldots, A_d) \in \mathcal{P}_{g_2} \times \ldots \times \mathcal{P}_{g_d}$ and each $(x_2, \ldots, x_d) \in X_2 \times \ldots \times X_d$ in the sense of the preceding Remark. This implies:

A) For each $(A_2, \ldots, A_d) \in \mathcal{P}_{g_2} \times \ldots \times \mathcal{P}_{g_d}$ and each $(x_2, \ldots, x_d) \in X_2 \times \ldots \times X_d$ the measure $\int_{(A_2, \ldots, A_d)} (\cdot, x_2 \cdot x_{A_2}, \ldots, x_d \cdot x_{A_d}) \, d\Gamma: \mathcal{P}_{g_1} \to L(X_1, Y)$, countably additive in the strong operator topology, has $\sigma$-finite semivariation on $\mathcal{P}_{g_1}$, $g_1$ is integrable with respect to this measure, $(g_1, x_2 \cdot x_{A_2}, \ldots, x_d \cdot x_{A_d}) \in \mathcal{M}(\Gamma)$, and

$$
\int_{A_1} g_1 \, d\left( \int_{(A_2, \ldots, A_d)} (\cdot, x_2 \cdot x_{A_2}, \ldots, x_d \cdot x_{A_d}) \, d\Gamma \right)
$$

for each $A_1 \in \sigma(\mathcal{P}_{g_1})$;

B) for each $A_1 \in \sigma(\mathcal{P}_{g_1})$ we have $\int_{(A_1, \ldots, A_d)} (g_1, \ldots) \, d\Gamma: \mathcal{P}_{g_2} \times \ldots \times \mathcal{P}_{g_d} \to L^{d-1}(X_2, \ldots, X_d; Y)$, and it is separately countably additive in the strong operator topology.

2) $g_2 \in \mathcal{L}_1((A_1, A_3, \ldots, A_d) (g_1, \cdot, x_3 \cdot x_{A_3}, \ldots, x_d \cdot x_{A_d}) \, d\Gamma$ for each $(A_1, A_3, \ldots, A_d) \in \sigma(\mathcal{P}_{g_1}) \times \mathcal{P}_{g_3} \times \ldots \times \mathcal{P}_{g_d}$ and each $(x_3, \ldots, x_d) \in X_3 \times \ldots \times X_d$ in the sense of the preceding Remark. This implies:

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d) $g_d \in \mathcal{L}_1((A_1, A_d-1, \ldots) (g_1, \ldots g_{d-1}, \cdot) \, d\Gamma$ for each $(A_1, \ldots, A_d-1) \in \sigma(\mathcal{P}_{g_1}) \times \ldots \times \sigma(\mathcal{P}_{g_{d-1}})$ in the sense of the preceding Remark. This implies
A) For each \((A_1, \ldots, A_d) \in \sigma(\mathcal{P}_g) \times \cdots \times \sigma(\mathcal{P}_g)\) the measure 
\[ \int_{(A_1, \ldots, A_{d-1})} (g_1, \ldots, g_{d-1}, \cdot) \, d\Gamma: \mathcal{P}_g \to L(X_d, Y), \] 
countably additive in the strong operator topology, has \(\sigma\)-finite semivariation on \(\mathcal{P}_g\), \(g_d\) is integrable with respect to this measure, \((g_1) \in \mathcal{F}(\Gamma)\), and 
\[ \int_{A_d} g_d \, d\left( \int_{(A_1, \ldots, A_{d-1})} (g_1, \ldots, g_{d-1}, \cdot) \, d\Gamma \right) = \int_{(A_d)} (g_1) \, d\Gamma \]
for each \(A_d \in \sigma(\mathcal{P}_g)\).

If a) holds, then the analogs of b) hold for all permutations \(p\) of \(\{1, \ldots, d\}\).

If \(X = K\) is the space of scalars, and \(m: \mathcal{P} \to L(K, Y) = Y\) is a countably additive vector measure, then \(\mathcal{F}(m) = L_1(m)\), see Part II. Hence we easily obtain

**Corollary 3.** Let each \(X_i, i = 1, \ldots, d\) be a finite dimensional Banach space. Then in b) of the preceding Corollary 2 we may replace the requirements \(g_i \in \mathcal{L}_1(\mathcal{P})\), \(i = 1, \ldots, d\) by the requirements \(g_i \in \mathcal{F}(\mathcal{P})\), \(i = 1, \ldots, d\).

