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LIFTS OF 1-FORMS TO THE TANGENT BUNDLE OF HIGHER ORDER

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0. INTRODUCTION

Let $M$ be a manifold. We denote by $T^r M$ the tangent bundle of order $r$. A mapping
\[ \mathcal{L} : \mathcal{X}^r(M) \to \mathcal{X}^r(T^r M) \]
where $\mathcal{X}^r(M)$ and $\mathcal{X}^r(T^r M)$ denote the modules of 1-forms on $M$ and $T^r M$ respectively, is called a lift of 1-forms from $M$ to $T^r M$ if the following conditions hold:
(i) $\mathcal{L}$ is linear over $\mathbb{R}$, (ii) $\mathcal{L}$ is local, (iv) $\mathcal{L}$ is natural, and (ix) $\mathcal{L}$ is regular.

All the $\lambda$-lifts defined by A, Morimoto [4], [5], [6] are lifts in the sense of the proposed definition. In this paper we shall define a new lift of 1-forms from $M$ to $T^r M$ called the $\square$-lift. The main theorem of this paper says that if $\dim M \geq 2$, then every lift $\mathcal{L}$ of 1-forms from $M$ to $T^r M$ is a linear combination (with constant coefficients) of the $\square$-lift and the $\lambda$-lifts for $\lambda = 0, \ldots, r$. (If $\dim M = 1$, then every lift $\mathcal{L}$ of 1-forms from $M$ to $T^r M$ is a linear combination (with constant coefficients) of the $\lambda$-lifts for $\lambda = 0, \ldots, r$.)

In this paper the differentiability means always the differentiability of class $C^\infty$.

1. PRELIMINARIES

Let $M$ be a manifold of dimension $n$ and $r$ be a natural number. Denote by $T^r M = T^r 0(R, M)$ the set of $r$-jets at 0 of mappings $R \to M$. This bundle is called the tangent bundle of order $r$. We denote by $\pi : T^r M \to M$ the bundle projection defined by
\[ \pi(j_0^r \gamma) = \gamma(0) \]
If $\varphi : M \to N$ is a differentiable mapping, then the induced mapping $T^r \varphi : T^r M \to T^r N$ is defined by the formula;
\[ T^r \varphi(j_0^r \gamma) = j_0^r(\varphi \circ \gamma) \]

If $(U, x^i)$ is a chart on $M$, then the induced chart $(\pi^{-1}(U), x^{i,\varphi})$ is given by
\[ x^{i,\varphi}(j_0^r \gamma) = \frac{1}{v!} D_v(x^{i, \gamma})(0) \]
for $i = 1, \ldots, n = \dim M$ and $v = 0, \ldots, r$. 

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If \( f \) is a differentiable function on \( M \), then for \( \lambda = 0, \ldots, r \) the \( \lambda \)-lift of \( f \) to the bundle \( T^\ast M \) is the differentiable function \( f^{(\lambda)} \) on \( T^\ast M \) defined by the formula:

\[
(1.2) \quad f^{(\lambda)}(j^\ast y) = \frac{1}{\lambda!} D_\lambda (f \circ y)(0).
\]

The \( \lambda \)-lifts of functions have the following properties:

\[
(1.3) \quad (af + bg)^{(\lambda)} = af^{(\lambda)} + bg^{(\lambda)},
\]

\[
(1.4) \quad (fg)^{(\lambda)} = \sum_{\nu=1}^{\lambda} f^{(\nu)} g^{(\lambda-\nu)}.
\]

Formulas (1.1) and (1.2) imply immediately

\[
(1.5) \quad x^{i,v} = (x^i)^{(v)}.
\]

The family of functions \( f^{(\lambda)} \), where \( f \) is a function on \( M \) and \( \lambda = 0, \ldots, r \) is an important family of functions on \( T^\ast M \) because we have the following proposition:

**Proposition 1.1.** If \( V \) and \( W \) are vector fields on \( T^\ast M \) such that

\[
V(f^{(\lambda)}) = W(f^{(\lambda)})
\]

for every functions \( f \) on \( M \) and \( \lambda = 0, \ldots, r \), then \( V = W \).

