Ján Jakubík
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CONVERGENCES AND HIGHER DEGREES OF DISTRIBUTIVITY
OF LATTICE ORDERED GROUPS AND OF BOOLEAN ALGEBRAS

Ján Jakubík, Košice

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All lattice ordered groups dealt with in the present note are assumed to be abelian.
The partially ordered set of all convergences of a lattice ordered group $G$ will be
denoted by Conv $G$ (cf. [2], [3]). Similarly, Conv $B$ denotes the partially ordered
set of all convergences of a Boolean algebra $B$ (cf. [8]). In general, neither Conv $G$ nor
Conv $B$ need be a lattice; Conv $G$ is a lattice iff it possesses a greatest element and
in such a case it is a complete lattice. An analogous result holds for Conv $B$.

In [7] it was shown that the existence of the greatest element in Conv $G$ depends
merely from the lattice properties of $G$ and that the class of all lattice ordered groups
having the largest convergence is a radical class (in the sense of [5]).

The following results were proved in [6] and [8]:

(A) If $G$ is a completely distributive archimedean lattice ordered group, then
Conv $G$ is a complete lattice.

(B) If $B$ is a completely distributive Boolean algebra, then Conv $B$ is a complete
lattice.

In the present note these results will be sharpened as follows:

(A$_1$) If $G$ is an $(\mathfrak{N}_0, 2)$-distributive lattice ordered group, then Conv $G$ is a complete
lattice.

(B$_1$) If $B$ is an $(\mathfrak{N}_0, 2)$-distributive Boolean algebra, then Conv $B$ is a complete
lattice.

The notion of bounded convergence in a lattice ordered group was introduced
in [7]. Let Conv$_b$ $G$ be the set of all bounded convergences in $G$.

If $0 < e \in G$ and $e$ is a singular element, then the interval $[0, e]$ of $G$ is Boolean
algebra. Put $[0, e] = B$. It will be shown that if $e$ is, at the same time, a strong unit
in $G$, then the partially ordered sets Conv$_b$ $G$ and Conv $B$ are isomorphic. Next,
Conv $B$ is a complete lattice if and only if Conv $G$ is a complete lattice.
1. THE CASE OF LATTICE ORDERED GROUPS

Let $G$ be a lattice ordered group. We recall briefly the basic notions concerning sequential convergences in $G$.

Let $N$ be the set of all positive integers and let $G_n = G$ for each $n \in N$. We denote
\[
\prod_{n \in N} G_n = G^N.
\]
The elements of $G^N$ (denoted, e.g., by $(g_n)$) are called sequences in $G$.

If $g \in G$ and $g_n = g$ for each $n \in N$, then we denote $(g_n) = \text{const} \, g$.

Let $\alpha$ be a convex subsemigroup of the semigroup $(G^N)^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \alpha$, then each subsequence of $(g_n)$ belongs to $\alpha$.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of $(g_n)$ has a subsequence belonging to $\alpha$, then $(g_n) \in \alpha$.

(III) Let $g \in G$. Then $\text{const} \, g$ belongs to $\alpha$ if and only if $g = 0$.

Under these assumptions $\alpha$ is said to be a convergence in $G$. The system of all convergences in $G$ (partially ordered by inclusion) will be denoted by $\text{Conv} \, G$.

For $\alpha \in \text{Conv} \, G$, $(g_n) \in G^N$ and $g \in G$ we put $g_n \to_\alpha g$, if $|g_n - g| \in \alpha$.

A sequence $(g_n) \in (G^N)^+$ is said to be regular if there exists $\alpha \in \text{Conv} \, G$ such that $(g_n) \in \alpha$.

From the convexity of $\alpha$ in $(G^N)^+$ and from (II), (III) we obtain immediately:

1.1. Lemma. Let $(g_n)$ be a regular sequence in $G$ and let $(g_m)$ be a subsequence of $(g_n)$. Then $\bigwedge g_m = 0$.

Next, from the lemmas 3.2, 3.3 and 2.4 of [7] we obtain:

1.2. Lemma. Let $G$ be a lattice ordered group. The following conditions are equivalent:

(i) $\text{Conv} \, G$ has no greatest element.

(ii) There are regular sequences $(g_n), (h_n)$ in $G$ and $0 < c \in G$ such that $g_n \lor \lor h_n \geq c$ for each $n \in N$.

