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CAUCHY SEQUENCES IN $\mathcal{L}$-GROUPS

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The relationship between Cauchy sequences in an $\mathcal{L}$-group $G$ and Cauchy filters in the first countable filter modification $\gamma G$ of $G$ (introduced by R. Beattie and H.-P. Butzmann in [4]) is investigated. In particular, an $\mathcal{L}$-group $G$ (without the Urysohn axiom of convergence) and a Cauchy sequence $S$ in $G$ such that the corresponding elementary filter of sections of $S$ fails to be a Cauchy filter in $\gamma G$ is constructed.

1.

In what follows, $N$ denotes the positive integers, $MON$ the set of all strictly monotone mappings of $N$ into $N$ and $FTON$ the set of all finite-to-one mappings of $N$ into $N$ (i.e., $\{n \in N; s(n) = k\}$ is a finite set whenever $s \in FTON$ and $k \in N$). Let $G$ be a nonempty set; a sequence $S = \langle S(n) \rangle$ of points of $G$ is a mapping of $N$ into $G$, and for $s \in MON$ the composition $S \circ s$ denotes the subsequence of $S$ the $n$-th term of which is $S(s(n))$; for $x \in G$, $\langle x \rangle$ denotes the constant sequence each term of which is $x$; if $S, T$ are sequences in $G$, then $S \land T$ is defined by $(S \land T) (2n - 1) = S(n)$ and $(S \land T) (2n) = T(n)$, $n \in N$; if $S$ is a sequence in $G$ then the sets $\{S(n); n > k\}, k \in N$, form a base of the so-called elementary (Fréchet) filter $\mathcal{F}(S)$ of sections of $S$; by a sequential convergence on $G$ we understand a subset $\mathcal{G} \subset G^N \times G$ satisfying certain axioms of convergence (throughout the paper we assume that every constant sequence $\langle x \rangle$ converges to $x$, each subsequence of a convergent sequence converges to the same limit, and, with the exception of Proposition 1 and Proposition 2, every convergent sequence has a unique limit), $(S, x) \in \mathcal{G}$ means that $S$ converges (i.e. $\mathcal{G}$-converges) to $x$, and for $x \in G$ the set of all sequences converging to $x$ is denoted by $\mathcal{G}^{-1}(x)$. Let $G$ be a group equipped with a sequential convergence $\mathcal{G}$ such that $(ST^{-1}, xy^{-1}) \in \mathcal{G}$ whenever $(S, x) \in \mathcal{G}$ and $(T, y) \in \mathcal{G}$. Then $(G, \mathcal{G})$, or simply $G$, is said to be an $\mathcal{L}$-group (cf. [7]). We are mainly interested in abelian groups and in such cases the additive notation will be used.

Besides the basic axioms of convergence, we consider the following ones (cf. [6]):

($\mathcal{F}$) if $(S, x) \in \mathcal{G}$ and $\mathcal{F}(S) = \mathcal{F}(T)$, then $(T, x) \in \mathcal{G}$;

($\mathcal{M}$) if $(S, x), (T, x) \in \mathcal{G}$, then $(S \land T, x) \in \mathcal{G}$. 

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Starting with a filter convergence $A$ on a set $X$ (we assume that for each $x \in X$ the ultrafilter $\hat{x}$ converges to $x$, and if a filter converges to $x$, then each finer filter converges to $x$), the most natural way to define a sequential convergence on $X$ is to let a sequence $S$ converge to a point $x$ whenever the elementary filter $F(S)$ $A$-converges to $x$; denote by $\mathcal{L}(A)$ the resulting sequential convergence. As shown in [4], [2], [1] and [3], among all known opposite functors (assigning to suitable sequential convergences certain filter convergences) the one introduced by R. Beattie and H.-P. Butzmann plays a fundamental role: starting with a sequential convergence $\mathcal{L}$ on $X$, a filter $F$ on $X$ converges to a point $x$ whenever there is a finer filter $\mathcal{F}$ with a countable basis such that every sequence $\mathcal{L}$-converges to $x$ whenever $F(S) \supseteq \mathcal{F}$; denote by $\gamma(\mathcal{L})$ the resulting filter convergence.