The proof is an easy verification (see [1] or [3]).

From Proposition 1.1 we can obtain:

**Proposition 1.2.** (A. Morimoto [4], [6]) If \( X \) is a vector field on \( M \) and \( \lambda = 0, \ldots, r \), then there exists one and only one vector field \( X^{(\lambda)} \) on \( T^\ast M \) such that for every function \( f \) on \( M \) and every \( \mu = 0, \ldots, r \) we have

\[
(1.6) \quad X^{(\lambda)}(f^{(\mu)}) = (Xf)^{(\lambda+\mu-r)}.
\]

The vector field \( X^{(\lambda)} \) on \( T^\ast M \) is called the \( \lambda \)-lift of \( X \) from \( M \) to \( T^\ast M \). The \( r \)-lift is called the complete lift of vector field and we will write \( X^C \) instead of \( X^{(r)} \). The family of vector fields \( X^C \), where \( X \) is a vector field on \( M \), is important because we have (see [3]):

**Proposition 1.3.** If \( t \) and \( t' \) are differentiable fields of tensors of type \( (\varepsilon, p) \) on \( T^\ast M \), where \( \varepsilon = 0, 1 \), such that for every vector fields \( X_1, \ldots, X_p \) on \( M \) we have

\[
t(X_1^C, \ldots, X_p^C) = t'(X_1^C, \ldots, X_p^C)
\]

then \( t = t' \).

Now we can define the \( \lambda \)-lift of 1-forms from \( M \) to the tangent bundle \( T^\ast M \). Namely we have:

**Proposition 1.4.** (Morimoto [4], [6]) If \( \omega \) is an 1-form on \( M \) and \( \lambda = 0, \ldots, r \), then there exists one and only one 1-form \( \omega^{(\lambda)} \) on \( T^\ast M \) such that for each vector field \( X \) on \( M \) and each \( \mu = 0, \ldots, r \) the following formula holds

\[
(1.7) \quad \omega^{(\lambda)}(X^{(\mu)}) = (\omega X)^{(\lambda+\mu-r)}.
\]
Observe that according to Proposition 1.3 formula (1.7) determines uniquely the $\lambda$-lift of an 1-form $\omega$. By using Propositions 1.2 and 1.4 it is not difficult to show the following formulas (see [4], [6]):

\begin{equation}
(fX)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)}X^{(\lambda-\mu)},
\end{equation}

\begin{equation}
(f\omega)^{(\lambda)} = \sum_{\mu=0}^{\lambda} f^{(\mu)}\omega^{(\lambda-\mu)}.
\end{equation}

If $(U, x^i)$ is a chart on $M$, then we have

\begin{equation}
\frac{\partial}{\partial x^{i,v}} = \left(\frac{\partial}{\partial x^i}\right)^{(r-v)},
\end{equation}

\begin{equation}
(dx^i)^{(v)} = dx^i.
\end{equation}

If $\omega = \omega_i dx^i$, then

\begin{equation}
\omega^{(\lambda)} = \sum_{i=0}^{n} \sum_{v=0}^{\lambda} \omega_i^{(v)} dx^{i,\lambda-v}.
\end{equation}

2. Lifts of 1-Forms

We propose the following definition of lifts of 1-forms from $M$ to the tangent bundle $T^*M$ of order $r$.

**Definition 2.1.** We denote by $\mathfrak{X}^*(M)$ and $\mathfrak{X}^*(T^*M)$ the modules of 1-forms on $M$ and $T^*M$ respectively. A mapping $\mathcal{L}: \mathfrak{X}^*(M) \to \mathfrak{X}^*(T^*M)$ is called a **lift of 1-forms from $M$ to $T^*M$** if the following conditions hold:

(a) $\mathcal{L}$ is linear over $\mathbb{R}$, that is, for every 1-forms $\omega$, $\omega'$ on $M$ and every real numbers $a, b$,

\[ \mathcal{L}(a\omega + b\omega') = a\mathcal{L}(\omega) + b\mathcal{L}(\omega'). \]

