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ABSOLUTELY CONTINUOUS SELECTIONS FROM ABSOLUTELY
CONTINUOUS SET VALUED MAP

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INTRODUCTION

In this paper we prove that the barycentric selection from an absolutely continuous set valued map $F: (0, T) \rightsquigarrow \mathbf{R}^n$ with nonempty convex values is absolutely continuous. Moreover we prove using the barycentric selection that under certain conditions for every $x_0 \in F(t_0)$ there exists an absolutely continuous selection $f(\cdot)$ from a set valued map $F(\cdot)$ such that $f(t_0) = x_0$.

The existence of an absolutely continuous selection plays an important role in the viability theory (see [1]) if the viability map $K(\cdot)$ depends only measurably on time. Then the necessary condition for the existence of a viable solution is the existence of an absolutely continuous selection from $K(\cdot)$.

NOTATION

\mathbf{R}^n is the Euclidian n -dimensional space; $d(x, y)$ is the Euclidian distance from x to y . $B(x, M)$ denotes the open ball of radius M about x and $B := B(0, 1)$. S denotes the unit sphere. If A, B are subsets of \mathbf{R}^n , $d(x, A) := \inf \{d(x, y) \mid y \in A\}$, $\delta(A, B) := \sup \{d(x, B) \mid x \in A\}$ denotes the separation of A from B and $d^*(A, B) := \sup (\delta(A, B), \delta(B, A))$ is Hasudorff distance of the sets A and B . For $x, y \in \mathbf{R}^n$, $\langle x, y \rangle$ denotes the scalar product. Let $A \subset \mathbf{R}^n$, $A \neq \emptyset$, $e \in S$ then $\sigma_A(e) := \sup_{a \in A} \langle a, e \rangle$ is the support function of the set A . By $\text{ri}(A)$ we denote the relative interior of the set A .

MAIN RESULTS

Definition 1. Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with convex and compact values. We say that F is an *absolutely continuous map* if the following condition is fulfilled

$$\forall \varepsilon > 0, \quad \exists \delta > 0 \quad \text{such that for every system of intervals} \\ [t_1, \tau_1], \dots, [t_m, \tau_m], \quad (0 \leq t_1 \leq \tau_1 \leq \dots \leq t_m \leq \tau_m \leq T)$$

the following holds

$$\sum_{j=1}^m (\tau_j - t_j) < \delta \Rightarrow \max \left(\sum_{j=1}^m \mu_n((F(t_j) + B) \setminus (F(\tau_j) + B)), \right. \\ \left. \sum_{j=1}^m \mu_n((F(\tau_j) + B) \setminus (F(t_j) + B)) \right) < \varepsilon,$$

where μ_n denotes n -dimensional Lebesgue measure.

Let $A \subset \mathbf{R}^n$ be a convex compact set with nonempty interior. Then we define (see [1])

$$b(A) := \frac{1}{\mu_n(A)} \int_A x \, d\mu_n.$$

Theorem 1. Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be an absolutely continuous set valued map with nonempty convex and compact values. Let $F(\cdot)$ be bounded, i.e. there exists $M > 0$ such that

$$\forall t \in (0, T), \quad F(t) \subset M \cdot B.$$

Then the map $f: (0, T) \mapsto \mathbf{R}^n$

$$f(t) := b(F(t) + B)$$

is an absolutely continuous selection from $F(\cdot)$.

To prove this theorem we use the following lemma.

Lemma 1 (see Aubin and Cellina, 1984, p. 78). Let $A \subset \mathbf{R}^n$ be a convex and compact set and $A_1 := A + B$. Then $b(A_1) \in A$.