The importance of $\gamma$ follows, for instance, from the fact that the Novák completion of an abelian sequential convergence (the convergence is maximal, i.e., satisfies the Urysohn axiom) group $G$ (cf. [11], [8]) can be constructed via the completion of the filter convergence group $\gamma G$ (see Corollary 3.16 in [1]) and, for every sequentially determined filter convergence group $H$ (i.e. $H = \gamma \mathcal{H}$) with a maximal sequential convergence, the completion of $H$ can be constructed via the Novák completion of $\mathcal{L}H$ (see Corollary 3.18 in [1], cf. Theorem 8 in [3]). This is partly due to the fact that in case of a maximal sequential convergence a sequence $S$ is Cauchy in $G$ iff $F(S)$ is a Cauchy filter in $\gamma G$. In view of Proposition 3.11 in [1], if the sequential convergence in $G$ is not maximal then this might be not true any more. Indeed, answering a question by R. Beattie and H.-P. Butzmann, we construct an $\mathcal{L}$-group $G$ and a Cauchy sequence $S$ such that $F(S)$ fails to be a Cauchy filter in $\gamma G$.

Our construction is based on the fact that in a group $G$ every compatible sequential convergence on $G$ can be identified with a certain subgroup of $G^N$. The straightforward proofs of the next two propositions are omitted. Similar propositions (with different axioms of convergence) can be found in [9] and [12].

**Proposition 1.** Let $(G, \mathfrak{B})$ be an $\mathcal{L}$-group and let $e$ be the neutral element of $G$. Then $\mathfrak{B}(e)$ has the following properties:

(i) $\mathfrak{B}(e)$ is a subgroup of $G^N$;
(ii) $\mathfrak{B}(x) = \langle x \rangle$ $\mathfrak{B}(e) = \mathfrak{B}(e) \langle x \rangle$ for all $x \in G$;
(iii) if $S \in \mathfrak{B}(e)$ and $s \in MON$, then $S \cdot s \in \mathfrak{B}(e)$;
(iv) $\mathfrak{B}$ has unique limits iff $\langle e \rangle$ is the only constant sequence in $\mathfrak{B}(e)$;
(v) $\mathfrak{B}$ satisfies axiom $(\mathcal{M})$ iff the following implication holds: if $S \in \mathfrak{B}(e)$, then $S \wedge \langle e \rangle \in \mathfrak{B}(e)$;
(vi) $\mathfrak{B}$ satisfies axiom $(\mathcal{F})$ iff the following implication holds: if $S \in \mathfrak{B}(e)$, $T \in G^N$ and $F(T) = F(S)$, then $T \in \mathfrak{B}(e)$.

Let $G$ be a group. Identifying $x \in G$ with $\langle x \rangle \in G^N$, we can consider $G$ to be a subgroup of $G^N$. A subgroup $H$ of $G^N$ is said to be normal with respect to $G$ if $gSg^{-1} = \langle g \rangle S \langle g^{-1} \rangle \in H$ whenever $g \in G$ and $S \in H$. Let $\mathcal{A}$ be a subset of $G^N$. Let $\mu \mathcal{A}$ be the set of all sequences $S \wedge \langle e \rangle$ such that $S \in \mathcal{A}$, let $\delta \mathcal{A}$ be the set of all sequences
such that $S \in \mathcal{A}$ and $s \in MON$, and let $\varphi \mathcal{A}$ be the set of all sequences $T \in G^N$ such that $\mathcal{F}(T) = \mathcal{F}(S)$ for some $S \in \mathcal{A}$. Consider the set of all subgroups of $G^N$ containing $\mathcal{A}$ and normal with respect to $G$. Denote by $[\mathcal{A}]_G$ the intersection of all such subgroups. Then $G^N$ is the largest and $[\mathcal{A}]_G$ the smallest element of the set.