(b) $\mathcal{L}$ is local, that is, if $U$ is an open subset of $M$ and $\omega, \omega'$ are 1-forms on $M$ such that $\omega \mid U = \omega' \mid U$, then

\[ \mathcal{L}\omega \mid \pi^{-1}(U) = \mathcal{L}\omega' \mid \pi^{-1}(U). \]

(c) $\mathcal{L}$ is natural, that is, if $\varphi: U \to V$ is a diffeomorphism of open subsets $U, V$ of $M$ and $\omega$ is an 1-form on $M$, then

\[ \mathcal{L}(\varphi^*\omega) = (T^*\varphi)^{\ast} (\mathcal{L}\omega) \]

where $\ast$ denotes the pull-back of an 1-form by a diffeomorphism.

(d) $\mathcal{L}$ is regular, that is, if $K$ is an open subset of $\mathbb{R}^k$ and

\[ \omega: K \times M \to T^*M \]

is a differentiable mapping such that for every $t \in K \omega_t$ is an 1-form on $M$, then the mapping

\[ K \times T^*M \ni (t, p) \to (\mathcal{L}\omega_t)(p) \in T^*(T^*M) \]

is differentiable.
Now we have the following theorem:

**Theorem 2.2.** For every $\lambda = 0, \ldots, r$ the mapping

$$\mathcal{X}^\lambda(M) \ni \omega \mapsto \omega^{(2)} \in \mathcal{X}^\lambda(T^*M)$$

is a lift of 1-forms from $M$ to $T^*M$.

**Proof.** The conditions (a), (b) and (d) are evident. We need only to show that the above mapping is natural. In order to do this, let $\omega$ be an 1-form on $M$ and $\varphi: U \to V$ be a diffeomorphism of two open subsets of $M$. Now by using (1.7) and the formula

$$(\varphi^*\omega)(X) = \omega(\varphi_*X) \circ \varphi$$

where $X$ is a vector field on $M$ and $\varphi_*X$ is the image of $X$ by $\varphi$, we obtain

$$(\varphi^*\omega)^{(2)}(X^c) = ((\varphi^*\omega)(X))^{(2)} = ((\omega)(\varphi_*X) \circ \varphi)^{(2)} =$$

$$= (\omega(\varphi_*X))^{(2)} \circ T^*\varphi = \omega^{(2)}(\varphi_*X)^c \circ T^*\varphi = \omega^{(2)}((T^*\varphi)_*X^c) \circ T^*\varphi =$$

$$= ((T^*\varphi)^* \omega^{(2)})(X^c).$$

In the above calculation we have used two facts proved in [1], namely, we have used that for every function $f$ on $M$ and every vector field $X$ on $M$ two formulas hold

$$(f \circ \varphi)^{(2)} = f^{(2)} \circ T^*\varphi, \quad (\varphi_*X)^c = (T^*\varphi)_* X^c.$$

The proof of Theorem 2.2 is now finished.

Now we shall define a new lift of 1-forms. For any $r \geq 1$ we consider the projection

$$\pi^r_1: T^rM \ni j^r_{0\gamma} \to j^r_{1\gamma} = \gamma(0) \in TM.$$

Let $\omega$ be an 1-form on $M$. We consider the vertical lift $\omega^V$ of $\omega$ to the tangent bundle $TM$, $\omega^V: TM \to R$ is the differentiable function on $TM$ given by $\omega^V(v) = \omega_{\pi^r_1}(v)$. Immediately from the definition of $\omega^V$ we obtain

$$(\varphi^*\omega)^V = \omega^V \circ d\varphi.$$

We define

$$\omega^\square = d(\omega^V \circ \pi^r_1)$$

$\omega^\square$ is an 1-form on $T^rM$ and it is called the $\square$-lift of $\omega$ from $M$ to $T^rM$. This definition implies immediately the following formula

$$\omega^\square = \sum_{j=1}^n \left\{ \sum_{i=1}^n x^{i,1} \frac{\partial w_i}{\partial x^j} dx^{j,0} + \omega_j dx^{j,1} \right\}.$$

From (1.11) and (2.2) we obtain that if $\omega$ is a closed 1-form on $M$, then $\omega^\square = \omega^{(1)}$.