Proof of theorem 1. Let

$$\Phi(t) := F(t) + B.$$

Let $\varepsilon > 0$. Since $F(\cdot)$ is an absolutely continuous set valued map there exists $\delta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (0 \leq t_1 \leq \tau_1 \leq \dots \leq t_m \leq \tau_m \leq T)$$

holds

$$\sum_{j=1}^m (\tau_j - t_j) < \delta \Rightarrow \max \left(\sum_{j=1}^m \mu_n(\Phi(t_j) \setminus \Phi(\tau_j)), \right. \\ \left. \sum_{j=1}^m \mu_n(\Phi(\tau_j) \setminus \Phi(t_j)) \right) < \varepsilon \mu_n(B) / (4(M + 1)).$$

It follows that

$$\sum_{i=1}^m \|f(t_i) - f(\tau_i)\| = \sum_{i=1}^m \left\| \frac{1}{\mu_n(\Phi(t_i))} \int_{\Phi(t_i)} x \, d\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))} \int_{\Phi(\tau_i)} x \, d\mu_n \right\| \leq \\ \leq \sum_{i=1}^m \left(\left\| \left(\frac{1}{\mu_n(\Phi(t_i))} - \frac{1}{\mu_n(\Phi(\tau_i))} \right) \int_{\Phi(t_i) \cap \Phi(\tau_i)} x \, d\mu_n \right\| + \right. \\ \left. + \left\| \frac{1}{\mu_n(\Phi(t_i))} \int_{\Phi(t_i) \setminus \Phi(\tau_i)} x \, d\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))} \int_{\Phi(\tau_i) \setminus \Phi(t_i)} x \, d\mu_n \right\| \right).$$

Using lemma 1 and boundedness of the map $F(\cdot)$ we get

$$\begin{aligned} \mu_n(\Phi(t_i)) &\geq \mu_n(B), \quad \mu_n(\Phi(\tau_i)) \geq \mu_n(B), \quad i = 1, \dots, m, \\ \left\| \left(\frac{1}{\mu_n(\Phi(t_i))} - \frac{1}{\mu_n(\Phi(\tau_i))} \right) \int_{\Phi(t_i) \cap \Phi(\tau_i)} x \, d\mu_n \right\| &\leq \\ &\leq |\mu_n(\Phi(\tau_i)) - \mu_n(\Phi(t_i))| (M + 1) / \mu_n(B), \\ \left\| \frac{1}{\mu_n(\Phi(t_i))} \int_{\Phi(t_i) \setminus \Phi(\tau_i)} x \, d\mu_n - \frac{1}{\mu_n(\Phi(\tau_i))} \int_{\Phi(\tau_i) \setminus \Phi(t_i)} x \, d\mu_n \right\| &\leq \\ &\leq (\mu_n(\Phi(t_i) \setminus \Phi(\tau_i)) + \mu_n(\Phi(\tau_i) \setminus \Phi(t_i))) (M + 1)^2 / \mu_n(B). \end{aligned}$$

Since $F(\cdot)$ is an absolutely continuous map

$$\begin{aligned} \sum_{i=1}^m |\mu_n(\Phi(t_i)) - \mu_n(\Phi(\tau_i))| &= \\ = \sum_{i=1}^m |\mu_n(\Phi(t_i) \setminus \Phi(\tau_i)) - \mu_n(\Phi(\tau_i) \setminus \Phi(t_i))| &< \mu_n(B) \varepsilon / (2(M + 1)^2). \end{aligned}$$

Using these estimates we get

$$\sum_{i=1}^m \|f(t_i) - f(\tau_i)\| < \varepsilon.$$

We proved that $f(\cdot)$ is absolutely continuous on the interval $(0, T)$. □

Lemma 2. *Let $M > 0$. Then there exists $k > 0$ such that for every two nonempty convex and compact sets, $C, D \subset \mathbf{R}^n$ such that $C, D \subset M \cdot B$ holds*

$$kd^*(C, D) \geq \max [\mu_n((C + B) \setminus (D + B)), \mu_n((D + B) \setminus (C + B))].$$

Proof. We prove that there exists $k_1 > 0$ such that

$$k_1 \delta(C, D) = k_1 \delta(C + B, D + B) \geq \mu_n((C + B) \setminus (D + B)).$$

There exists $k_1 > 0$ (see [1], p. 80) such that

$$\begin{aligned} \mu_n((C + B) \setminus (D + B)) &\leq \mu_n(B(D + B, \delta(C + B, D + B))) - \\ &- \mu_n(D + B) \leq k_1 \delta(C + B, D + B). \end{aligned}$$

Similarly we prove that there exists $k_2 > 0$ such that

$$k_2 \delta(D, C) = k_2 \delta(D + B, C + B) \geq \mu_n((D + B) \setminus (C + B)).$$

Let

$$k := \max(k_1, k_2).$$

Then

$$kd^*(C, D) \geq \max [\mu_n((C + B) \setminus (D + B)), \mu_n((D + B) \setminus (C + B))].$$

□

The following definition was used by Kikuchi and Tomita, [3].

