Josef Niederle
Frame tolerances are directly decomposable

_Czechoslovak Mathematical Journal_, Vol. 40 (1990), No. 3, 422–423

Persistent URL: http://dml.cz/dmlcz/102394

Terms of use:

© Institute of Mathematics AS CR, 1990

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
FRAME TOLERANCES ARE DIRECTLY DECOMPOSABLE

JOSEF NIEDERLE, BRNO

(Received June 17, 1988)

A frame is a complete lattice satisfying the Join Infinite Distributive Identity \( a \land \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \land b_i) \). A lattice tolerance on a frame (or, generally, on a lattice) is a reflexive symmetric relation on its support compatible with finite meets and joins. A frame tolerance on a frame is a lattice tolerance compatible with arbitrary joins (suprema). A lattice (frame) tolerance on the product of frames \( \mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i \) is directly decomposable if there exist lattice (frame) tolerances \( T_i \) on \( \mathfrak{F}_i \) such that \( T = \prod_{i \in I} T_i = \{([a, b] \in \mathfrak{F} \times \mathfrak{F} \mid \forall (i \in I) [p_i(a), p_i(b)] \in T_i \} \) where \( p_i \) are projections of \( \mathfrak{F} \) onto \( \mathfrak{F}_i \) (i \( \in \) I).

We know that lattice tolerances on products of finite number of lattices are directly decomposable while those on products of infinite number of nontrivial lattices are not (cf. [1]). The same statement is obviously valid for lattice tolerances on frames. For frame tolerances, we shall prove a stronger result.

**Theorem.** Frame tolerances on arbitrary products of frames are directly decomposable.

**Proof.** Let \( T \) be a frame tolerance on the product of frames \( \mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i \). Denote by \( 0_i \) the least and by \( 1_i \) the greatest elements of \( \mathfrak{F}_i \), put \( \beta_i(T) = \{[x, y] \in \mathfrak{F}_i \times \mathfrak{F}_i \mid \exists e_i(x), e_i(y) \in T \} \) where \( e_i(x) \) is defined by \( p_i(e_i(x)) = 0_i \) if \( i \neq j \), and \( p_i(e_i(x)) = x \) (i \( \in \) I). They are obviously frame tolerances. We shall prove \( T = \prod_{i \in I} \beta_i(T) \). Let \( [a, b] \in T \). Then \( e_i(p_i(a)), e_i(p_i(b)) \) \( = \) \( e_i(1_i) \land a, e_i(1_i) \land b \) \( \in \) \( T \), and so \( p_i(a), p_i(b) \in \beta_i(T) \) (i \( \in \) I). Hence \( [a, b] \in \prod_{i \in I} \beta_i(T) \). Conversely, let \( [a, b] \in \prod_{i \in I} \beta_i(T) \), i.e. \( p_i(a), p_i(b) \in \beta_i(T) \) (i \( \in \) I). Then \( e_i(p_i(a)), e_i(p_i(b)) \) \( \in \) \( T \) (i \( \in \) I), and consequently \( [a, b] = \bigvee_{i \in I} [e_i(p_i(a)), e_i(p_i(b))] \in T \). Q.E.D.

Frame tolerances on a frame form a complete lattice (cf. [2]).

**Corollary.** The lattice of all frame tolerances on the product of frames \( \mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i \) is isomorphic to the product of lattices of all frame tolerances on the frames \( \mathfrak{F}_i \) (i \( \in \) I).

**Proof.** In fact, the theorem assigns to any frame tolerance \( T \) on \( \mathfrak{F} \) an element of the product of lattices of frame tolerances on \( \mathfrak{F}_i \) (i \( \in \) I). This assignment is obviously an injective isotone mapping. It remains to prove its surjectivity. Let \( T_i \) (i \( \in \) I)
be frame tolerances on $\mathcal{Q}_i$ respectively. Put $T = \prod_{i=1}^{n} T_i$. Then $\beta_i(T) = \{[x, y] \in \mathcal{Q}_i \times \mathcal{Q}_i \mid [e_i(x), e_i(y)] \in T_i\} = \{[x, y] \in \mathcal{Q}_i \times \mathcal{Q}_i \mid [p_i(e_i(x)), p_i(e_i(y))] \in T_i\} = \{[x, y] \in \mathcal{Q}_i \times \mathcal{Q}_i \mid [x, y] \in T_i\} = T_i$. Q.E.D.

Added in proof. Analogous statements may be a fortiori proved for frame congruences. Both proofs work also for frame-compatible reflexive relations.

References


Author’s address: 615 00 Brno 15, Viniční 60, Czechoslovakia.