Now we prove:

**Theorem 2.3.** The mapping $\square: \mathcal{X}^\lambda(M) \ni \omega \mapsto \omega^\square \in \mathcal{X}^\lambda(T^*M)$ is a lift of 1-forms from $M$ to $T^*M$.

**Proof.** The conditions (a), (b) and (d) of Definition 2.1 are evident. To show the condition (c), let $\omega$ be an 1-form on $M$ and $\varphi: U \to V$ be a diffeomorphism of open
subsets of $M$. Now, the formula $(\varphi^*\omega)^V = \omega^V \circ d\varphi$ implies

$$(\varphi^*\omega)^V = d((\varphi^*\omega)^V \circ \pi'_1) = d(\omega^V \circ d\varphi \circ \pi'_1) = d(\omega^V \circ \pi'_1 \circ T^*\varphi) = (T^*\varphi)^* (d(\omega^V \circ \pi'_1)) = (T^*\varphi)^* (\omega^\square)$$

because $d\varphi \circ \pi'_1 = \pi'_1 \circ T^*\varphi$. The proof is now complete.

From (2.1) we obtain immediately:

**Corollary 2.4.** For any 1-form $\omega$ on $M \omega^\square$ is a closed 1-form on $T^*M$.

From (1.7) and (1.11) we have immediately:

**Corollary 2.5.** If $\omega$ is a closed 1-form on $M$, then for every $\lambda = 0, \ldots, r$, $\omega^{(\lambda)}$ is a closed 1-form on $T^*M$.

To end this paragraph we formulate the following proposition which it is easy to verify.

**Proposition 2.6.** If $\mathcal{L}$ is a lifting of 1-forms from $M$ to $T^*M$, then for any 1-form $\omega$ and any vector field $X$ on $M$ we have

$$\mathcal{L}(L_X\omega) = L_{X^\perp}(\mathcal{L}\omega).$$

3. THE MAIN THEOREM

In this section we formulate and we prove the main theorem of this paper.

**Theorem 3.1.** Let $M$ be a manifold such that dim $M \geq 2$. If $\mathcal{L}$ is a lift of 1-forms from $M$ to the tangent bundle of order $r$, then $\mathcal{L}$ is a linear combination with constant real coefficients of $\square$-lift and $\lambda$-lifts for $\lambda = 0, \ldots, r$, that is, there exist real numbers $a_{\square}, a_0, \ldots, a_r$ such that for every 1-form $\omega$ on $M$ the following formula holds:

$$(3.1) \quad \mathcal{L}\omega = \sum_{\lambda=0}^{\lambda} a_{\lambda} \omega^{(\lambda)} + a_{\square} \omega^\square.$$

The proof of this theorem is based on the following sequence of lemmas and propositions. At first, we formulate an elementary lemma which is an immediate consequence of Euler's identity.

**Lemma 3.2.** Let $f: \mathbb{R}^q \rightarrow \mathbb{R}$ be a differentiable function.

(i) If $f$ satisfies the condition

$$\sum_{j=1}^{q} v^j \frac{\partial |f|}{\partial v^j} = 0$$

then $f$ is constant.
(ii) If $f$ satisfies the condition
\[
\sum_{j=1}^{q} \frac{\partial f}{\partial v^j} + f = 0
\]
then $f$ is identically zero on $\mathbb{R}^q$.

Lemma 3.3. Let $(U, x^i)$ be a chart on $M$ and $x_0$ be a point of $U$. If $\omega$ is a closed 1-form on $M$, then there exists a vector field $X$ on $M$ such that
\[
\omega = L_X(dx^1)
\]
in some neighborhood of $x_0$.