Definition 2. Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with nonempty compact

values. We say that F is d^* -absolutely continuous if for $\forall \varepsilon > 0, \exists \delta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (0 \leq t_1 \leq \tau_1 \leq \dots \leq t_m \leq \tau_m \leq T)$$

the following holds

$$\sum_{j=1}^m (\tau_j - t_j) < \delta \Rightarrow \sum_{j=1}^m d^*(F(t_j), F(\tau_j)) < \varepsilon.$$

From lemma 2 follows:

Lemma 3. *Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a bounded, d^* -absolutely continuous set valued map with nonempty convex compact values. Then $F(\cdot)$ is an absolutely continuous map.*

Lemma 4. *Let $F: (0, T) \rightsquigarrow \mathbf{R}^n$ be a set valued map with nonempty convex and compact values and $h: (0, T) \mapsto \mathbf{R}^n$ be an absolutely continuous function such that*

$$\forall e \in S, \quad \forall t, \tau \in (0, T), \quad \sigma_{F(t)}(e) - \sigma_{F(\tau)}(e) \leq |h(t) - h(\tau)|.$$

Then $F(\cdot)$ is d^ -absolutely continuous set valued map.*

Proof. Using the minimax theorem (see [2]) we get

$$\begin{aligned} \delta(F(t) + B, F(\tau) + B) &= \sup_{e \in S} \sup_{y \in F(t) + B} \inf_{x \in F(\tau) + B} \langle e, y - x \rangle = \\ &= \sup_{e \in S} \left(\sup_{y \in F(t) + B} \langle e, y \rangle - \sup_{x \in F(\tau) + B} \langle e, x \rangle \right) = \sup_{e \in S} (\sigma_{F(t) + B}(e) - \sigma_{F(\tau) + B}(e)) = \\ &= \sup_{e \in S} (\sigma_{F(t)}(e) - \sigma_{F(\tau)}(e)) = \delta(F(t), F(\tau)). \end{aligned}$$

It follows that

$$\delta(F(t), F(\tau)) \leq |h(t) - h(\tau)|, \quad \delta(F(\tau), F(t)) \leq |h(\tau) - h(t)|,$$

i.e.

$$d^*(F(t), F(\tau)) \leq |h(t) - h(\tau)|.$$

Since $h(\cdot)$ is an absolutely continuous function then $F(\cdot)$ is d^* -absolutely continuous set valued map. \square

Theorem 2. *Let $H: (0, T) \rightsquigarrow \mathbf{R}^n$ be a bounded set valued map with nonempty convex and compact values and let $t_0 \in (0, T), x_0 \in H(t_0)$. Let $h: (0, T) \mapsto \mathbf{R}^n$ be an absolutely continuous function such that*

$$\forall e \in S, \quad \forall t, \tau \in (0, T), \quad \sigma_{H(t)}(e) - \sigma_{H(\tau)}(e) \leq |h(t) - h(\tau)|.$$

Then there exists $\delta > 0$ and an absolutely continuous selection $r: [t_0, t_0 + \delta) \mapsto \mathbf{R}^n$ from $H(\cdot)$ such that

$$r(t_0) = x_0.$$

To prove theorem 2 we will use the following definition and lemma.

