Albert J. Milani

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GLOBAL EXISTENCE FOR QUASI-LINEAR DISSIPATIVE HYPERBOLIC EQUATIONS WITH LARGE DATA AND SMALL PARAMETER

ALBERT J. MILANI, Milwaukee

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1. INTRODUCTION

Consider the following dissipative quasi-linear hyperbolic initial-value problem

$$(\mathrm{H}_\varepsilon) \qquad \varepsilon u_{tt} + u_t - \sum_{i,j=1}^{n} a_{ij}(\nabla u)\,\partial_i\partial_j u = 0$$

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)$$

where $\varepsilon \in \boldsymbol{R}^+$, $u = u(x,t) \in \boldsymbol{R}^1$, $t \geqq 0$, $x \in \boldsymbol{R}^n$, $\partial_i = \partial/\partial x_i$ and ∇ is the gradient with respect to the space variables only. The coefficients a_{ij} are smooth symmetric, and satisfy the strong ellipticity condition

$$(1) \qquad \forall p \in \boldsymbol{R}^n\,, \quad \forall q \in \boldsymbol{R}^n\,, \quad \sum a_{ij}(p)\,q^i q^j \geqq a(p)\,|q|^2\,,$$

where $a\colon \boldsymbol{R}^n \to \boldsymbol{R}$ is a continuous function, such that $a(0) > 0$. We are interested to smooth solutions to (H_ε), corresponding to smooth initial data, and to conditions that ensure that such solutions exist globally in time. Local existence results for (H_ε) are well known; we refer mainly to Kato's general theory of [1], in the framework of the Sobolev spaces $H^s = H^s(\boldsymbol{R}^n)$, of sufficiently high order s $(s > 1 + n/2)$. Such results are generally established under some smallness assumptions on the initial data; since (H_ε) is dissipative, a further smallness assumption on the data allows us to extend these local solutions to all later times (Matsumura, [2]). These smallness conditions on the data refer to their H^s norm, so that all their derivatives are usually required to have a small L^2 norm. We have recently been interested in the question, whether such smallness requirements on the data can be replaced by a smallness condition on the parameter ε; in particular, making essential use of the assumption that the coefficients a_{ij} do not depend on u_t (they might, however, depend on x, t and u), we have been able to prove that (H_ε) is globally solvable for small ε in the following circumstances:

1) a smallness condition is required on the norm of u_0 only, and not on that of u_1 ([3]);

2) if a uniformly strong ellipticity condition holds, that is if in (1) we have that $a(p) \geqq a > 0$, then no smallness condition has to be required of u_0 as well, provided that the reduced parabolic problem

$$\text{(P)} \qquad v_t - \sum a_{ij}(\nabla u)\, \partial_i \partial_j v = 0\,, \qquad v(x, 0) = u_0(x)$$

has a global smooth solution ([4]);

3) for the equation in divergence form and in one space variable

$$\text{(W)} \qquad \varepsilon u_{tt} + u_t - \sigma(u_x)_x = 0\,, \quad \sigma'(0) > 0\,,$$

a smallness condition is required only on the L^∞ norm of u_0 and u_{0x}, with no restriction on the higher order derivatives ([5]).

We remark that this last result is somehow to be expected: in fact, since $\sigma'(0) > 0$, (W) remains hyperbolic for as long as $|u_{0x}|$ is close to 0. The purpose of this paper is to extend this last result to the higher dimensional case $n > 1$; we shall not require (H_ε) to be in divergence form, although we do still require that the a_{ij} do not depend on u_t.

2. NOTATIONS AND STATEMENTS OF RESULTS

We refer to Kato's classical framework of Sobolev spaces $H^s(\boldsymbol{R}^n)$, with (integer) $s > (1 + n)/2$. For $m \in \boldsymbol{N}$ we set $H^m = H^m(\boldsymbol{R}^n)$, and note $\|\cdot\|_m$ and $(\cdot, \cdot)_m$ its norm and scalar product (we shall drop the index $m = 0$); we note $|\cdot|$ the L^∞ norm, and if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, we write $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. As in [3], we assume that the coefficients a_{ij} satisfy (1) and some growth conditions near the origin, of the form

$$\text{(2)} \qquad |D^\alpha a_{ij}(p)| \leqq h_\alpha(|p|)$$

where $h_\alpha\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a continuous non decreasing function (we shall actually denote by h any such generic function, not necessarily the same, just like c will denote any generic positive constant, not depending on ε or on t).