Proof. Let $\omega = \omega_i dx^i$. Since $\omega$ is closed, thus $\partial_i \omega_j = \partial_j \omega_i$ for $i, j = 1, \ldots, n$. If $X = X^i(\partial/\partial x^i)$, then the condition (3.2) is equivalent to the following one:
\[
\omega_j = \partial X^1/\partial x^j.
\]
Now we define $X^j = 0$ for $j \neq 1$ and $X^1$ is a solution of (3.3). The condition $\partial_i \omega_i = 0$ implies the existence of solutions of (3.3).

Proposition 3.4. If $\mathcal{L}$ is a lift of 1-forms from $M$ to the tangent bundle of order $r$, then there exist real numbers $c_0, \ldots, c_r$ such that for every closed 1-form $\omega$ on $M$ we have
\[
\mathcal{L}\omega = \sum_{\mu=0}^{r} c_{\mu} \omega^{(\mu)}.
\]

Proof. Let $(U, x^i)$ be a chart on $M$. Then there exist differentiable functions $c_{j,\mu}$ ($j = 1, \ldots, n$, $\mu = 0, \ldots, r$) defined on $T^r M \mid U$ such that
\[
\mathcal{L}(dx^1) = \sum_{j=1}^{n} \sum_{\mu=0}^{r} c_{j,\mu} dx^j \omega^{(\mu)} = \sum_{j=1}^{n} \sum_{\mu=0}^{r} c_{j,\mu} (dx^j)^{(\mu)}.
\]
Using Proposition 2.5 from the formula
\[
\delta_q^1 dx^k = L_{x^k/\partial x^q} dx^1
\]
we obtain
\[
\delta_q^1 \mathcal{L}(dx^k) = L_{(x^k/\partial x^q)c} \mathcal{L}(dx^1).
\]
Next according to Lemma 1.4 from [6] we obtain
\[
(x^k \partial/\partial x^q)^c = \sum_{v=0}^{r} x^{k,v} \partial/\partial x^{v,q}.
\]
By (3.4) it implies
\[
\delta_q^1 \mathcal{L}(dx^k) = \sum_{j=1}^{n} \sum_{\mu=0}^{r} \left( \sum_{v=0}^{r} x^{k,v} \frac{\partial c_{j,\mu}}{\partial x^{v,q}} + \delta_j^v c_{j,\mu} \right) dx^{j,\mu}.
\]
For $k = 1$, from (3.4) and (3.5) we have
\[
\delta_q^1 c_{j,\mu} = \sum_{v=0}^{r} x^{1,v} \frac{\partial c_{j,\mu}}{\partial x^{v,q}} + \delta_j^1 c_{q,\mu}.
\]
If we set \( k = q + 1 \) and \( j = 1 \) in (3.5) then

\[
(3.7) \quad \sum_{v=0}^{r} x^{q,v} \frac{\partial c_{1,\mu}}{\partial x^{q,v}} = 0.
\]

On the other hand, if we set \( q = 1 \) and \( j = 1 \) in (3.6), then we obtain

\[
(3.8) \quad \sum_{v=0}^{r} x^{1,v} \frac{\partial c_{1,\mu}}{\partial x^{1,v}} = 0.
\]

Now (3.7) and (3.8) imply

\[
\sum_{q=1}^{n} \sum_{v=0}^{r} x^{q,v} \frac{\partial c_{1,\mu}}{\partial x^{q,v}} = 0.
\]

According to Lemma 3.2 it implies \( c_{1,\mu} \) is constant.

Next we observe that from (3.6) for \( j = 1 \) and \( q \neq 1 \) we obtain

\[
c_{q,\mu} = - \sum_{v=0}^{r} x^{1,v} \frac{\partial c_{1,\mu}}{\partial x^{1,v}} = 0.
\]

We set \( c_{v} = c_{1,v} \). Now from (2.5) we have

\[
(3.9) \quad \mathcal{L}(dx^{k}) = \sum_{\mu=0}^{r} c_{\mu} dx^{1,\mu}.
\]

If \( X \) is a vector field on \( U \), then Proposition 2.6 and formula (3.9) imply

\[
\mathcal{L}(L_{X}dx^{1}) = L_{Xc}(\mathcal{L}(dx^{1})) = L_{Xc} \left( \sum_{\mu=0}^{r} c_{\mu} dx^{1,\mu} \right) =
\]

\[
= \sum_{\mu=0}^{r} c_{\mu} L_{Xc}(dx^{1,\mu}) = \sum_{\mu=0}^{r} c_{\mu} (L_{X}dx^{1})(^{(\mu)}).
\]

According to Lemma 3.3 every 1-form \( \omega \) on \( M \) can be written in the form \( \omega = L_{X}(dx^{1}) \), thus from the last formula we obtain Proposition 3.4. The proof is finished.