**Proposition 2.** Let $G$ be a group and let $\mathcal{A}$ be a subset of $G^N$.

(i) $[\mathcal{A}]_G$ consists precisely of the finite products of sequences of the form $gS^g^{-1} = \langle g S(n)^e g^{-1} \rangle$, where $g \in G$, $S \in \mathcal{A}$ and $e = \pm 1$.

(ii) $[\varphi \delta \mathcal{A}]_G$ is the smallest subgroup of $G^N$ containing $\mathcal{A}$, normal with respect to $G$ and closed with respect to $\delta$ and $\varphi$.

(iii) There is a sequential convergence $\mathcal{S}_{\mathcal{A}}$ on $G$ satisfying axiom ($\mathcal{F} \mathcal{L}$) such that $(G, \mathcal{S}_{\mathcal{A}})$ is an $\mathcal{L}$-group and $\mathcal{A} \subseteq [\varphi \delta \mathcal{A}]_G = \mathcal{S}_{\mathcal{A}}(e)$.

(iv) If $(G, \mathcal{S})$ is an $\mathcal{L}$-group such that $\mathcal{S}$ satisfies axiom ($\mathcal{F} \mathcal{L}$) and $\mathcal{A} \subseteq \mathcal{S}^-(e)$, then $\mathcal{S}_{\mathcal{A}} \subseteq \mathcal{S}$.

(v) $\mathcal{S}_{\mathcal{A}}$ has unique limits iff $[\varphi \delta \mathcal{A}]_G$ contains no constant sequence except $\langle e \rangle$.

(vi) $[\varphi \mu \mathcal{A}]_G$ is the smallest subgroup of $G^N$ containing $\mathcal{A}$, normal with respect to $\mathcal{G}$ and closed with respect to $\mu, \delta$ and $\varphi$.

(vii) There is a sequential convergence $\mathcal{S}_{\mathcal{A}}$ on $G$ satisfying axioms ($\mathcal{F} \mathcal{L}$) and ($\mathcal{M} \mathcal{L}$) such that $(G, \mathcal{S}_{\mathcal{A}})$ is an $\mathcal{L}$-group and $\mathcal{A} \subseteq [\varphi \mu \mathcal{A}]_G = \mathcal{S}_{\mathcal{A}}(e)$.

(viii) If $(G, \mathcal{S})$ is an $\mathcal{L}$-group such that $\mathcal{S}$ satisfies axioms ($\mathcal{F} \mathcal{L}$) and ($\mathcal{M} \mathcal{L}$) and $\mathcal{A} \subseteq \mathcal{S}^-(e)$, then $\mathcal{S}_{\mathcal{A}} \subseteq \mathcal{S}$.

(ix) $\mathcal{S}_{\mathcal{A}}$ has unique limits iff $[\varphi \mu \mathcal{A}]_G$ contains no constant sequence except $\langle e \rangle$.

2.

Cauchy sequences in $\mathcal{L}$-groups have been studied, e.g., in [5] and [10]. Recall that a sequence $S$ in an $\mathcal{L}$-group is Cauchy if $S \circ s - S \circ t$ converges to 0 for all $s, t \in MON$.

**Definition.** Let $G$ be an $\mathcal{L}$-group. A sequence $S$ of points of $G$ is said to be FTON-Cauchy if $S \circ s - S \circ t$ converges to 0 for all $s, t \in$ FTON.

By Proposition 3.11 in [1], in an $\mathcal{L}$-group $G$ a sequence $S$ is FTON-Cauchy iff $\mathcal{F}(S)$ is a Cauchy filter in $\gamma G$. Further (cf. Corollary 3.12 in [1]), if the sequential convergence in $G$ is maximal, then each Cauchy sequence in $G$ is FTON-Cauchy. In this section we construct (Example 1) an $\mathcal{L}$-group $G$ satisfying axiom ($\mathcal{F} \mathcal{L}$) in which a Cauchy sequence need not be FTON-Cauchy. The construction is then modified (Example 2) so that $G$ satisfies axioms ($\mathcal{F} \mathcal{L}$) and ($\mathcal{M} \mathcal{L}$).