Definition 3. Let L, K be linear subspaces in \mathbf{R}^n . Let $\Pi_L(\cdot)$ denote the projection of the best approximation on the set L . We define

$$\alpha(L, K) := \sup \{1 - \|\Pi_L(x)\| \mid x \in K, \|x\| = 1\}.$$

Lemma 5. Let $H \subset B(0, R)$, ($R > 0$) be a convex compact set, L be a linear subspace of \mathbf{R}^n , $L \subset \text{aff}(H) - \text{aff}(H)$ ($\text{aff}(H)$ denotes the affine hull of the set H , see [4]) and there exists $x_0 \in \mathbf{R}^n$ and $\delta > 0$ such that

$$B(x_0, \delta) \cap (L + x_0) \subset H.$$

Let K be a linear subspace in \mathbf{R}^n such that $K + L = \mathbf{R}^n$ and $K \cap L = \{0\}$. Let $L_0 := N_K(0)$ ($N_K(0)$ denotes the normal cone to K at 0, see [1]) and $\alpha(L, L_0) < 1$. Then there exists a constant $r > 0$ such that

$$\sigma_{H \cap K}(e) = \inf \{ \sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e, \|e'\| + \|e''\| \leq r \}, \forall e \in S,$$

where r depends only on n, α, R, δ .

To prove lemma 5 we use the following two lemmas.

Lemma 6. Let L, L_0 be linear subspaces in \mathbf{R}^n and let $\dim(L_0) = \dim(L)$, $\alpha := \alpha(L, L_0) < 1$. Then the projection map $\Pi_L: L_0 \rightarrow L$ has an inverse $\Pi_L^{-1}: L \rightarrow L_0$ and

$$\|\Pi_L^{-1}\| := \sup \{ \|\Pi_L^{-1}(y)\| \mid y \in L, \|y\| = 1 \} \leq 1/(1 - \alpha).$$

Proof. Let Q be an subspace in \mathbf{R}^n orthogonal to L such that $Q + L = \mathbf{R}^n$. If $\dim(L_0 \cap Q) \geq 1$, then there exists $q \in Q \cap L_0$, $\|q\| = 1$. Since $\Pi_L(q) = 0$ it follows that $\alpha = 1$. This contradicts with the assumption $\alpha < 1$. We proved that $L_0 \cap Q = \{0\}$. For given $y \in L$ since $\dim L + \dim Q = \dim L_0 + \dim Q = n$ there exists exactly one $x \in L_0$ such that $\Pi_L(x) = y$. From the definition of α follows that $\|\Pi_L(x)\| \geq (1 - \alpha) \|x\|$ and therefore $\|\Pi_L^{-1}\| \leq 1/(1 - \alpha)$. \square

Lemma 7. Let $L_0 := \{x \in \mathbf{R}^n \mid x_1 = \dots = x_k = 0\}$, $K := \{x \in \mathbf{R}^n \mid x_{k+1} = \dots = x_n = 0\}$, L be a linear subspace in \mathbf{R}^n , $\dim(L) = \dim(L_0)$ and $\alpha := \alpha(L, L_0) < 1$, $c > 0$. Let

$$Z := \{x \in \mathbf{R}^n \mid \sum_{i=1}^k x_i^2 \leq c^2, \|\Pi_L(x)\| \leq c\}.$$

Then for every $y \in Z$ the following holds

$$\|y\| \leq \frac{3c(n + 2 - \alpha)}{(1 - \alpha)^2}.$$

Proof. Let $y \in Z$. Due to lemma 6 there exists only one $x^y \in L_0$ such that

$$\Pi_L(x^y) = \Pi_L(y) \quad \text{and} \quad \|x^y\| \leq \frac{c}{1 - \alpha}.$$

Let Q be an subspace in \mathbf{R}^n orthogonal to L such that $Q + L = \mathbf{R}^n$. Let a^1, \dots, a^k