From Proposition 3.4 and corollary 2.5 we obtain:

**Corollary 3.5.** Let \( \mathcal{L} \) be a lift of 1-forms from \( M \) to \( T^{*}M \). If \( \omega \) is a closed 1-form on \( M \), then \( \mathcal{L}\omega \) is closed.

If \( \dim M = 1 \), then every 1-form on \( M \) is closed. Thus from Proposition 3.4 we obtain

**Theorem 3.6.** Let \( M \) be a manifold such that \( \dim M = 1 \). If \( \mathcal{L} \) is a lift of 1-forms from \( M \) to \( T^{*}M \), then \( \mathcal{L} \) is a linear combination with real coefficients of the \( \lambda \)-lifts for \( \lambda = 0, \ldots, r \).

Next we prove the following proposition:

**Proposition 3.7.** Let \( \dim M \geq 2 \). If \( \mathcal{L}^{\lambda} \) is a lift of 1-forms from \( M \) to \( T^{*}M \) such that \( \mathcal{L}^{\lambda}(\omega) = 0 \) for every closed 1-form \( \omega \), then there exists a real number \( b \)
such that

\begin{equation}
\mathcal{L}^{-}(\omega) = b(\omega^\square - \omega^{(1)})
\end{equation}

for every 1-form \( \omega \) on \( M \), where \( \omega^\square \) is defined by (2.1).

Proof. Let \((U, x^i)\) be a chart on \( M \). At first, we shall show the formula (3.10) for \( w_0 = x^2\;dx^1 \) (\( w_0 \) is well-defined because \( \dim M \geq 2 \)).

There exist differentiable functions \( c_{i,v} \) such that

\begin{equation}
\mathcal{L}(\omega_0) = \sum_{j=1}^n \sum_{v=0}^r c_{j,v}\;dx^{j,v}.
\end{equation}

Let \( X = \partial_q = \partial/\partial x^q \). Since \( L_X\omega_0 \) is closed, thus from Proposition 2.6 we obtain

\[ 0 = \mathcal{L}(L_q\omega_0) = L_q\mathcal{L}(\omega_0) = \sum_{j=1}^n \sum_{v=0}^r \frac{\partial^2 c_{j,v}}{\partial x^{q,0}}\;dx^{j,v} \]

because \( \partial_q^C = \partial_{q,0} = \partial/\partial x^{q,0} \). The last formula implies that \( \partial c_{j,v}/\partial x^{q,0} = 0 \), that is,

\begin{equation}
c_{j,v} \text{ does not depend of } x^{q,0}.
\end{equation}

Let \( Y = x^k \partial_q \). Now for any \( k \) and \( q \geq 3 \) the 1-form \( L_Y\omega_0 \) is closed, and from Proposition 2.6, we obtain

\[ L_Y\mathcal{L}(\omega_0) = \mathcal{L}(L_Y\omega_0) = 0. \]

It means that for \( q \geq 3 \) and \( k = 1, \ldots, n \) we have

\[ \sum_{v=0}^r x^k \frac{\partial^2 c_{i,v}}{\partial x^{q,0}} + \frac{\partial^2 c_{i,v}}{\partial x^{q,k}} = 0. \]

Setting \( k = q \) and summing with respect to \( k \) from 3 to \( n \) we obtain

\[ \sum_{k=3}^n \sum_{v=0}^r x^k \frac{\partial^2 c_{i,v}}{\partial x^{q,0}} + c_{i,v} = 0 \quad \text{for} \quad i \geq 3, \]

\[ \sum_{k=3}^n \sum_{v=0}^r x^k \frac{\partial^2 c_{i,v}}{\partial x^{q,k}} = 0 \quad \text{for} \quad i \leq 2. \]