For the particular case \(d = 2\), \(X_1 = X_2 = Y = C\) — the field of complex numbers, and a separately countably additive bimeasure \(\beta: \mathcal{P}_1 \times \mathcal{P}_2 \to C\) in Definition 2.6 in [1], see also [2], the concept of the strict \(\beta\)-integrability of a pair \((g_1, g_2)\) of \(\mathcal{P}_1\)-measurable functions \(g_i: T_i \to C\), \(i = 1, 2\) was introduced by the following three requirements. In our notation:

1. \((g_1) \in \mathcal{F}(\beta('), (A_2))\) for each \(A_2 \in \mathcal{P}_g\), and \(g_2 \in \mathcal{F}(\mathcal{F}(A_1, \cdot) (g_1, \cdot) \, d\beta)\) for each \(A_1 \in \sigma(\mathcal{P}_g)\).
2. \((g_2) \in \mathcal{F}(\beta(A_1, \cdot))\) for each \(A_1 \in \mathcal{P}_g\), and \(g_1 \in \mathcal{F}(\mathcal{F}(\cdot, g_2) (\cdot, g_2) \, d\beta)\) for each \(A_2 \in \sigma(\mathcal{P}_g)\).
3. \(\int A_d g_d \, d\left( \int_{(A_1, \ldots, A_{d-1})} (g_1, \ldots, g_{d-1}, \cdot) \, d\Gamma \right) = \int_{(A_d)} (g_1) \, d\Gamma \)

By Corollary 3 above (i) \(\Leftrightarrow (g_1, g_2) \in \mathcal{L}_1(\beta) \Leftrightarrow\) (ii), and they imply (iii). Note further that according to Theorem 5 and Theorem 8 below, \((g_1, g_2) \in \mathcal{L}_1(\beta)\) if and only if \(\beta((g_1, g_2), (T_1, T_2)) < +\infty\).

We shall need the following useful theorem, which by virtue of Theorem 5 is a generalization of Theorem VIII.5.

**Theorem 7.** Let \((g_i) \in \mathcal{L}_1(\Gamma)\), let \(A_{i,n} \in \sigma(\mathcal{P}_i), n = 1, 2, \ldots, i = 1, \ldots, d\), and let \(A_{i,n} \to 0\) for each \(i = 1, \ldots, d\). Then \(\hat{\Gamma}[(g_i), (A_{i,n})] \to 0\) as \(n \to \infty\).

**Proof.** Since \(A_{i,n} \to 0\) if and only if \(\limsup A_{i,n} \to 0\), we may and will suppose that \(A_{i,n} \to 0\) for each \(i = 1, \ldots, d\). Suppose \(\hat{\Gamma}[(g_i), (A_{i,n})] > a > 0\) for each \(n = 1, 2, \ldots\). Put \(n_0 = 1\). By the Fatou property of the multiple \(L_1\)-gauge \(\hat{\Gamma}[(\cdot), (\cdot)]\), see Theorem VIII.4, there is an \(n_1 > n_0\) such that

\[ \hat{\Gamma}[(g_i), (A_{i,n_0} - A_{i,n_1})] > a. \]

By the definition of \(\hat{\Gamma}[(\cdot), (\cdot)]\), see Definition VIII.3, there are \(u_{i,1} \in S(\mathcal{P}_i, X_i)\) with \(\|u_{i,1}\| \leq |g_i| \cdot L_{A_{i,n_0} - A_{i,n_1}}, i = 1, \ldots, d\), such that

\[ \left| \int_{T_i} (u_{i,1}) \, d\Gamma \right| > a. \]
Repeating the above consideration we obtain a subsequence \( \{n_k\} \subset \{n\} \), and for each \( k = 2, 3, \ldots \) we obtain functions \( u_{i,k} \in \mathcal{S}(\mathcal{P}_i, X_i) \), \( i = 1, \ldots, d \), such that 
\[ |u_{i,k}| \leq |g_i| \cdot \chi_{A_{i,n_k-1} - A_{i,n_k}} \]
for each \( i = 1, \ldots, d \), and 
\[ \left| \int (r_i) (u_{i,k}) \, d\Gamma \right| > a \].