1.3. Lemma. Let $G$ be $(\aleph_0, 2)$-distributive. Then $\text{Conv} \, G$ possesses the greatest element.

Proof. By way of contradiction, assume that $\text{Conv} \, G$ has no greatest element. Then in view of 1.2 there are sequences $(g_n)$ and $(h_n)$ in $G$ such that the condition (ii) from 1.2 is satisfied. Put $g_{n0} = c \land g_n$ and $h_{n0} = c \land h_n$ for each $n \in N$. Then $c = g_{n0} \lor h_{n0}$ for each $n \in N$. Hence in view of $(\aleph_0, 2)$-distributivity of $G$ we obtain
\[
0 < c = (g_{10} \lor h_{10}) \land (g_{20} \lor h_{20}) \land \ldots.
\]
Let $I$ be the set of all mappings $t_i$ of the set $N$ into $\bigcup_{n \in N} \{g_{n0}, h_{n0}\}$ such that for each $n \in N$ we have $t_i(n) \in \{g_{n0}, h_{n0}\}$. Let us write $t_{in}$ instead of $t_i(n)$. Let $i \in N$ be fixed. Then some of the following conditions is valid:

(a) the set $\{j \in N: t_{ij} = g_{ij}\}$ is infinite;

(b) the set $\{j \in N: t_{ij} = h_{ij}\}$ is infinite.
According to 2.1, in both the cases (a) and (b) we have
\[ t_{11} \land t_{12} \land t_{13} \land \ldots = 0 , \]
hence
\[ \forall_{t_{i1}} (t_{11} \land t_{12} \land t_{13} \land \ldots) = 0 . \]
The relation (1) and (2) show that \( G \) is not \( (\mathbb{N}_0, 2) \)-distributive, which is contradiction.

From 1.3 and from [4] we infer that \( (A_4) \) holds.

Let us remark that if \( G \) is \( (\mathbb{N}_0, 2) \)-distributive, then it need not be archimedean (e.g., it suffices to take a non-archimedean linearly ordered group).

2. THE CASE OF BOOLEAN ALGEBRAS

Let \( B \) be a Boolean algebra. For each \( n \in \mathbb{N} \) let \( B_n = B \). The direct product (in lattice-theoretic sense) of lattices \( B_n \) \( (n \in \mathbb{N}) \) will be denoted by \( B^\mathbb{N} \). The elements of \( B^\mathbb{N} \) are denoted, e.g., as \( (b_n) \) and they will be called sequences in \( B \).

The notion of sequential convergence in \( B \) was introduced in [8] (Definition 1.1). Let \( \text{Conv} B \) be the system of all sequential convergences in \( B \); this system is partially ordered by inclusion.

For \( x \in \text{Conv} B \) we denote by \( x_0 \) the set of all \( (x_n) \in x \) such that \( x_n \to 0 \). Let \( \text{Conv}_0 B \) be the set of all \( x_0 \), where \( x \) runs over the system \( \text{Conv} B \). The set \( \text{Conv}_0 B \) is partially ordered by inclusion. In [8] it was shown that the mapping \( x \to x_0 \) \( (x \in \text{Conv} B) \) is an isomorphism of \( \text{Conv} B \) onto \( \text{Conv}_0 B \). The elements of \( \text{Conv}_0 B \) are called 0-convergences in \( B \).

From 1.5 in [8] it follows that for a subset \( \beta \) of \( B^\mathbb{N} \) the following conditions are equivalent:

(i) \( \beta \in \text{Conv}_0 B \).
(ii) \( \beta \) is an ideal of the lattice \( B^\mathbb{N} \) such that the condition (I), (II) and (III) are satisfied (where \( \alpha \) and \( G \) are replaced by \( \beta \) or \( B \), respectively).

Since \( \text{Conv} B \) and \( \text{Conv}_0 B \) are isomorphic, by proving \( (B_4) \) it suffices to prove the corresponding assertion for \( \text{Conv}_0 B \).

A sequence \( (x_n) \) in \( B \) will be called regular in \( B \) if there is \( \beta \in \text{Conv}_0 B \) such that \( (x_n) \in \beta \).

The assertion of Lemma 1.1 remains valid if \( G \) is replaced by \( B \) (let us denote this modified assertion as 2.1). Similarly, we can formulate the assertion 2.2 which is analogous to 1.2.

2.2. Lemma. Let \( B \) be a Boolean algebra. The following conditions are equivalent:

(i) \( \text{Conv}_0 B \) has no greatest element.
(ii) There are regular sequences \( (g_n) \) and \( (h_n) \) in \( B \) and \( 0 \leq c \in B \) such that \( g_n \vee h_n \geq c \) for each \( n \in \mathbb{N} \).

Proof. The implication (ii) \( \Rightarrow \) (i) is obvious. The implication (i) \( \Rightarrow \) (ii) is contained in the proof of 3.4 in [8].
Next, by replacing 1.1 and 1.2 in the proof of 1.3 by 2.1 and 2.2 respectively we obtain that the following assertion analogous to 1.3 holds:

2.3. Lemma. Let $B$ be $(\mathbb{N}_0, 2)$-distributive. Then $\text{Conv} \ B$ possesses the greatest element.

The above lemma and Theorem 3.6 of [8] yield that $(B_1)$ is valid.