According to Lemma 3.2 it implies

\begin{equation}
c_{i,v} = 0 \quad \text{for} \quad i \geq 3, \quad (3.13)
\end{equation}

\begin{equation}
c_{1,v} \quad \text{and} \quad c_{2,v} \quad \text{do not depend of} \quad x^{k,k} \quad \text{for} \quad k \geq 3. \quad (3.14)
\end{equation}

From (3.11)–(3.14) we obtain

\begin{equation}
\mathcal{L}\omega_0 = \sum_{v=0}^r \{ c_{1,v}\;dx^{1,v} + c_{2,v}\;dx^{2,v} \},
\end{equation}

\[ c_{1,v} = c_{1,v}(x^{1,1}, \ldots, x^{1,r}, x^{2,1}, \ldots, x^{2,r}), \]

\[ c_{2,v} = c_{2,v}(x^{1,1}, \ldots, x^{1,r}, x^{2,1}, \ldots, x^{2,r}). \]

Next by using \( V = (x^1)^k \partial_2 \) we observe that \( L_Y\omega_0 \) is closed for any natural number \( k \),

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and hence, by Proposition 2.5 we have
\[ \sum_{k=1}^r \sum_{\lambda_1 + \ldots + \lambda_k = \lambda} x^{1,\lambda_1} \ldots x^{1,\lambda_k} \frac{\partial c_{2,y}}{\partial x^{2,\lambda}} + \frac{\partial c_{1,y}}{\partial x^{2,\lambda}} = 0 \]
\[ + \sum_{k=1}^r \sum_{\lambda_1 + \ldots + \lambda_k = \lambda - v} x^{1,\lambda_1} \ldots x^{1,\lambda_k-1} \delta_{1}\frac{\partial c_{2,y}}{\partial x^{2,\lambda}} = 0 \]
for \( i = 1, 2 \). If \( i = 2 \), then we obtain
\[ (3.17) \sum_{k=1}^r \sum_{\lambda_1 + \ldots + \lambda_k = \lambda} x^{1,\lambda_1} \ldots x^{1,\lambda_k} \frac{\partial c_{2,y}}{\partial x^{2,\lambda}} = 0. \]

We fix \( \lambda \) and we consider the system of linear equations (3.17) for \( k = 1, \ldots, r \), where \( \xi_{\lambda} = \frac{\partial c_{2,y}}{\partial x^{2,\lambda}} \) are unknown. Since the determinant \( W \) of this system is a polynomial of \( r \) variables \( x^{1,1}, \ldots, x^{1,r} \), thus \( W \neq 0 \) on some dense and open subset \( \mathcal{U} \) of \( R^r \). Now \( \xi_{\lambda} = 0 \) on \( \mathcal{U} \). Since \( c_{2,y} \) is a differentiable functions of \( x^{1,1}, \ldots, x^{1,r} \), thus
\[ \frac{\partial c_{2,y}}{\partial x^{2,\lambda}} = 0. \]

It means that \( c_{2,y} \) does not depend of \( x^{2,1}, \ldots, x^{2,r} \). We can write
\[ (3.18) c_{2,y} = c_{2,y}(x^{1,1}, \ldots, x^{1,r}). \]

Analogously, if we will use \( V = (x^2)^k \delta_1 \) instead of \( (x^1)^k \delta_2 \), then we obtain that \( c_{1,y} \) does not depend of \( x^{1,1}, \ldots, x^{1,r} \), that is,
\[ (3.19) c_{1,y} = c_{1,y}(x^{2,1}, \ldots, x^{2,r}). \]

Now from (3.16) for \( k = 1 \) and \( i = 1 \) we obtain
\[ (3.20) \sum_{k=1}^r x^{1,\lambda} \frac{\partial c_{1,y}}{\partial x^{2,\lambda}} + c_{2,y} = 0. \]