**Example 1.** Let $X$ be a countably infinite set arranged into a one-to-one sequence $S = \langle S(n) \rangle$. Let $G$ be the free abelian group generated by $X$. Denote by $\mathcal{A}$ the set of all sequences of the form $S \circ s - S \circ t$, where $s, t \in MON$. Observe that for each $T \in \mathcal{A}$ and each $s \in MON$ we have $-T \in \mathcal{A}$ and $T \circ s \in \mathcal{A}$. We shall define a sequential convergence $\mathcal{S}$ on $X$ satisfying axiom ($\mathcal{F} \mathcal{L}$) in such a way that, first, each sequence
in $\mathcal{S}$ converge to 0 (hence $S$ will be a Cauchy sequence), and, secondly, for $u \in FTON$ defined by $u(1) = 1$, $u(2) = u(3) = 2$, $u(4) = u(5) = u(6) = 3$, ..., the sequence $S \circ u$ will not $\mathcal{S}$-converge to 0 (hence $S$ will not be a FTON-Cauchy sequence).

In view of Proposition 2 it suffices to construct $\mathcal{N} \subset G^N$ such that:

(i) $\mathcal{N}$ is a subgroup of $G^N$;

(ii) $\mathcal{A} \subset \mathcal{N}$;

(iii) $T \circ s \in \mathcal{N}$ whenever $T \in \mathcal{N}$ and $s \in MON$;

(iv) if $T \in \mathcal{N}$, $U \in G^N$ and $\mathcal{F}(T) = \mathcal{F}(U)$, then $U \in \mathcal{N}$;

(v) $\langle x \rangle \notin \mathcal{N}$ whenever $x \neq 0$;

(vi) $S - S \circ u \notin \mathcal{N}$;

and then put $(T, x) \in \mathcal{S}$ iff $T - \langle x \rangle \in \mathcal{N}$. Observe that $\mathcal{N} = \mathcal{S}^-(0)$.

Define $\mathcal{N}$ as follows: $T \in \mathcal{N}$ iff there are $k \in \mathbb{N}$, $T_i \in \mathcal{A}$, $s_i \in FTON$, $i = 1, \ldots, k$, such that $T(n) = (T_1 \circ s_1 + \ldots + T_k \circ s_k)(n)$ for all but finitely many $n \in \mathbb{N}$.

Claim. $\mathcal{N}$ satisfies all conditions (i)-(vi).

Proof. Clearly, $\mathcal{N}$ satisfies conditions (i), (ii) and (iii). Condition (iv) follows immediately from Proposition 2 in [2] which asserts that (in sequential convergence spaces in which the convergence of a sequence does not depend on finitely many terms of the sequence) axiom (\textit{FP}) is equivalent to the fact that a sequence $T \circ t$ converges to $x$ whenever $T$ converges to $x$ and $t \in FTON$. Since $G$ is a free group over the set $\{s(n); n \in \mathbb{N}\}$, $\langle x \rangle \notin \mathcal{N}$ for all $x \in G$, $x \neq 0$, and hence $\mathcal{N}$ satisfies condition (v). Finally, given $k \in \mathbb{N}$ and $s_i, t_i \in MON$, $u_i \in FTON$, $i = 1, \ldots, k$, consider for each $n \in \mathbb{N}$ the following proposition:

$$(S - S \circ u)(n) = ((S \circ s_1 - S \circ t_1) \circ u_1)(n) + \ldots + ((S \circ s_k - S \circ t_k) \circ u_k)(n)$$

and then put $(T, x) \in \mathcal{S}$ iff $T - \langle x \rangle \in \mathcal{N}$. Observe that $\mathcal{N} = \mathcal{S}^-(0)$.