Put \( u_i = \sum_{k=1}^{\infty} u_{i,k} \), \( i = 1, \ldots, d \). Then \( |u_i| \leq |g_i| \) for each \( i = 1, \ldots, d \), hence \( (u_i) \in \mathcal{F}(\mathcal{I}) \). Since \( u_{i,k} = u_i \cdot \chi_{A_{i,n_k-1} - A_{i,n_k}} \) for each \( i = 1, \ldots, d \) and each \( k = 1, 2, \ldots \), and since \( A_{i,n_k-1} - A_{i,n_k} = \emptyset \) as \( k \to \infty \) for each \( i = 1, \ldots, d \), according to Theorems VIII.1, IX.3 and IX.4 we obtain that 
\[ a < \left| \int (r_i) (u_{i,k}) \, d\Gamma \right| = \left| \int (A_{i,n_k-1} - A_{i,n_k}) (u_i) \, d\Gamma \right| \to 0 \]
as \( k \to \infty \), a contradiction. The theorem is proved.

In the forthcoming Part XIII we will prove the analog of the next result for arbitrary \( d \).

**Theorem 8.** Let \( d = 2 \) and let \((f_1, f_2) \in \mathcal{L}_2(\mathcal{I})\). Then \( \hat{\mathcal{F}}[(f_1, f_2), (T_1, T_2)] < +\infty \).

**Proof.** For \( i = 1, 2 \) put \( F_i = \{ t_i \in T_i, f_i(t_i) \neq 0 \} \in \sigma(\mathcal{P}_i) \). By the assumed local \( \sigma \)-finiteness of the semivariation \( \hat{\mathcal{F}} \) on \( F_1 \cap \sigma(\mathcal{P}_1) \times F_2 \cap \sigma(\mathcal{P}_2) \) there are \( F_{i,r} \in \mathcal{P}_i \), \( r = 1, 2, \ldots, i = 1, 2 \) such that \( F_{i,r} \subset F_i \) as \( r \to \infty \) for both \( i = 1, 2 \), and \( \hat{\mathcal{F}}(F_{i,r}, F_{2,r}) < +\infty \) for each \( r = 1, 2, \ldots \). Put 
\[ F_{i,r} = \{ t_i \in F_i, |f_i(t_i)| \leq r \} \cap F_{i,r} \]
for \( r = 1, 2, \ldots \) and \( i = 1, 2 \). Obviously 
\[ \hat{\mathcal{F}}[(f_1, f_2), (T_1, T_2)] = \hat{\mathcal{F}}[(f_1, f_2), (F_1, F_2)] \leq \hat{\mathcal{F}}[(f_1, f_2), (F_{1,r}, F_{2,r})] + \hat{\mathcal{F}}[(f_1, f_2), (F_1 - F_{1,r}, F_{2,r})] + \hat{\mathcal{F}}[(f_1, f_2), (F_{1,r}, F_2 - F_{2,r})] + \hat{\mathcal{F}}[(f_1, f_2), (F_1 - F_{1,r}, F_2 - F_{2,r})] \]
for each \( r = 1, 2, \ldots \). Clearly \( \hat{\mathcal{F}}[(f_1, f_2), (F_{1,r}, F_{2,r})] \leq r^2 \cdot \hat{\mathcal{F}}(F_{1,r}, F_{2,r}) < +\infty \) for each \( r = 1, 2, \ldots \). Since \( F_i - F_{i,r} \subset \emptyset \) as \( r \to \infty \) for both \( i = 1, 2 \), according to Theorem 7 there is an \( r_0 \) such that 
\[ \hat{\mathcal{F}}[(f_1, f_2), (F_1 - F_{1,r_0}, F_{2,r_0})] \leq 1 \]
for each \( r \geq r_0 \). Hence to prove the theorem it suffices to show that there is an \( r_0 \) such that 
\[ \hat{\mathcal{F}}[(f_1, f_2), (F_1 - F_{1,r_0}, F_{2,r_0})] + \hat{\mathcal{F}}[(f_1, f_2), (F_{1,r_0}, F_2 - F_{2,r_0})] < +\infty \]. Suppose the contrary. Then either 
\[ \hat{\mathcal{F}}[(f_1, f_2), (F_1 - F_{1,r_0}, F_{2,r_0})] = +\infty \]
for an infinite subsequence \( r_k \), \( k = 1, 2, \ldots \) with \( r_1 \leq r_0 \), or 
\[ \hat{\mathcal{F}}[(f_1, f_2), (F_{1,r_0}, F_2 - F_{2,r_0})] = +\infty \]
for an infinite subsequence \( r_k \), \( k = 1, 2, \ldots \) with \( r_1 \geq r_0 \).