We set $s_0 = E(n/2) + 1$, $s_1 = s_0 + 1$ and, for $s \geqq s_1$, $T \in]0, +\infty]$, $r \in \boldsymbol{R}^+$,

$$X(T) = C([0, T]; H^{s+1}) \cap C^1([0, T]; H^s)\,,$$

$$B(r) = \{u \in H^{s+1} \mid \|u\|_{s_1} \leqq r\}\,.$$

Since the function a is continuous and $a(0) > 0$, there exist two constants $v_0 > 0$ and $v \in]0, a(0)]$ such that

$$a(p) \geqq v > 0 \quad \text{if} \quad |p| \leqq v_0 :$$

by Sobolev's imbedding theorems, $H^{s_0} \hookrightarrow L^\infty$ because $2s_0 > n$, so that a third constant $\delta_0 > 0$ exists such that if $u \in H^{s_1}$, then

$$|\nabla u| \leqq v \quad \text{if} \quad \|\nabla u\|_{s_0} \leqq \delta_0\,,$$

In particular, this means that if for $w \in B(\delta_0)$ and $m \in N$ we define

$$Q_m(w; \nabla u) = \sum_{|\alpha| \leqq m} \sum_{i,j} (a_{ij}(\nabla w)\, \partial_i D^\alpha u, \partial_j D^\alpha u),$$

$Q_m(w; \cdot)$ is a quadratic form on H^{m+1} and actually, because of (1) and (2), an equivalent seminorm: in fact we have, $\forall u \in H^{m+1}$:

$$\text{(3)} \qquad \nu \|\nabla u\|_m^2 \leqq Q_m(w; \nabla u) \leqq h(\delta_0) \|\nabla u\|_m^2 .$$

Throughout the paper we shall always assume that $u_0 \in H^{s+1}$ and $u_1 \in H^s$, with $s \geqq s_1$, and that $\varepsilon \in]0, 1]$. Given $m \in N$ and $u \in H^{s+1}$, we abbreviate $Q_m(\nabla u)$ for $Q_m(u; \nabla u)$ and define

$$E_m(u, t) = (s\|u_t\|_m^2 + \tfrac{1}{2}\|u\|_m^2 + s(u, u_t)_m + Q_m(\nabla u))(t) + \\ + \tfrac{1}{2} \int_0^t (\|u_t\|_m^2 + Q_m(\nabla u)),$$

under these conditions, the following uniformly local existence result was proved in [3].

Theorem 1: *There exist $\delta_1 \in]0, \delta_0[$ and $\varepsilon_1 \in]0, 1]$ such that if $\|u_0\| \leqq \delta_1$ and $\varepsilon \leqq \varepsilon_1$, (H_ε) has a unique local solution $u \in X(\theta)$, for some $\theta > 0$ depending on $\|u_0\|_{s+1}$ but not on ε. Moreover, u satisfies the bound*

$$\text{(4)} \qquad \forall t \in [0, 0], \quad E_s(u, t) \leqq \nu\, \delta_0^2 .$$

We recall that ε_1 depends on $\|u_1\|_s$ and δ_1 (and therefore on ν through δ_0 and ν_0); θ depends on δ_0, but not on ε if $\varepsilon \leqq \varepsilon_1$: this is what we mean by "uniformly local". Also, (4) assures that (H_ε) remains hyperbolic in all of $[0, \theta]$: in fact, since $\varepsilon \leqq 1$,

$$\varepsilon\|u_t\|^2 + \tfrac{1}{2}\|u\|^2 + \varepsilon(u, u_t) \geqq 0,$$

so that from (3)

$$\|\nabla u(t)\|_{s_0}^2 \leqq \nu^{-1} Q_{s_0}(\nabla u(t)) \leqq \nu^{-1} E_s(u, t) \leqq \delta_0^2 ,$$

and therefore $|\nabla u(t)| \leqq \nu_0$. This indicates, however, that to guarantee the local existence of solutions of (H_ε) it should not be necessary to require an a priori bound on the derivatives of u_0 of order higher than s_1: in fact, a straightforward modification of the proof of Theorem 1, based on the use of the estimates we shall established later on in Proposition 3, would allow us to prove the following.