The equation whether the $(\mathbb{N}_0, 2)$-distributivity of $B$ is necessary for $\text{Conv} \ B$ to be complete remains open. The corresponding question for lattice ordered groups remains open as well.

3. SINGULAR STRONG UNIT

Again, let $G$ be an abelian lattice ordered group, $G \neq \{0\}$. We recall the following definitions (cf. [1]):

An element $0 < x \in G$ is called singular if, whenever $y \in G$, $0 < y < x$, then $(x - y) \land y = 0$.

Let $0 < e \in G$. The element $e$ is said to be a weak unit in $G$, if whenever $0 < y \in G$, then $e \land y > 0$. Next, $e$ is called a strong unit in $G$ if for each $y \in G$ there is $n \in \mathbb{N}$ such that $y < ne$. Every strong unit in $G$ is a weak unit in $G$.

It is easy to verify that an element $0 < x \in G$ is singular if and only if the interval $[0, x]$ of $G$ is a Boolean algebra.

A subset $\alpha_1$ of $(G^N)^+$ will be called regular if there exists $\alpha \in \text{Conv} \ G$ such that $\alpha_1 \subseteq \alpha$. Analogously we define the regularity of a subset of $B^N$, where $B$ is a Boolean algebra.

3.1. Lemma. Let $0 < e \in G$ such that (i) $e$ is a weak unit in $G$, and (ii) $e$ is singular. Denote $B = [0, e]$ and let $\alpha_1 \subseteq B^N$. Then the following conditions are equivalent:

(a) $\alpha_1$ is regular with respect to $G$.
(b) $\alpha_1$ is regular with respect to $B$.

Proof. The equivalence (a) ⇔ (b) follows from 1.2 and 2.2.

Let $\text{Conv}_G$ be the set of all $\alpha \in \text{Conv} \ G$ having the property that whenever $(x_n) \in \alpha$, then $(x_n)$ is bounded in $G$. The set $\text{Conv}_G$ is partially ordered by inclusion.

3.2. Proposition. (Cf. [7], Theorem 4.8.) The following conditions are equivalent:

(i) $\text{Conv} \ G$ has a greatest element.
(ii) $\text{Conv}_G$ has a greatest element.

Let $e$ and $B$ be as in 3.1 and let $\alpha_1 \in \text{Conv}_0 \ B$. We denote by $T(\alpha_1)$ the least element of $\text{Conv} \ G$ which is larger or equal to $\alpha_1$; such an element does exist in view of 3.1. Then we have

3.3. Lemma. Let $(x_0) \in (G^N)^+$. Under the above assumptions and denotations, the following conditions are equivalent:

(i) $(x_n) \in T(\alpha_1)$.
(ii) There are \( m \in \mathbb{N} \) and \((z_n) \in \alpha_1\) such that \( x_n \leq mz_n \) for each \( n \in \mathbb{N} \).

Proof. The implication (ii) \( \Rightarrow \) (i) is obvious. Let (i) be valid. By similar reasoning as in the proof of Lemma 2.5 in [7] we obtain that there are \( m_1 \in \mathbb{N} \) and \((y_n^1), (y_n^2), \ldots, (y_n^k) \in \alpha_1\) such that

\[
x_n \leq m_1(y_n^1 + y_n^2 + \ldots + y_n^k) \quad \text{for each} \quad n \in \mathbb{N}.
\]

Thus in view of Lemma 2.4 in [7] there is \( m \in \mathbb{N} \) such that

\[
x_n \leq m(y_n^1 \lor y_n^2 \lor \ldots \lor y_n^k) \quad \text{for each} \quad n \in \mathbb{N}.
\]

Since \((y_n^1 \lor y_n^2 \lor \ldots \lor y_n^k) \in \alpha_1\), it suffices to put \( z_n = y_n^1 \lor y_n^2 \lor \ldots \lor y_n^k \).

Throughout this section, the above denotations will be applied.

3.4. Corollary. Let \( x_1 \in \text{Conv}_0 B \). Then \( T(x_1) \in \text{Conv}_b G \).

Proof. Let \((x_n) \in T(x_1)\) and let \( m \) be as in 3.3 (ii). Then \( x_n \leq me \) for each \( n \in \mathbb{N} \), hence \((x_n)\) is bounded in \( G \).

3.5. Lemma. Let \( x, y \in [0, e], m \in \mathbb{N}, x \leq my \). Then \( x \leq y \).

Proof. By way of contradiction, assume that \( x \nleq y \). Then (since \([0, e]\) is a Boolean algebra) there is \( x_1 \in [0, e] \) such that \( 0 < x_1 \leq x \) and \( x_1 \land y = 0 \). Hence \( x_1 \land \land my = 0 \), which is a contradiction.