By symmetry in coordinates it is enough to suppose that 
\[ \hat{\mathcal{F}}[(f_1, f_2), (E_{1,k}, F_2 - F_{2,k})] = +\infty \]
for each \( k = 1, 2, \ldots \), where \( E_{i,k} = F_{i,r_0} \) and \( r_1 \leq r_0 \), \( i = 1, 2 \), \( k = 1, 2, \ldots \). Put \( k_0 = 1 \). By the definition of the multiple \( L_1 \)-gauge, see Definition VIII.3, there is a pair \((u_{1,1}^*, u_{2,1}^*) \in \mathcal{S}(\mathcal{P}_1, X_1) \times \mathcal{S}(\mathcal{P}_2, X_2)\) such that 
\[ |u_{1,1}^*| \leq |f_1| \cdot \chi_{E_{1,k_0}} \leq r_1, \ |u_{2,1}^*| \leq |f_2| \cdot \chi_{E_2 - E_{2,k_0}} \] and 
\[ \left| \int (r_1, f_2 - E_{2,k_0}) (u_{1,1}^*, u_{2,1}^*) \, d\Gamma \right| > 3.4. r_1 \].
Put \[ u_{1,1} = u_{1,1}^* \frac{1}{2r_1} \cdot \]

Let \[ \mathcal{P}_1 = \bigcup_{k=1}^{\infty} E_{1,k} \cap \mathcal{P}_1 \] and \[ \mathcal{P}_2 = \bigcup_{k=1}^{\infty} E_{2,k} \cap \mathcal{P}_2. \]

For \( E_2 \in \mathcal{P}_2 \) and \( x_2 \in X_2 \) put \[ m_{u_{1,1}}(E_2) x_2 = \int_{(E_1,1,E_2)} (u_{1,1}, x_2 \cdot \chi_{E_2}) d\Gamma. \]

Clearly \( m_{u_{1,1}} : \mathcal{P}_2 \to L(X_2, Y) \), and it is countably additive in the strong operator topology. Further, since \( u_{1,1} \) is a \( \mathcal{P}_1 \) - simple function, and since the semivariation \( \mathcal{m} \) is finite on \( \mathcal{P}_1 \times \mathcal{P}_2 \), the semivariation \( m_{u_{1,1}} \) is finite on \( \mathcal{P}_2 \). Similarly as we showed that \( g_1 \in L_1(m_{1,1,1}) \) in the proof of Theorem 6 we conclude that \( f_2 \in L_1(m_{u_{1,1}}). \)

Put \[ u_1 = m_{u_{1,1}}(f_2, T_2) < + \infty. \]

Since \[ \left| \int_{(F_1,F_2-E_{2,k_2})} (u_{1,1}, u_{2,1}) d\Gamma \right| > 2.3 \] and \( E_{2,k} \to F_2 \) as \( k \to \infty \), by the separate countable additivity of \( \mathcal{m} \) in the strong operator topology there is a \( k_1 > k_0 = 1 \) such that \[ \left| \int_{(F_1,E_{2,k_1}-E_{2,k_2})} (u_{1,1}, u_{2,1}) d\Gamma \right| > 2.3. \]

Put \[ u_2,1 = u_{2,1}^* \cdot \chi_{E_{2,k_1}-E_{2,k_2}}. \]

Let \[ l_{u_2,1}(E_1) x_1 = \int_{(E_1,F_2)} (x_1 \cdot \chi_{E_1}, u_{2,1}) d\Gamma \]

for \( E_1 \in \mathcal{P}_1 \) and \( x_1 \in X_1 \). Then \( l_{u_2,1} : \mathcal{P}_1 \to L(X_1, Y) \), it is countably additive in the strong operator topology, and has finite semivariation \( m_{u_{2,1}} \) on \( \mathcal{P}_1 \). Now similarly as above we obtain that \( f_1 \in L_1(l_{u_2,1}) \).