Theorem 2: *There exist $\delta_2 \in]0, \delta_0[$ and $\varepsilon_2 \in]0, 1]$ such that if $u_0 \in B(\delta_2)$ and $\varepsilon \leqq \varepsilon_2$, (H_ε) has a unique local solution $u \in X(\theta)$, for some θ not dependent of ε. Moreover, given $\Delta^2 > E_s(u, 0)$, θ can be chosen so that u satisfies the bounds*

$$\text{(5)} \qquad \forall t \in [0, \theta], \quad E_{s_0}(u, t) \leqq \nu\, \delta_0^2 , \quad E_s(u, t) \leqq \Delta^2 .$$

We remark that here ε_2 depends on $\|u_1\|_{s_0}$ and δ_2, and θ depends on δ and Δ but, again, not on ε if $\varepsilon \leqq \varepsilon_2$. Also, a similar result would hold when $\varepsilon = 1$, providing local existence under a smallness requirement on $\|u_0\|_{s_1}$ and $\|u_1\|_{s_0}$ only, and this would hold even if the dissipation term u_t is missing; this last result, however, would not be uniform with respect to variations of ε.

We propose now to extend the local solutions of (H_ε) provided by Theorem 2 to all later times. We claim

Theorem 3: *There exist $\delta_3 \in]0, \delta_2]$ and $\varepsilon_3 \in]0, \varepsilon_2]$ such that if $u_0 \in B(\delta_3)$ and $\varepsilon \leqq \varepsilon_3$, (H_ε) has a unique solution $u \in X(+\infty)$, satisfying the bound*

$$(6) \qquad \forall t \geqq 0\,, \quad \|u(t)\|_{s_1} \leqq \delta_3\,.$$

Moreover, there exists $M^2 > E_s(u, 0)$, depending on δ_3 but not on ε, such that

$$(7) \qquad \forall t \geqq 0\,, \quad E_s(u, t) \leqq M^2\,.$$

We consider this result a generalization of the one given in [5] for (W): in fact, if $n = 1$ then $s_1 = 2$ and, by Sobolev's imbedding theorem, $H^{s_1} = H^2 \hookrightarrow W^{1,\infty}$, so that

$$|u_0| + |u_{0x}| \leqq \|u_0\|_2 = \|u_0\|_{s_1}\,.$$

3. PROOF OF THEOREM 3

We shall establish the global existence Theorem 3 by proving that solutions of (H_ε) in $X(T)$ can be bounded a priori independently of T if $\|\nabla u_0\|_{s_0}$ and ε are sufficiently small. For $m \geqq s_1$ and $u \in X(T)$, $T > 0$ given, we define

$$N_m(u, t) = (\varepsilon^2\|u_t\|_m^2 + \tfrac{1}{2}\|u\|_m^2 + \varepsilon(u, u_t)_m +$$
$$+ \varepsilon\, Q_m(\nabla u))(t) + \tfrac{1}{2}\int_0^t (\varepsilon\|u_t\|_m^2 + Q_m(\nabla u))\,,$$

and claim.

Proposition 1: *Assume (H_ε) has a solution $u \in X(T)$ such that $\forall t \in [0, T]$, $N_{s_1}(u, t) \leqq \omega^2$, for some $T > 0$ and $\omega > 0$. There exists $\omega_0 > 0$, independent of T and ε, such that if $\omega \leqq \omega_0$ and $s_1 \leqq m \leqq s$, u satisfies the estimates*

$$(8) \qquad \forall t \in [0, T]\,, \quad N_m(u, t) \leqq N_m(u, 0)\,.$$

Proposition 2: *Assume (H_ε) has a solution $u \in X(T)$, for some $T > 0$, such that $\forall t \in [0, T]$, $N_{s_1}(u, t) \leqq \omega_0^2$. Then if $s_1 \leqq m \leqq s$, u satisfies the estimates*

$$(9) \qquad \forall t \in [0, T]\,, \quad E_m(u, t) \leqq E_m(u, 0) \exp h(\omega_0) \int_0^t Q_m(\nabla u)\,.$$