be an orthonormal basis of the space Q such that

$$a^1 = \frac{x^y - y}{\|x^y - y\|}.$$

For every $x \in \mathbf{R}^n$

$$\Pi_L(x) = x - \sum_{s=1}^k \langle x, a^s \rangle a^s.$$

Let

$$\delta := \frac{\alpha - 1}{\alpha - n - 2}.$$

We prove

$$\sum_{i=1}^k (a_i^1)^2 \geq \delta^2.$$

Let us suppose that

$$(1) \quad \sum_{i=1}^k (a_i^1)^2 < \delta^2.$$

Let

$$b_i := a_i^1, \quad i = 1, \dots, k$$

$$b_i := 0, \quad i = k + 1, \dots, n$$

and

$$\hat{x} := a^1 - b \in L_0.$$

It follows

$$\|b\| < \delta, \quad \|\hat{x}\| > 1 - \delta,$$

$$\Pi_L(\hat{x}) = -b + \sum_{s=1}^k \langle b, a^s \rangle a^s.$$

Since

$$\|\Pi_L(\hat{x})\| \leq (n + 1) \delta$$

then

$$\alpha \geq 1 - \frac{\|\Pi_L(\hat{x})\|}{\|\hat{x}\|} > 1 - \frac{(n + 1) \delta}{1 - \delta}.$$

Since

$$1 - \frac{(n + 1) \delta}{1 - \delta} = \alpha$$

we get the contradiction with the assumption (1). It follows

$$\sum_{i=1}^k (a_i^1)^2 \geq \delta^2.$$

Since

$$y = x^y + t a^1$$

then

$$\sum_{i=1}^k (x_i^y + t a_i^1)^2 \leq c^2$$

and consequently

$$|t| \leq \frac{\sqrt{(\sum_{i=1}^k (x_i^y)^2) + c}}{\sqrt{(\sum_{i=1}^k (a_i^1)^2)}} \leq \frac{2c}{(1-\alpha)\delta}.$$

It follows

$$\|y\| \leq \|x^y\| + \frac{2c}{(1-\alpha)\delta} \leq \frac{3c}{(1-\alpha)\delta} = \frac{3c(n+2-\alpha)}{(1-\alpha)^2}. \quad \square$$

Proof of lemma 5. By translation and unitary transformation we can achieve $x_0 = 0$ and $K = \{x \in \mathbf{R}^n \mid x_{k+1} = \dots = x_n = 0\}$ where $n - k = \dim(L)$. Let

$$L_0 := \{x \in \mathbf{R}^n \mid x_1 = \dots = x_k = 0\}.$$

From the assumptions we get

$$\sigma_H(e) \geq \sigma_{L \cap H}(e) = \sigma_{L \cap H}(\Pi_L(e)) \geq \delta \|\Pi_L(e)\|.$$

Moreover (see [4] and $e \in S$)

$$\begin{aligned} R &\geq \sigma_{H \cap K}(e) = \inf \{ \sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e \} \geq \\ &\geq \inf \{ \delta \|\Pi_L(e')\| \mid e' + e'' = e, e'' \in L_0 \}. \end{aligned}$$

We get

$$\|\Pi_L(e')\| \leq R/\delta \text{ whenever } e - e' \in L_0.$$

Since

$$\sum_{i=1}^k e_i''^2 = 0$$

then

$$\sum_{i=1}^k e_i'^2 \leq 1.$$

From lemma 7 follows

$$\begin{aligned} \sigma_{H \cap K}(e) &= \inf \{ \sigma_H(e') + \sigma_K(e'') \mid e' + e'' = e, \|e'\| + \|e''\| \leq r \}, \\ &\forall e \in S, \end{aligned}$$

where

$$r := \frac{6R(n+2-\alpha)}{\delta(1-\alpha)^2} + 1. \quad \square$$

Proof of theorem 2. A) Let $\dim(\text{aff } H(t_0)) = 0$, i.e. $H(t_0) = \{x_0\}$. Then for barycentric selection holds

$$b(H(t_0) + B) = x_0.$$

Therefore we may define due to theorem 1 an absolutely continuous selection

$$r(t) := b(H(t) + B).$$

B) Let $x_0 \in \text{ri}(H(t_0))$, $\dim \text{aff } H(t_0) = n - k \geq 1$ and K be a linear subspace such that $K = N_{\text{aff}(H(t_0))}(x_0)$. By translation and unitary transformation we can achieve $x_0 = 0$ and

$$K := \{x \in \mathbf{R}^n \mid x_{k+1} = \dots = x_n = 0\}.$$

From Carathéodory theorem (see [1]) follows that there exist points $a_i \in H(t_0)$, $i = 1, \dots, n - k + 1$ such that

$$x_0 = \sum_{i=1}^{n-k+1} \lambda_i a_i, \quad \sum_{i=1}^{n-k+1} \lambda_i = 1, \quad \lambda_i > 0.$$