3.6. Lemma. Let \( \alpha_1, \beta_1 \in \text{Conv}_0 B \). Then we have

\[
\alpha_1 \leq \beta_1 \iff T(\alpha_1) \leq T(\beta_1).
\]

Proof. The implication \( \alpha_1 \leq \beta_1 \Rightarrow T(\alpha_1) \leq T(\beta_1) \) is obvious. Hence it suffices to verify that if \( \alpha_1 \nleq \beta_1 \), then \( T(\alpha_1) \nleq T(\beta_1) \).

Assume that \( \alpha_1 \nleq \beta_1 \). Hence there exists \( (t_n) \in \alpha_1 \setminus \beta_1 \). Clearly \( (t_n) \in T(\alpha_1) \). We shall show that \( (t_n) \) does not belong to \( T(\beta_1) \). By way of contradiction, suppose that \( (t_n) \in T(\beta_1) \). Thus in view of 3.3 there are \( m \in \mathbb{N} \) and \((z_n) \in \beta_1\) such that

\[
t_n \leq mz_n \quad \text{for each} \quad n \in \mathbb{N}.
\]

By applying 3.5 we obtain that

\[
t_n \leq z_n \quad \text{for each} \quad n \in \mathbb{N}.
\]

Hence \((t_n) \in \beta_1\), which is a contradiction.

From 3.4 and 3.5 we obtain:

3.7. Theorem. Let \( 0 < e \leq G \) be a singular weak unit in \( F \). Then the mapping \( \alpha_1 \to T(\alpha_1) \) (where \( \alpha_1 \) runs over \( \text{Conv}_0 [0, e] \)) is an isomorphism of the partially ordered set \( \text{Conv}_0 [0, e] \) into the partially ordered set \( \text{Conv}_b G \).

3.8. Lemma. Let \( 0 < e \leq G \) be a singular strong unit in \( G \). Let \( \alpha \in \text{Conv}_b G \). Put \( \alpha_1 = \alpha \cap B^\alpha \), where \( B = [0, e] \). Then \( \alpha_1 \in \text{Conv}_0 B \) and \( T(\alpha_1) = \alpha \).

Proof. The verification of the relation \( \alpha_1 \in \text{Conv}_0 B \) is easy. Since \( \alpha_1 \leq \alpha \), we have \( T(\alpha_1) \leq T(\alpha) = \alpha \). Let \((x_n) \in \alpha \). Because of \( \alpha \in \text{Conv}_b (G) \), there is \( 0 < g \in G \)
such that \( x_n \leq g \) for each \( n \in N \). Next, \( e \) is a strong unit in \( G \) and thus there is \( m \in N \) such that \( g \leq me \). Therefore
\[
x_n \leq me \quad \text{for each} \quad n \in N.
\]
Let \( n \) be fixed. There are \( x_{n_1}, x_{n_2}, \ldots, x_{n_m} \) in \( G \) such that

1. \( 0 \leq x_{n_j} \leq e \) for \( j = 1, 2, \ldots, m \),
2. \( x_n = x_{n_1} + x_{n_2} + \cdots + x_{n_m} \).

Thus according to Lemma 2.4, [7] there is \( m_1 \in N \) such that

3. \( x_n \leq m_1 (x_{n_1} \lor x_{n_2} \lor \cdots \lor x_{n_m}) \).

In view of (1) and (2) we have \( (x_{n_1}), (x_{n_2}), \ldots, (x_{n_k}) \in \alpha_1 \), whence \( (z_n) \in \alpha_1 \), where
\[
z_n = x_{n_1} \lor x_{n_2} \lor \cdots \lor x_{n_m} \quad \text{for each} \quad n \in N.
\]
Thus (3) yields that \( (x_n) \) belongs to \( T(\alpha_1) \), completing the proof.

The following theorem is a consequence of 3.7 and 3.8.

**Theorem 3.9.** Let \( 0 < e \in G \) be a singular strong unit in \( G \). Then the mapping \( \alpha_1 \rightarrow T(\alpha_1) \) (where \( \alpha_1 \) runs over \( \text{Conv}_0 [0, e] \)) is an isomorphism of the partially ordered set \( \text{Conv}_0 [0, e] \) onto the partially ordered set \( \text{Conv}_e G \).

Next, 3.9 and 3.2 yield:

**Corollary 3.10.** Let \( 0 < e \in G \) be a singular strong unit in \( G \). Then \( \text{Conv}_0 [0, e] \) is a complete lattice iff \( \text{Conv} G \) is a complete lattice.

**References**


*Author's address:* 040 01 Košice, Grešáková 6, Czechoslovakia (Matematický ústav SAV, dislokované pracovisko v Košiciach).