We postpone the proof of Propositions 1 and 2 to the next sections, and proceed to prove Theorem 3 assuming that these Propositions hold. We have that

$$(10) \qquad N_m(u, 0) = \varepsilon^2\|u_1\|_m^2 + \tfrac{1}{2}\|u_0\|_m^2 + \varepsilon(u_0, u_1)_m + \varepsilon\, Q_m(\nabla u_0) \leqq \|u_0\|_m^2$$

if ε is sufficiently small (this defines ε_3). Define now $\delta_3 = \min(\omega_0, \tfrac{1}{2}\delta_2)$, and assume that $\|u_0\|_{s_1} \leqq \delta_3$: from (8) and (10) with $m = s_1$ we have

$$(11) \qquad \tfrac{1}{2}\|u(t)\|_{s_1}^2 \leqq 2N_{s_1}(u, t) \leqq 2N_{s_1}(u, 0) \leqq 2\|u_0\|_{s_1}^2 \leqq \tfrac{1}{2}\delta_2^2\,,$$

that is, (6). This means that $u(t) \in B(\delta_3)\ \forall t \in [0, T]$: in view of the local existence Theorem 2, it follows that u can be extended to a global solution. To obtain the global bound (7), we see from (8) and (10) that

$$\int_0^t Q_m(\nabla u) \leqq 2N_m(u, t) \leqq 2N_m(u, 0) \leqq 2\|u_0\|_m^2 \leqq 4E_m(u, 0)$$

thus, (7) follows from (9) for $m = s$, with $M^2 \equiv E_s(u, 0) \exp 4h(\omega_0)\, E_s(u, 0)$.

4. PROOF OF PROPOSITION 1

We establish estimates on u which hold if $\|\nabla u_0\|_{s_0}$ is sufficiently small; our procedure is rather formal, since u is in general not regular enough to allow for all the differentiations we are going to carry out. This difficulty can, however. be overcome by an appropriate regularization procedure, by means of Friedrichs' mollifiers, as shown by Matsumura in [2]. For $|\alpha| \leqq s$ we set $\varphi = D^\alpha u$ and derive from (H_ε)

(12)
$$\varepsilon\varphi_{tt} + \varphi_t - \sum a_{ij}(\nabla u)\, \partial_i\partial_j\varphi = G_\alpha \equiv$$
$$\equiv \sum\{D^\alpha[a_{ij}(\nabla u)\, \partial_i\partial_j u] - a_{ij}(\nabla u)\, D^\alpha\, \partial_i\partial_j u\}$$

(note that $G_0 = 0$). Multiplying this by $2\varphi_t$ and $(1/\varepsilon)\,\varphi$ we obtain

(13)
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|\varphi_t\|^2 + (\varphi_t, \varphi) + \frac{1}{2\varepsilon}\|\varphi\|^2 + Q_0(u; \nabla\varphi)\right\} +$$
$$+ \|\varphi_t\|^2 + \frac{1}{\varepsilon}\, Q_0(u; \nabla\varphi) = \sum(\partial_t[a_{ij}(\nabla u)]\, \partial_i\varphi, \partial_j\varphi) +$$
$$+ 2(\|G_\alpha\| + \|\sum \partial_j[a_{ij}(\nabla u)]\, \partial_i\varphi\|)\left(\|\varphi_t\| + \frac{1}{2\varepsilon}\|\varphi\|_2\right).$$

We can estimate the right side of (13) by means of the following

Proposition 3: *Let $u, w \in H^{s+1}$. For $s_1 \leqq m \leqq s$ and $0 < |\alpha| \leqq m$, we have:*

(14)
$$\text{(a)}\quad \sum \|\partial_j[a_{ij}(\nabla w)]\, \partial_i D^\alpha u\| \leqq h(\|\nabla w\|_{s_0})\, \|\nabla w\|_{s_1}\, \|\nabla D^\alpha u\|\,,$$
$$\text{(b)}\quad \sum |(\partial_t[a_{ij}(\nabla w)]\, \partial_i D^\alpha u, \partial_j D^\alpha u)| \leqq h(\|\nabla w\|_{s_0})\, \|w_t\|_{s_1}\, \|\nabla D^\alpha u\|^2\,,$$
$$\text{(c)}\quad \|G_\alpha\| \leqq h(\|\nabla w\|_{s_0})\, (\|\nabla w\|_m\, \|\nabla u\|_{|\alpha|})\,.$$

