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BOUNDARY VALUE PROBLEM FOR A DIFFERENCE EQUATION

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In the paper there is considered a difference equation

$$a_k w_{k+2} + b_k w_{k+1} + c_k w_k = f_k$$

with conditions

$$w_{k_0} = a\,, \quad w_{k_1} = b\,.$$

Suppose we are given the operational calculus $\mathrm{CO}(L^0, L^1, S, T(q), s(q), Q)$, where L^0 and L^1 are linear spaces. S, $T(q)$, $s(q)$ are linear operations called derivative, integral and limit condition, so that: $S: L^1 \to L^0$ (onto), $T(q): L^0 \to L^1$, $s(q): L^1 \to \operatorname{Ker} S$, $q \in Q$ where Q is the set of indices. Let us assume that the operations S, $T(q)$, $s(q)$ satisfy the following properties:

$$ST(q)f = f \quad \text{for} \quad f \in L^0\,,$$

$$T(q)\,Sg = g - s(q)\,g \quad \text{for} \quad g \in L^1\,.$$

(Axioms, properties of operational calculus etc. see [1, 2, 3, 5]).

The operational calculus $\mathrm{CO}(C(N), C(N), S = \Delta, T(k_0), s(k_0), Q = N)$ is a known example of operational calculus, where $C(N)$ is a space of the sequences of real numbers $x = \{x_k\}$, $k = 0, 1, 2, \ldots$. The derivative Δ, the integral $T(k_0)$ and the limit condition $s(k_0)$ are defined by the formulas

$$(1) \qquad \Delta\{x_k\} \overset{\mathrm{df}}{=} \{x_{k+1} - x_k\}\,,$$

$$(2) \qquad T(k_0)\,\{f_k\} \overset{\mathrm{df}}{=} \begin{cases} 0 & \text{for} \quad k = k_0\,, \\ \sum\limits_{i=k_0}^{k-1} f_i & \text{for} \quad k > k_0\,, \\ -\sum\limits_{i=k}^{k_0-1} f_i & \text{for} \quad k < k_0\,, \end{cases}$$

$$(3) \qquad s(k_0)\,\{x_k\} \overset{\mathrm{df}}{=} \{x_{k_0}\}\,, \quad \{x_k\}, \{f_k\} \in C(N) \quad \text{(see [4])}.$$

Definition 1. [7]. Operation $\Delta_{p_k}: C(N) \to C(N)$ defined by the formula

$$(4) \qquad \Delta_{p_k}\{u_k\} \overset{\mathrm{df}}{=} \{u_{k+1} - p_k u_k\}$$

where $\{p_k\}$ is an arbitrarily fixed sequence such that $p_k \neq 0$ for $k = 0, 1, 2, \ldots$

will be called a *difference derivative* of the base $\{p_k\}$.

Remark. We will be assuming that $\prod_{i=0}^{-1}(\cdot) =^{\mathrm{df}} 1$.

Theorem 1. [7]. *For a difference derivative of the base* $\{p_k\}$, *the operation* $T_{p_k}(k_0)$: $C(N) \to C(N)$ *defined by the formula*

$$(5) \qquad T_{p_k}(k_0)\{f_k\} =^{\mathrm{df}} \{\prod_{i=0}^{k-1} p_i\}\, T(k_0) \left\{\frac{f_k}{\prod_{i=0}^{k} p_i}\right\}$$

is an integral and the operation $s(k_0)$: $C(N) \to C(N)$ defined by the formula

$$(6) \qquad s_{p_k}(k_0)\{x_k\} =^{\mathrm{df}} \left\{\frac{\prod_{i=0}^{k-1} p_i}{\prod_{i=0}^{k_0-1} p_i}\right\} s(k_0)\{x_k\}$$

is a limit condition.

Remark 2. Instead of writing "sequence $\{u_k\}$" we will often write "u_k".

Corollary. *If* $\alpha_k \neq 0$ *for* $k = 0, 1, 2, \ldots$ *then the operations defined by the following formulas*

$$(7) \qquad \Delta_{p_k}^{\alpha_k} x_k =^{\mathrm{df}} \alpha_k(\Delta_{p_k} x_k)\,,$$

$$(8) \qquad T_{p_k}^{\alpha_k}(k_0) f_k =^{\mathrm{df}} T_{p_k}(k_0)\left(\frac{f_k}{\alpha_k}\right),$$

$$(9) \qquad s_{p_k}^{\alpha_k}(k_0)\, x_k =^{\mathrm{df}} s_k(k_0)\, x_k\,, \quad x_k, f_k \in C(N)$$

satisfy the axioms of the operational calculus.