Put \[ b_1 = l_{u_2,1}(f_1, T_1) < + \infty. \]

By assumption \( \mathcal{m}([f_1,f_2], (E_{1,k_1}, F_2 - E_{2,k_2}]) = + \infty. \)

For \( n = 2, 3, \ldots \) we proceed successively in the following way: given \( (u_{1,n-1}, u_{2,n-1}) \in S(\mathcal{P}_1, X_1) \times \times S(\mathcal{P}_2, X_2) \), \( a_{n-1} \) and \( b_{n-1} \), we have \( \mathcal{m}([f_1,f_2], (E_{1,k_{n-1}}, F_2 - E_{2,k_{n-1}}]) = + \infty \) by assumption. Hence there are \( (u_{1,n}, u_{2,n}) \in S(\mathcal{P}_1, X_1) \times S(\mathcal{P}_2, X_2) \) and \( k_n > k_{n-1} \) such that \[ |u_{1,n}^*| \leq |f_1| \cdot \chi_{E_{1,k_{n-1}-1}} \]

\[ |u_{2,n}^*| \leq |f_2| \cdot \chi_{E_{2,k_{n}-E_{2,k_{n}-1}} - 1} \]

and \[ \left| \int_{(F_1,F_2)} (u_{1,n}^*, u_{2,n}^*) d\Gamma \right| > 2^n \cdot r_{k_{n-1}} \cdot 3 \cdot (1 + a_{n-1}) \cdot (1 + b_1) \cdot \ldots \cdot (1 + b_{n-1}) \cdot \]

\[ \ldots \cdot (1 + b_{n-1}) \cdot \]

Put \[ u_{1,n} = 2^{-n} \cdot r_{k_{n-1}}^{-1} \cdot (1 + b_1)^{-1} \cdot \ldots \cdot (1 + b_{n-1})^{-1} \cdot u_{1,n}^* \]

and \[ u_{2,n} = u_{2,n}^* \cdot \chi_{E_{2,k_{n}-E_{2,k_{n}-1}} - 1}. \]

Clearly \[ |u_{1,n}| \leq 2^{-n} \cdot (1 + b_1)^{-1} \cdot \ldots \cdot (1 + b_{n-1})^{-1}. \]
Similarly as above,
\[ a_n = \hat{m}(\sum_{j=1}^{n} u_{1,j})(f_2, T_2) < +\infty \quad \text{and} \quad b_n = \hat{m}(u_{2,n})(f_1, T_1) < +\infty. \]

Obviously \[ \sum_{n=1}^{\infty} |u_{1,n}(t_i)| \leq |f_1(t_i)| < +\infty \] for each \( t_i \in T_i \), for both \( i = 1, 2 \). Put \( u_i = \sum_{n=1}^{\infty} u_{i,n}, \) \( i = 1, 2 \). Then obviously \( u_i: T_i \rightarrow X_i \) is \( \mathcal{P}' \)-measurable and \( |u_i| \leq |f_i| \) for both \( i = 1, 2 \). Hence \((u_1, u_2) \in \mathcal{F}(\Gamma)\) by the definition of \( \mathcal{L}_1(\Gamma)\). Let \( \gamma(A_1, A_2) = \int_{(A_1, A_2)} (u_1, u_2) d\Gamma'\), \( (A_1, A_2) \in \sigma(\mathcal{P}_1') \times \sigma(\mathcal{P}_2') \). Then \( \gamma: \sigma(\mathcal{P}_1') \times \sigma(\mathcal{P}_2') \rightarrow Y \) is a separately countably additive vector bimeasure, see Theorem IX.4. Put \( A_{2,n} = E_{2,k_n} - E_{2,k_{n-1}}, n = 1, 2, \ldots \). Then \( A_{2,n}, n = 1, 2, \ldots \) are pairwise disjoint sets from \( \sigma(\mathcal{P}_2')\), hence \( \gamma(F_1, A_{2,n}) \rightarrow 0 \) as \( n \rightarrow \infty \). Let \( n_0 \) be such that \( |\gamma(F_1, A_{2,n_0})| < 1 \).

Then
\[ 3(1 + a_{n_0-1}) < |\int_{(A_1, A_2, n_0)} (u_{1,n_0}, u_{2,n_0}) d\Gamma| = \]
\[ = |\int_{(A_1, A_2, n_0)} (u_1 - \sum_{j=n_0+1}^{\infty} u_{1,j} - \sum_{j=1}^{n_0-1} u_{1,j}, u_{2,n_0}) d\Gamma| < \]
\[ < 1 + |\int_{(A_1, A_2, n_0)} (\sum_{j=n_0+1}^{\infty} u_{1,j}, u_{2,n_0}) d\Gamma| + \]
\[ + |\int_{(A_1, A_2, n_0)} (\sum_{j=1}^{n_0-1} u_{1,j}, u_{2,n_0}) d\Gamma| < \]
\[ < 1 + 2^{-n_0}b_{n_0}(1 + b_{n_0})^{-1} + a_{n_0-1}, \]
a contradiction. The theorem is proved.