Proof. The first two estimates are immediate: we just have to recall (2) and that $H^{s_0} \subset L^\infty$. The proof of (c) is based on the following well known product estimates for functions in Sobolev spaces: "if $f \in H^m$, $g \in H^r$ and $p \leqq m$, $p \leqq r$ and $p < m + r - n/2$, the pointwise product $fg \in H^p$, and $\|fg\|_p \leqq \|f\|_m\, \|g\|_r$. Moreover, if F is a smooth function and $u \in H^m$, $m \geqq s_0$, then $F(u) \in H^m$ and $\forall\alpha$, $|\alpha| \leqq m$ $\|D^\alpha F(u)\| \leqq h(\|u\|_{s_0})\, \|D^\alpha u\|$". We now compute that

$$G_\alpha = \sum_{ij} \sum_{|\beta|=1}^{|\alpha|} C_{\alpha,\beta}\, D^\beta[a_{ij}(\nabla w)]\, D^{\alpha-\beta}\, \partial_i\partial_j u\,,$$

for suitable constants $c_{\alpha,\beta}$, so that considering $\nabla w \in H^m$ and $u \in H^{|\alpha|+1}$, we have

$$\|D^\beta[a_{ij}(\nabla w)]\, D^{\alpha-\beta}\, \partial_i \partial_j u\| \leqq D^\beta[a_{ij}(\nabla w)]_{m-|\beta|}\, \|D^{\alpha-\beta+1}\nabla u\|_{|\beta|-1} \leqq$$
$$\leqq \|a_{ij}(\nabla w)\|_m\, \|\nabla u\|_{|\alpha|} \leqq h(\|\nabla w\|_{s_0})\, \|\nabla w\|_m\, \|\nabla u\|_{|\alpha|},$$

since $m \geqq |\beta| \geqq 1$ and $(m - |\beta|) + (|\beta| - 1) - n/2 = m - 1 - n/2 \geqq s_1 - 1 - n/2 = s_0 - n/2 > 0$.

We now go back to (12): summing for $|\alpha| \leqq m$, $s_1 \leqq m \leqq s$, and using (14) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|u_t\|_m^2 + (u_t, u)_m + \frac{1}{2\varepsilon}\|u\|_m^2 + Q_m(\nabla u)\right\} + \tag{15}$$
$$+ \|u_t\|_m^2 + \frac{1}{\varepsilon} Q_m(\nabla u) \leqq h_1(\|\nabla u\|_{s_0})\, \|\nabla u\|_{s_1}\, \|\nabla u\|_m\, \|u_t\|_m +$$
$$+ h_2(\|\nabla u\|_{s_0})\, \|u_t\|_{s_1}\, \|\nabla u\|_m^2 + \frac{1}{\varepsilon} h_3(\|\nabla u\|_{s_0})\, \|\nabla u\|_{s_1}\, \|\nabla u\|_m\, \|\nabla u\|_{m-1}.$$

By interpolation inequalities (Niremberg, [6]), we have

$$\|\nabla u\|_{s_1} \leqq c\|\nabla u\|_m^\beta\, \|\nabla u\|_{s_0}^{1-\beta}, \tag{16}$$
$$\|\nabla u\|_{m-1} \leqq c\|\nabla u\|_m^{1-\beta}\, \|\nabla u\|_{s_0}^\beta,$$

where $\beta = 1/(m - s_0)$ (recall that $m \geqq s_0 + 1 = s_1$). Thus, from (15)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varepsilon} N_m(u, t) + \tfrac{1}{2}\|u_t\|_m^2 + \frac{1}{2\varepsilon} Q_m(\nabla u) \leqq h_1^2(\|\nabla u\|_{s_0})\, \|\nabla u\|_{s_1}^2\, \|\nabla u\|_m^2 +$$
$$+ \tfrac{1}{2}\|u_t\|_m^2 + h_2(\|\nabla u\|_{s_0})\, \|u_t\|_{s_1}\, \|\nabla u\|_m^2 +$$
$$+ \frac{1}{\varepsilon} h_3(\|\nabla u\|_{s_0})\, \|\nabla u\|_{s_0}\, \|\nabla u\|_m^2,$$