Theorem 2 [8]. *If* $a_k \neq 0$ *and* $c_k \neq 0$ *for* $k = 0, 1, 2, \ldots$ *then the difference equation*

$$(10) \qquad a_k w_{k+2} + b_k w_{k+1} + c_k w_k = f_k$$

can be always presented in the equivalent form

$$(11) \qquad (\Delta_{p_k}^{\alpha_k})^2\, w_k = \alpha_k \alpha_{k+1} \frac{f_k}{a_k}$$

where

$$(12) \qquad \alpha_k = \prod_{i=0}^{k-1} \frac{-p_i}{\frac{b_i}{a_i} + p_{i+1}}$$

and the sequence p_k *is a solution of the difference equation*

$$(13) \qquad p_k(b_k + a_k p_{k+1}) = -c_k\,.$$

Definition 2 [3]. If $L^1 \subset L^0$ then we denote

$$L^2 =^{\mathrm{df}} \{x \in L^1 \colon Sx \in L^1\} .$$

Assuming $n = 2$ in [9] we will get

Theorem 3. *A solution of an abstract differential equation*

$$S^2 x = f \tag{14}$$

with conditions

$$s(q_1)\, x = x_{q_1} \tag{15}$$

$$s(q_2)\, x = x_{q_2}\,, \quad x \in L^2\,, \quad f \in L^0 \supset L^1 \tag{16}$$

(a) *exists if the operation* $[T(q_1) - T(q_2)]\,|_{\mathrm{Ker}S} =^{\mathrm{df}} U$ *is a surjection onto* Ker S;

(b) *exists and there is exactly one such solution if the operation U is a bijection onto* Ker S.

Proof. (In [9] you can find the proof for an equation of the n-order.)

On the basis of axioms of operational calculus and condition (15) we have

$$x = T^2(q_1) f + T(q_1)\, c + x_{q_1}\,, \quad c \in \mathrm{Ker}\, S\,.$$

Applying condition (16) we will get the equation

$$[T(q_1) - T(q_2)]\, c = x_{q_2} - x_{q_1} - [T(q_1) - T(q_2)]\, T(q_1) f \tag{17}$$

from which we can find $c \in \mathrm{Ker}\, S$.

The analysis of the equation (17) gives us the thesis of theorem 3.

Theorem 4. *If sequence* p_k *satisfies the difference equation* (13) *and*

$$A =^{\mathrm{df}} \sum_{k=k_0}^{k_1-1} \frac{1}{p_k} \prod_{i=0}^{k-1} \frac{c_i}{a_i p_i^2} \neq 0 \tag{18}$$

then the difference equation

$$a_k w_{k+2} + b_k w_{k+1} + c_k w_k = f_k\,, \quad a_k \neq 0\,, \quad c_k \neq 0 \quad \textit{for} \quad k = 0, 1, 2, \ldots \tag{19}$$

with conditions

$$s(k_0)\, w_k = w_{k_0} = a\,, \tag{20}$$

$$s(k_1)\, w_k = w_{k_1} = b\,, \quad k_0 < k_1 \tag{21}$$

has only one solution.

Proof. From the theorem 2 it follows that the difference equation (19) is equivalent to the difference equation (11).

From formula (6) it follows that conditions (20), (21) are equivalent to conditions

$$s_{p_k}(k_0)\, w_k = \frac{\prod_{i=0}^{k-1} p_i}{\prod_{i=0}^{k_0-1} p_i}\, w_{k_0}\,, \tag{22}$$

$$(23)\qquad s_{p_k}(k_1)\, w_k = \frac{\prod_{i=0}^{k-1} p_i}{\prod_{i=0}^{k_1-1} p_i}\, w_{k_1}\,.$$

We can apply now theorem 3 to the equation (11) with conditions (22), (23).

We will show when operation

$$(24)\qquad \left[T_{p_k}^{\alpha_k}(k_0) - T_{p_k}^{\alpha_k}(k_1)\right]\Big|_{\mathrm{Ker}\Delta_{p_k}^{\alpha_k}=\mathrm{Ker}\Delta_{p_k}}$$

is a bijection onto Ker $\Delta_{p_k}^{\alpha_k}$.

It is known that

$$d_k \in \mathrm{Ker}\,\Delta_{p_k}^{\alpha_k} \quad \text{if and only if} \quad d_k = \gamma \prod_{i=0}^{k-1} p_i\,, \quad \gamma \in R\,.$$