The analog of the next theorem for \( \mathcal{L}_1(\Gamma) \) is evidently valid.

**Theorem 9.** Let \((f_i) \in \mathcal{L}_1(\Gamma)\), for each \( i = 1, \ldots, d \), let \( \mathcal{P}'_i \subset \mathcal{P}_i \) be a \( \delta \)-subring and suppose \( f_i \) is \( \mathcal{P}'_i \)-measurable. Denote by \( \Gamma' \) the restriction \( \Gamma' = \Gamma: X_{\mathcal{P}_1'} \rightarrow L^d(X; Y) \), and suppose that the semivariation \( \hat{\Gamma} \) is locally \( \sigma \)-finite on \( X_{\sigma(\mathcal{P}_i')} \). Then \((f_i) \in \mathcal{L}_1(\Gamma')\), and
\[ (1) \quad \int_{(A_i')} (f_i) d\Gamma' = \int_{(A_i')} (f_i) d\Gamma \]
for each \((A_i) \in X_{\sigma(\mathcal{P}_i')}\). (\( \hat{\Gamma} \) is also locally \( \sigma \)-finite on \( X_{\sigma(\mathcal{P}_i')} \).)

**Proof.** Put \( F'_i = \{ t_i \in T_i, f_i(t_i) \neq 0 \} \in \sigma(\mathcal{P}_i'), i = 1, \ldots, d \). By the assumed local \( \sigma \)-finiteness of the semivariation \( \hat{\Gamma} \) on \( X_{\sigma(\mathcal{P}_i')} \) there are \( F''_{i,r} \in \mathcal{P}_i', r = 1, 2, \ldots, i = 1, \ldots, d \) such that \( F''_{i,r} \neq F'_i \) as \( r \rightarrow \infty \) for each \( i = 1, \ldots, d \), and \( \hat{\Gamma}(F''_{i,r}) < +\infty \) for each \( r = 1, 2, \ldots \). Define \( F'_{i,r} = \{ t_i \in T_i, |f_i(t_i)| \leq r \} \cap F''_{i,r}, i = 1, \ldots, d \) and \( r = 1, 2, \ldots \). Then \((F'_{i,r}) \in X_{\sigma(\mathcal{P}_i')}, \hat{\Gamma}(F'_{i,r}) < +\infty \) and \( \hat{\Gamma}[ (f_i'), (F'_{i,r}) ] \leq r^d \hat{\Gamma}(F'_{i,r}) < +\infty \) for each \( r = 1, 2, \ldots \), and \( F'_{i,r} \neq F_i' \) for each \( i = 1, \ldots, d \).

Let \( u_i: T_i \rightarrow X_i \) be \( \mathcal{P}'_i \)-measurable, \( i = 1, \ldots, d \), and let \( |u_i| \leq |f_i| \) for each \( i \). To prove the theorem we have to show that \((u_i') \in \mathcal{F}(\Gamma')\) and that (1) holds. Since \((f_i') \in \mathcal{F}(\Gamma')\), we have \((u_i') \in \mathcal{F}(\Gamma')\). Hence \( \gamma: X(F'_i \cap \sigma(\mathcal{P}_i')) \rightarrow Y, \gamma(A_i) = \int_{(A_i)} (u_i') d\Gamma', \) is a separately countably additive vector \( d \)-polymeasure. Now, to show that \((u_i') \in \mathcal{F}(\Gamma')\), according to Corollary 2 of Theorem IX.4 it suffices to prove that \((u_i': X_{F_i'}') \in \mathcal{F}(\Gamma') \).
For each \( r = 1, 2, \ldots \), and that for a given \( r \), \( \gamma(A_i' \cap F_{i,r}) = \int_{(A_i')'} (u'_i \cdot \chi_{F_{i,r}}) \, d\Gamma' \) for each \( (A_i') \in (F_i' \cap \sigma(\mathcal{P}_i')) \), hence for each \( (A_i') \in \mathcal{X}(F_i' \cap \mathcal{P}_i') \).