so that by (3) if $N_{s_1} \leqq \omega^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varepsilon} N_m(u. t) + \frac{1}{2\varepsilon} Q_m(\nabla u) \leqq h_4(\omega)\, \omega^2 \frac{1}{\varepsilon} Q_m(\nabla u) + \tag{17}$$
$$+ h_5(\omega)\, \omega \frac{1}{\sqrt{\varepsilon}} Q_m(\nabla u) + \frac{1}{\varepsilon} h_6(\omega)\, \omega\, Q_m(\nabla u) \leqq \frac{1}{\varepsilon} h_7(\omega)\, \omega\, Q_m(\nabla u).$$

Define now $\omega_0 > 0$ by $2h_7(\omega_0)\, \omega_0 = 1$: then (8) follows from (17) if $\omega \leqq \omega_0$, since then

$$\frac{\mathrm{d}}{\mathrm{d}t} N_m(u, t) \leqq 0$$

(note that ω_0 depends only on the coefficients a_{ij}).

5. PROOF OF PROPOSITION 2

We derive a similar set of estimates from (12): multiplying by $2\varphi_t$ and φ we have

$$(18)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\{\varepsilon\|\varphi_t\|^2 + \varepsilon(\varphi_t, \varphi) + \tfrac{1}{2}\|\varphi\|^2 + Q_0(u; \nabla\varphi)\} + \|\varphi_t\|^2 + Q_0(u; \nabla\varphi) \leqq$$
$$\leqq 2(\|G_\alpha\|^2 + \|\sum \partial_j[a_{ij}(\nabla u)]\,\partial_i\varphi\|)(\|\varphi_t\| + \|\varphi\|) +$$
$$+ \sum (\partial_t[a_{ij}(\nabla u)]\,\partial_i\varphi, \partial_j\varphi).$$

We estimate the right side of (18) with Proposition 3, and sum for $|\alpha| \leqq m$, $s_1 \leqq \leqq m \leqq s$: using the interpolation inequalities (16) again, we obtain

$$(19)\qquad \frac{\mathrm{d}}{\mathrm{d}t} E_m(u, t) + \tfrac{1}{2}\|u_t\|_m^2 + \tfrac{1}{2}Q_m(\nabla u) \leqq h_1^2(\|\nabla u\|_{s_0})\,\|\nabla u\|_{s_1}^2\,\|\nabla u\|_m^2 + \tfrac{1}{4}\|u_t\|_m^2 +$$
$$+ h_2^2(\|\nabla u\|_{s_0})\,\|\nabla u\|_m^4 + \tfrac{1}{4}\|u_t\|_{s_1}^2 + h_3(\|\nabla u\|_{s_0})\,\|\nabla u\|_{s_0}\,\|\nabla u\|_m^2.$$

As in (11), we have by Proposition 1

$$\|\nabla u(t)\|_{s_0}^2 \leqq \|u(t)\|_{s_1}^2 \leqq 4N_{s_1}(u, t) \leqq 4\|u_0\|_{s_1}^2 \leqq 4\omega_0^2,$$

so that from (19)

$$\frac{\mathrm{d}}{\mathrm{d}t} E_m(u, t) + \tfrac{1}{2}Q_m(\nabla u) \leqq h_8(\omega_0)\,Q_m^2(\nabla u) + \omega_0\,h_7(\omega_0)\,Q_m(\nabla u).$$

Now, $Q_m(\nabla u(t)) \leqq E_m(u, t)$; thus, by the choice of ω_0,

$$\frac{\mathrm{d}}{\mathrm{d}t} E_m(u, t) \leqq h_8(\omega_0)\,Q_m(\nabla u)\,E_m(u, t),$$

whence (9) follows, with $h = h_8$, by Gronwall's inequality.

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Author's address: Department of Mathematics, University of Wisconsin-Milwaukee, 3200 North Cramer Rd., Milwaukee, WI 53201, USA.