According to (3.18) and (3.19) the last equality implies that \( \partial c_{1,y} / \partial x^{2,\lambda} \) is constant, that is, \( c_{1,y} \) is a linear function of \( x^{2,1}, \ldots, x^{2,r} \). Hence there exist real numbers \( a_{\lambda,y} \) such that
\[ (3.21) c_{1,y} = \sum_{\lambda=1}^r a_{\lambda,y} x^{2,\lambda} \]
for \( v = 0, \ldots, r \). From (3.20) and (3.21) we have also
\[ (3.22) c_{2,y} = -\sum_{\lambda=1}^r a_{\lambda,y} x^{1,\lambda}. \]

Now from (3.16) for \( k = 2 \) and \( i = 1 \) we obtain
\[ \sum_{\lambda=1}^r \sum_{\mu=0}^r x^{1,\mu} x^{1,\lambda-\mu} a_{\lambda,y} = \sum_{\lambda=0}^r \sum_{\mu=1}^r x^{1,\lambda} x^{1,\mu} a_{\mu,\lambda+v}. \]

By the comparing coefficients of polynomes from the last equality we obtain \( a_{\lambda,y} = 0 \).
provided \( \lambda \neq 1 \) or \( \nu \neq 0 \). We set \( b = a_{1,0} \). Now from (3.15), (3.21) and (3.22) we obtain

\[
(3.23) \quad \mathcal{L}(\omega_0) = b(x^{2,1} \, dx^{1,0} - x^{1,1} \, dx^{2,0}).
\]

According to (1.11) and (2.2) we have

\[
\omega^{(1)}_0 = x^{2,0} \, dx^{1,1} + x^{2,1} \, dx^{1,0},
\]

\[
\omega^\square = x^{1,1} \, dx^{2,0} + x^{2,0} \, dx^{1,1}.
\]

Now formula (3.23) implies

\[
(3.24) \quad \mathcal{L}(\omega_0) = b(\omega^{(1)}_0 - \omega^\square_0).
\]

By using a local diffeomorphism \( \varphi \) which permutes coordinates, by the naturality condition (see Definition 2.1), from (3.24) we obtain for any numbers \( i \neq j \)

\[
(3.25) \quad \mathcal{L}(x_i \, dx^j) = b((x_i \, dx^j)^{(1)} - (x_i \, dx^j)^\square).
\]

Now for an 1-form \( \omega = f \, dx^i \), where \( f \) is any function, we have

\[
\omega = L_{(f \circ \varphi)} x^j \, dx^i, \quad j \neq i
\]
and by Proposition 2.6 from (3.25) we obtain the formula (3.10) for \( \omega = f \, dx^i \). By the linearity of \( \mathcal{L} \), the formula (3.10) holds for every 1-form \( \omega \), and the proof of our proposition is finished.

**Proof of Theorem 3.1.** Let \( \mathcal{L} \) be a lift of 1-forms and let \( c_0, \ldots, c_r \) be the real numbers as in Proposition 3.3. It means that for every closed 1-form \( \omega \) the following equality holds:

\[
\mathcal{L} \omega = \sum_{\mu=0}^r c_\mu \omega^{(\mu)}.
\]

Now we define a new lift of 1-forms \( \mathcal{L}' : \mathcal{X}^* \rightarrow \mathcal{X}^*(T^*M) \) setting

\[
\mathcal{L}' \omega = \mathcal{L} \omega - \sum_{\mu=0}^r c_\mu \omega^{(\mu)}.
\]

\( \mathcal{L}' \) satisfies the assumption of Proposition 3.5. Thus, there exists a real number \( b \) such that

\[
\mathcal{L}' \omega = b(\omega^\square - \omega^{(1)}) = \mathcal{L} \omega - \sum_{\mu=0}^r c_\mu \omega^{(\mu)}.
\]

From the last formula we obtain Theorem 3.1 if we set

\[
a^\square = b, \quad a_1 = c_1 - b, \quad a_i = c_i \quad \text{for} \quad i \neq 1.
\]

The proof of Theorem 3.1 is now finished.
References


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