Let $\eta_1, \eta_2 > 0$ be such that

$$\begin{aligned} B(x_0, \eta_1) \cap \text{aff}(b_1, \dots, b_{n-k+1}) &\subset \\ &\subset \{x \in \mathbf{R}^n \mid x = \sum_{i=1}^{n-k+1} \mu_i b_i, \quad \sum_{i=1}^{n-k+1} \mu_i = 1, \mu_i \geq 0\} \end{aligned}$$

for every $\|b_i - a_i\| < \eta_2$, $i = 1, \dots, n - k + 1$.

Since $H(\cdot)$ is a continuous map then for $\eta_2 > 0$ there exists $\delta_1 > 0$ such that

$$H(t) \cap B(a_i, \eta_2/2) \neq \emptyset \quad \text{for} \quad |t - t_0| < \delta_1.$$

Let $b_i(t) \in H(t) \cap B(a_i, \eta_2/2)$ and

$$\begin{aligned} L_0 &:= \text{aff}\{a_1, \dots, a_{n-k+1}\} = \{x \in \mathbf{R}^n \mid x_1 = \dots = x_k = 0\}, \\ L(t) &:= \text{aff}\{b_1(t), \dots, b_{n-k+1}(t)\}. \end{aligned}$$

Let $R > 0$ be such that $H(t) \subset B(0, R)$ for $t \in [t_0, t_0 + \delta)$ and

$$G(t) := H(t) \cap K.$$

We find $\delta_2 > 0$ such that $\alpha(L(t), L_0) < 1/2$ for $t \in [t_0, t_0 + \delta_2)$, $\delta := \min(\delta_1, \delta_2)$. For $t \in [t_0, t_0 + \delta)$ are fulfilled the assumptions of lemma 5, where x_0 stands for $x(t) \in L(t) \cap K$, L stands for $L(t)$. Therefore there exists $r > 0$ such that

$$\sigma_{G(t)}(e) = \inf \{ \sigma_{H(t)}(e') \mid e' + e'' = e, e'' \in L_0, \|e'\| + \|e''\| \leq r \}.$$

We prove that

$$\sigma_{G(t)}(e) - \sigma_{G(\tau)}(e) \leq r|h(t) - h(\tau)|, \quad \forall t, \tau \in [t_0, t_0 + \delta), \quad \forall e \in S.$$

Since $\sigma_{K(t)}(\cdot)$ is lower semicontinuous function (see [4]) it follows that for every $\tau \in [t_0, t_0 + \delta)$ and every $e \in S$ there exists $\tilde{e} \in \mathbf{R}^n$, $\|\tilde{e}\| \leq r$ such that $e - \tilde{e} \in L_0$

$$\sigma_{H(\tau)}(\tilde{e}) = \sigma_{G(\tau)}(e) = \inf \{ \sigma_{H(\tau)}(e') \mid e' + e'' = e, e'' \in L_0, \|e'\| + \|e''\| \leq r \}.$$

It follows

$$\begin{aligned} \sigma_{G(t)}(e) - \sigma_{G(\tau)}(e) &\leq \sigma_{H(t)}(\tilde{e}/\|\tilde{e}\|) \|\tilde{e}\| - \sigma_{H(\tau)}(\tilde{e}/\|\tilde{e}\|) \|\tilde{e}\| \leq \\ &\leq r|h(t) - h(\tau)|. \end{aligned}$$