For each \( i = 1, \ldots, d \) take a sequence \( u_{i,n} \in S(\mathcal{P}_i, X_i) \), \( n = 1, 2, \ldots \) such that \( u_{i,n} \rightarrow u'_i \) and \( |u_{i,n} - u'_i| \rightarrow 0 \). Let \( r \in \{1, 2, \ldots \} \) be fixed. Since \( \tilde{F}[(u'_i), (F'_ii)] \leq r^d \tilde{F}(F_{i,r}) < +\infty \) and \( (u'_i \cdot \chi_{F_{i,r}}) \in \mathcal{L}_1(\Gamma) \), by the proof of Theorem 6 we obtain that

\[
\lim_{n_d \rightarrow +\infty} \ldots \lim_{n_1 \rightarrow +\infty} \int_{(A_i')} (u'_{i_1,n_1} \cdot \chi_{F_{i_1,r}}) \, d\Gamma' = \lim_{n_d \rightarrow +\infty} \ldots \lim_{n_2 \rightarrow +\infty} \int_{(A_i')} (u'_{i_2,n_2} \cdot \chi_{F_{i_2,r}}) \ldots u'_{i_d,n_d} \cdot \chi_{F_{i_d,r}}) \, d\Gamma' = \ldots \]

for each \( (A_i') \in \mathcal{X}(F_i' \cap \mathcal{P}_i) \), particularly for each \( (A_i') \in \mathcal{X}(F_i' \cap \mathcal{P}_i) \). But in the last case we may replace \( d\Gamma' \) by \( d\Gamma'' \), hence \( (u'_i \cdot \chi_{F_{i,r}}) \in \mathcal{F}_d(\Gamma') \), and the analog of (1) holds, which we wanted to show. Hence \( (u'_i) \in \mathcal{F}(\Gamma') \) and the analog of (1) holds for \( (u'_i) \). Taking \( (u'_i) = (f_i) \) we obtain (1). The theorem is proved.

Let us note that if \( \Gamma'(...) \colon \mathcal{X}\mathcal{P}_i \rightarrow Y \) has a locally control \( d \)-polymeasure for each \( (x_i) \in \mathcal{X}X_i \), then the assertion of the preceding theorem is a consequence of Theorem X.13.

We are now ready to prove

**Theorem 10.** (Lebesgue dominated convergence theorem in \( \mathcal{L}_1(\Gamma) \).) Let \( f_i, f_{i,n} \colon T_i \rightarrow X_i \), \( n = 1, 2, \ldots \) be \( \mathcal{P}_i \)-measurable for each \( i = 1, \ldots, d \), let the sequence of \( d \)-tuples \((f_{i,n})_n, n = 1, 2, \ldots \) converge \( \Gamma \)-almost everywhere to the \( d \)-tuple \((f_i), i = 1, \ldots, d \), and let there exist a \( d \)-tuple \((g_i) \in \mathcal{L}_1(\Gamma) \) such that \( |f_{i,n}| \leq |g_i|, i = 1, \ldots, d, \Gamma \)-almost everywhere for each \( n = 1, 2, \ldots \). Then \((f_i), (f_{i,n}) \in \mathcal{L}_1(\Gamma), n = 1, 2, \ldots \) and

\[
\lim_{n_1, \ldots, n_d \rightarrow +\infty} \int_{(A_i')} (f_{i,n}) \, d\Gamma' = \int_{(A_i')} (f_i) \, d\Gamma'
\]

for each \( (A_i') \in \mathcal{X}(\mathcal{P}_i) \).

If in each of the \( d \) coordinates either the convergence \( f_{i,n}(t_i) \rightarrow f_i(t_i) \) is uniform with respect to \( t_i \in T_i \), or the multiple \( L_1 \)-gauge \( \tilde{F}[(g_i), (\ldots, T_{i-1}, \ldots, T_{i+1}, \ldots)] \): \( \sigma(\mathcal{P}_i) \rightarrow [0, +\infty) \) is continuous in that coordinate, then the limit in (1) is uniform with respect to \((A_i') \in \mathcal{X}(\mathcal{P}_i) \).

**Proof.** Without loss of generality we may suppose that the second and third assumptions of the theorem hold everywhere. But then \((f_i), (f_{i,n}) \in \mathcal{L}_1(\Gamma), n = 1, 2, \ldots \) by the definition of \( \mathcal{L}_1(\Gamma) \).

Next, the last assertion of the theorem follows easily from the proof of Theorem IX.7 in [13].

Put \( G_i = \{ t_i \in T_i : g_i(t_i) \neq 0 \} \in \sigma(\mathcal{P}_i), i = 1, \ldots, d \). By the assumed local \( \sigma \)-finiteness of the semivariation \( \tilde{F} \) on \( \mathcal{X}\sigma(\mathcal{P}_i) \), see the beginning of Part IX, there are \((G_{i,n})_n \in \mathcal{X}\mathcal{P}_i, n = 1, 2, \ldots \) such that \( G_{i,n} \uparrow G_i \) as \( n \rightarrow \infty \) for each \( i = 1, \ldots, d \), and \( \tilde{F}(A_{i,n}) < +\infty \) for each \( n = 1, 2, \ldots \).