Applying (5), (8), (12), (13) and (2) we can notice that

$$\left[T_{p_k}^{\alpha_k}(k_0) - T_{p_k}^{\alpha_k}(k_1)\right] d_k = \left(\prod_{i=0}^{k-1} p_i\right)\left[T(k_0) - T(k_1)\right]\left(\frac{d_k}{\alpha_k \prod_{i=0}^{k} p_i}\right) =$$

$$= \gamma\left(\prod_{i=0}^{k-1} p_i\right)\left[T(k_0) - T(k_1)\right]\frac{1}{\left(\prod_{i=0}^{k-1}\dfrac{p_i}{-\dfrac{b_i}{a_i} - p_{i+1}}\right) p_k} =$$

$$= \gamma\left(\prod_{i=0}^{k-1} p_i\right)\left[T(k_0) - T(k_1)\right]\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right) =$$

$$= \gamma\left(\prod_{i=0}^{k-1} p_i\right)\left(\sum_{k=k_0}^{k_1-1}\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)$$

so

$$(25)\qquad \left[T_{p_k}^{\alpha_k}(k_0) - T_{p_k}^{\alpha_k}(k_1)\right] d_k = \gamma\left(\prod_{i=0}^{k-1} p_i\right)\left(\sum_{k=k_0}^{k_1-1}\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right).$$

It follows from the last equality that under assumption (18) that $A \neq 0$ the operation (24) is a bijection onto Ker $\Delta_{p_k}^{\alpha_k}$.

Theorem 5. *Difference equation* (19) *with conditions* (20), (21)

(a) *has infinitely many solutions if*

$$B =^{\mathrm{df}} \frac{b}{\prod_{i=0}^{k_1-1} p_i} - \frac{a}{\prod_{i=0}^{k_0-1} p_i} - \sum_{k=k_0+1}^{k_1-1}\left[\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)\left(\sum_{\iota=k_0}^{k-1}\frac{f_\iota}{a_\iota \prod_{i=0}^{\iota}\dfrac{c_i}{a_i p_i}}\right)\right] = 0$$

and $A = 0$,

(b) *has no solution if* $A = 0$ *and* $B \neq 0$.

Proof. It is known that the difference equation (19) with conditions (20), (21) is equivalent to the difference equation (11) with conditions (22), (23).

In the further part of the proof we will use the equation (17). Let us calculate its right side in the case of equation (11) with conditions (22), (23).

First let us calculate

$$[T_{p_k}^{\alpha_k}(k_0) - T_{p_k}^{\alpha_k}(k_1)]\left(T_{p_k}^{\alpha_k}(k_0)\left(\alpha_k\alpha_{k+1}\frac{f_k}{a_k}\right)\right).$$

Applying (5), (8), (12), (13) and (2) we can easily notice that

$$[T_{p_k}^{\alpha_k}(k_0) - T_{p_k}^{\alpha_k}(k_1)]\left(T_{p_k}^{\alpha_k}(k_0)\left(\alpha_k\alpha_{k+1}\frac{f_k}{a_k}\right)\right) =$$

$$= \left(\prod_{i=0}^{k-1} p_i\right)[T(k_0) - T(k_1)]\left(\frac{1}{\alpha_k p_k}T(k_0)\left(\frac{\alpha_{k+1}f_k}{a_k\prod_{i=0}^{k} p_i}\right)\right) =$$

$$= \left(\prod_{i=0}^{k-1} p_i\right)[T(k_0) - T(k_1)]\left\{\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)T(k_0)\left(\frac{f_k}{a_k\prod_{i=0}^{k}\frac{c_i}{a_i p_i}}\right)\right\} =$$

$$= \left(\prod_{i=0}^{k-1} p_i\right)\sum_{k=k_0+1}^{k_1-1}\left(\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)\left(\sum_{l=k_0}^{k-1}\frac{f_l}{a_l\prod_{i=0}^{l}\frac{c_i}{a_i p_i}}\right)\right).$$

The right side of the equation (17) will have the form

$$\frac{\prod_{i=0}^{k-1} p_i}{\prod_{i=0}^{k_1-1} p_i}b - \frac{\prod_{i=0}^{k-1} p_i}{\prod_{i=0}^{k_0-1} p_i}a - \left(\prod_{i=0}^{k-1} p_i\right)\sum_{k=k_0+1}^{k_1-1}\left(\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)\left(\sum_{\iota=k_0}^{k-1}\frac{f_\iota}{a_\iota\prod_{i=0}^{\iota}\frac{c_i}{a_i p_i}}\right)\right) =$$

$$= \left(\prod_{i=0}^{k-1} p_i\right)\left\{\frac{b}{\prod_{i=0}^{k_1-1} p_i} - \frac{a}{\prod_{i=0}^{k_0-1} p_i} - \sum_{k=k_0+1}^{k_1-1}\left(\left(\frac{1}{p_k}\prod_{i=0}^{k-1}\frac{c_i}{a_i p_i^2}\right)\left(\sum_{\iota=k_0}^{k-1}\frac{f_\iota}{a_\iota\prod_{i=0}^{\iota}\frac{c_i}{a_i p_i}}\right)\right)\right\}.$$

From the above facts and from (25) it follows that in this case the equation (17) has the form

(26) $\gamma A = B.$

From (26) there follows the thesis of theorem 5.

Remark 3. Let us notice that it isn't necessary to know all the sequence p_k to use theorem 4 or theorem 5.

It is enough to know its k_1 first elements.

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