Define $\mathcal{N}$ as follows: $T \in \mathcal{N}$ iff there are $k \in \mathbb{N}$, $T_i \in \mathcal{A}$, $s_i \in FTON$, $i = 1, \ldots, k$, such that $T(n) = (T_1 \circ s_1 + \ldots + T_k \circ s_k)(n)$ for all but finitely many $n \in \mathbb{N}$.

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$$(S - S \circ u)(n) = ((S \circ s_1 - S \circ t_1) \circ u_1)(n) + \ldots + ((S \circ s_k - S \circ t_k) \circ u_k)(n)$$

denote it by $P(n, (s_1, \ldots, s_k), (t_1, \ldots, t_k), (u_1, \ldots, u_k))$ or, simply by $P(n)$. To prove condition (vi) it suffices to prove that for each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $q > p$, such that proposition $P(q)$ is false. The proof is based on the so called box principle (if we place more than $n$ objects into $n$ boxes, then one of the boxes contains at least two objects) and the following observations.

\textit{(O₁)} For each $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $j < |\{p \in \mathbb{N}; u(p) = m\}|$ and $j < m < \min \{p \in \mathbb{N}; u(P) = m\}$; hence the sequence $S \circ u$ has arbitrarily long (finite) constant segments, while $\langle S - S \circ u \rangle(n + 2)$ is a one-to-one sequence.

\textit{(O₂)} For each $i \in \{1, \ldots, k\}$ we have $(S \circ s_i - S \circ t_i) \circ u_i = S \circ s_i \circ u_i - S \circ t_i \circ u_i$ and $(S \circ s_i \circ u_i)(n) = (S \circ s_i \circ u_i)(m)$ iff $(S \circ t_i \circ u_i)(n) = (S \circ t_i \circ u_i)(m)$, i.e., the sequences $S \circ s_i \circ u_i$ and $S \circ t_i \circ u_i$ are constant on the same segments of $\mathcal{N}$.

Now assume that, on the contrary, for some $k \in \mathbb{N}$ and for some $s_i, t_i \in MON$, $u_i \in FTON$, $i = 1, \ldots, k$, proposition $P(n)$ holds for all but finitely many $n \in \mathbb{N}$. We claim that then for each $p \in \mathbb{N}$ there exist $j_1, j_2 \in \mathbb{N}$ such that $p < j_1 < j_2$ and
proposition \((P_{ji})\) is of the form
\[ x_1 - x = (y_1 - z_1) + \ldots + (y_k - z_k) , \]
and at the same time proposition \(P(j_2)\) is of the form
\[ x_2 - x = (y_1 - z_1) + \ldots + (y_k - z_k) , \]
where \(x, x_1, x_2\) and also \(y_i, z_i, i = 1, \ldots, k\), are generators of the free group \(G\) and \(x_1 \neq x_2\). Since this is clearly impossible, either \(P(j_1)\) or \(P(j_2)\) is a false proposition. However, the claim is a straightforward consequence of \((O_1)\) and \((O_2)\) and the box principle. Indeed, using \((O_1)\), start with a sufficiently large set \(M \subset N\) such that propositions \(P(j), j \in M\), have the form
\[ x - x = (y_1 - z_1) + \ldots + (y_k - z_k) , \]
where \(x\) and \(x_i, i = 1, \ldots, k\), are generators of the free group \(G\) and \(x \neq x_i\). Using repeatedly \((O_2)\) and the box principle, we find a subset \(\{j_1, j_2\}\) of \(M\) such that for each \(i \in \{1, \ldots, k\}\) we have \(y_{j_i} = y_{j_1}, z_{j_i} = z_{j_1}\). This complete the proof.