Let \((A_i') \in \mathcal{X}(\mathcal{P}_i) \). From the definition of \( \mathcal{P} \)-measurable functions, see Section 1.2.
in Part I, we immediately see that for each \( i = 1, \ldots, d \) there is a countably generated \( \sigma \)-ring \( \mathcal{P}_{i,(A_i)} \) such that \( \{ G; \chi_i \} \) is \( \mathcal{P}_{i,(A_i)} \)-measurable and hence also the functions \( f_{i,n}, \chi_i \). Further, take separable closed subspaces \( X'_i \subset X_i \) for each \( i = 1, \ldots, d \). Denote by \( \Gamma_{i,(A_i)} \) the restriction \( \Gamma_{i,(A_i)} = \Gamma: X(G_i \cap \mathcal{P}_{i,(A_i)}) \to L^2(X'_i; Y) \). Evidently, the semivariation \( \dot{\Gamma}_{i,(A_i)}(G_i) \leq \Gamma(G_i) \) is \( \sigma \)-finite. Hence, according to Theorem 9, \( (g_i, f_{i,n}) \in L^1(\Gamma_{i,(A_i)}) \), \( n = 1, 2, \ldots \), and on both sides of (1) we may replace \( d\Gamma \) by \( d\Gamma_{i,(A_i)} \).

Owing to Corollary of Theorem VIII.11 and Theorems VIII.17 and VIII.19 of the proof of Theorem 3 in [18] yields (1). Namely, instead of (2) in that proof, by Lemma 2 and Theorem 7 there is an integer \( k_0 \) such that

\[
\left| \int_{(A_i - N_i - G_i', k_0)} (f_{i,n,t} - f_i) \, d\Gamma_{i,(A_i)} \right| \leq 2^d \cdot \dot{\Gamma}_{i,(A_i)} ((g_i), (A_i - N_i - G_i', k_0)) < \varepsilon /4 ,
\]

where \( G_i', k_0 = G_i' \cap \{ t_i \} \) \( \chi_i'(t_i) \leq k_0 \}, \) and \( G_i', k_0 \) is as in that proof. By Fubini’s theorem (Theorem 6) and the inductive assumption we obtain the analog of (3) from the original proof. The rest follows similarly as in [18]. Thus the theorem is proved.

Since for each \( \mathcal{P}_{i,(A_i)} \)-measurable \( g_i, i = 1, \ldots, d \), there is a sequence of \( \mathcal{P}_{i,(A_i)} \)-simple functions \( g_{i,n}, n = 1, 2, \ldots \), such that \( g_{i,n}(t_i) \to g_i(t_i) \) and \( |g_{i,n}(t_i)| < |g_i(t_i)| \) for each \( t_i \in T_i \), we immediately obtain

**Corollary 1.** \( L^1(G) \subset \mathcal{F}_1(G) \).

The next corollaries are also immediate.

**Corollary 2.** (Lebesgue bounded convergence theorem in \( L^1(G) \).) Let \( (x_i, \chi_i) \in L^1(G) \) for each \( (x_i) \in X \) and each \( (A_i) \in X \sigma(\mathcal{P}_i) \). Then the assertions of the theorem hold if \( \sup_{i,n} \| f_{i,n} \|_{T_i} < +\infty \).

**Corollary 3.** (Special case of LBCT in \( L^1(G) \), see Theorem 3 in [18].) Let each of \( X_i, i = 1, \ldots, d \), be a finite dimensional Banach space, and let each \( \mathcal{P}_i, i = 1, \ldots, d \), be a \( \sigma \)-ring. Then the assertions of the theorem hold if \( \sup_{i,n} \| f_{i,n} \|_{T_i} < +\infty \).

Evidently, our assertion (1) of the Lebesgue dominated convergence theorem in \( L^1(G) \) is stronger than the result of Corollary 2.9 in [1], obtained for scalar bimeasures, see the paragraph after Corollary 3 of Theorem 6. It is an interesting novelty that the proof of LDCT in \( L^1(G) \) requires the Fubini theorem in \( L^1(G) \), whose proof requires the weaker (iterated limit) version of LDCT.
References


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