Example 2. Let \(X, G, S\) and \(\mathcal{A}\) be the same as in Example 1. Let \(\mu, A\) be the set of all sequences \(T\) in \(G\) such that \(T = U \wedge \langle 0 \rangle\) for some \(U \in \mathcal{A}\). Define \(\mathcal{G}^\ast(0) \subset \mathcal{G}^N\) as follows: \(T\) belongs to \(\mathcal{G}^\ast(0)\) if there are \(k \in N, T_i \in \mu, s_i \in \text{FTON}, i = 1, \ldots, k\), such that \(T(n) = (T \circ s_1 + \ldots + T \circ s_k)(n)\) for all but finitely many \(n \in N\). Finally, define \(\mathcal{G} \subset \mathcal{G}^N \times G\) by putting \((T, x) \in \mathcal{G}\) if \((T - \langle x \rangle) \in \mathcal{G}^\ast(0)\). In a similar way as in Example 1 it can be proved that \(G\) equipped with \(\mathcal{G}\) is an \(\mathcal{L}\)-group satisfying axioms \((\mathcal{F} L)\) and \((\mathcal{M} L)\) in which \(S\) is a Cauchy sequence but fails to be \(\text{FTON-Cauchy}\).

Corollary 1. There exists an \(\mathcal{L}\)-group \(G\) satisfying axioms \((\mathcal{F} L)\) and \((\mathcal{M} L)\), and a Cauchy sequence \(S\) in \(G\) such that \(\mathcal{F}(S)\) fails to be a Cauchy filter in \(\gamma G\).

The following result has been announced in [3].

Corollary 2. There exists an incomplete \(\mathcal{L}\)-group \(H\) satisfying axioms \((\mathcal{F} L)\) and \((\mathcal{M} L)\) such that \(\gamma H\) is complete.

Proof. Consider the \(\mathcal{L}\)-group \((G, \mathcal{G})\) from Example 2. Then \(G\) equipped with \(\lambda = \gamma(\mathcal{G})\) is a sequentially determined convergence group. Let \((\hat{G}, \hat{\lambda}, e_\mathcal{G})\) be the categorical completion of \((G, \lambda)\). By Theorem 3.9 in [1], \((\hat{G}, \hat{\lambda})\) is sequentially determined. Put \(H = (\hat{G}, \mathcal{L}(\hat{\lambda}))\). Then \(\lambda H = (\hat{G}, \hat{\lambda})\). Clearly, \(\mathcal{L}(\hat{\lambda})\) satisfies axioms \((\mathcal{F} L)\) and \((\mathcal{M} L)\), and \(\mathcal{L}(\hat{\lambda})\) restricted to \(G\) equals \(\mathcal{G}\). Then \(S\) is a Cauchy sequence in \(H\) but fails to converge. Otherwise, \(\hat{\lambda}\) being sequentially determined, \(\mathcal{F}(S)\) would be \(\hat{\lambda}\)-convergent and hence \(\hat{\lambda}\)-Cauchy. But Proposition 3.11 in [1] would imply that \(S\) is \(\text{FTON-Cauchy}\) in \((G, \mathcal{G})\), a contradiction.
Motivated by Example 2 let us consider the following problem. Let \((G, \mathcal{G})\) be an \(L\)-group, let \(C\) be the set of all Cauchy sequences in \(G\) and let \(\sim\) be the usual equivalence for \(C\), i.e., \(S \sim T\) iff \(S - T\) converges to 0. Let \(f\) be a mapping of \(C\) into the set \(P(G^N)\) of all subsets of \(G^N\). Under what conditions is \(f(S)\) a set of Cauchy sequences each of which is equivalent to \(S\), \(S \in C\)? For instance, if \(f(S) = \{T \in G^N; \mathcal{F}(S) = \mathcal{F}(T)\}\) and \(\mathcal{G}\) is a maximal sequential convergence, then each \(T \in f(S)\) is a Cauchy sequence equivalent to \(S\). Is this true if \(\mathcal{G}\) satisfies axioms \((\mathcal{F}L)\) and \((\mathcal{ML})\) but fails to be maximal?

A similar question can be asked for general Cauchy structures, namely, given a Cauchy structure and an equivalence relation, under what conditions what operations on Cauchy objects preserve the equivalence classes?

References


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