From lemma 4 and the first part of this proof it follows that there exists an absolutely continuous selection $r(\cdot): [t_0, t_0 + \delta) \mapsto \mathbf{R}^n$ from the set valued map $G(\cdot)$,

$$r(t_0) = x_0.$$

C) Let $x_0 \in \text{bd}(H(t_0))$. Let us suppose that $\dim \text{aff } H(t_0) \geq 1$. Take $y \in \text{ri}(H(t_0))$ and let $\lambda_n \in \mathbf{R}$, $n = 1, \dots$ be a decreasing sequence such that $1 \geq \lambda_1$, $\lambda_n \rightarrow 0$. Let

$$y_n := x_0(1 - \lambda_n) + y\lambda_n.$$

Since $y_n \in \text{ri}(H(t_0))$ there exists an absolutely continuous selection $x_n(\cdot)$ defined on the interval $t_0 \leq t < t_0 + \eta_n$ such that

$$x_n(t_0) = y_n.$$

There exists $\delta_n < \min(1/n, \eta_n)$, $n = 1, \dots$ such that $\delta_{n+1} \leq \delta_n$,

$$\begin{aligned} \text{var}_{[t_0, t_0 + \delta_n]} x_n &< 1/n^2, \\ \text{var}_{[t_0, t_0 + \delta_{n+1}]} x_{n+1} &< 1/n^2. \end{aligned}$$

Let

$$\begin{aligned} x(t, \lambda) &:= \frac{\lambda - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} x_n(t) + \frac{\lambda_n - \lambda}{\lambda_n - \lambda_{n+1}} x_{n+1}(t) \\ \text{for } \lambda_{n+1} \leq \lambda \leq \lambda_n, \quad t_0 < t \leq t_0 + \delta_{n+1} \\ x(t_0, 0) &:= x_0. \end{aligned}$$

Since $x_n(\cdot)$, $n = 1, \dots$ are absolutely continuous and $H(\cdot)$ has convex values there exists an increasing continuously differentiable function $\hat{\delta} \in C^1[0, 1]$, $\hat{\delta}(\lambda_k) \leq \delta_{k+1}$, $\hat{\delta}(\lambda) > 0$ for $1 \geq \lambda > 0$, $\hat{\delta}(0) = 0$ such that

$$x(t, \lambda) \in H(t) \quad \text{for } t_0 \leq t \leq t_0 + \hat{\delta}(\lambda).$$

Let the function $\hat{\lambda}(t)$ be the inverse function for $t_0 + \hat{\delta}(\lambda)$. We prove that

$$\hat{x}(t) := x(t, \hat{\lambda}(t))$$

is absolutely continuous on the interval $[t_0, t_0 + \hat{\delta}(1)]$.

We prove that for every $\varepsilon > 0$ there exists $k \in \mathbf{N}$ such that $\hat{x}(\cdot)$ has variation on the interval $[t_0, t_0 + \hat{\delta}(\lambda_k)]$ less than ε . Let $\varepsilon > 0$. We choose $k \in \mathbf{N}$ such that

$$\lambda_k \|x_0 - y\| + 4 \sum_{n \geq k} 1/n^2 < \varepsilon/2.$$

There exist points $t_i \in (t_0, t_0 + \hat{\delta}(\lambda_k)]$, $i = 1, \dots, M + 1$ such that $t_i < t_{i+1}$

$$\left| \text{var}_{[t_0, t_0 + \hat{\delta}(\lambda_k)]} \hat{x} - \sum_{i=1}^M \|\hat{x}(t_{i+1}) - \hat{x}(t_i)\| \right| < \varepsilon/2.$$

We add the points $t_0 + \hat{\delta}(\lambda_s)$ for $s \geq k$ if $t_1 < t_0 + \hat{\delta}(\lambda_s)$ to the points t_i . Let $\lambda_{n+1} \leq \hat{\lambda}(t_i) \leq \hat{\lambda}(t_{i+1}) \leq \lambda_n$. Then

$$\begin{aligned} \hat{x}(t_{i+1}) - \hat{x}(t_i) &= -(\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i))(x_0 - y) + \\ &+ \frac{\hat{\lambda}(t_i) - \lambda_{n+1}}{\lambda_n - \lambda_{n+1}} (x_n(t_{i+1}) - x_n(t_i)) + \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda_n - \hat{\lambda}(t_{i+1})}{\lambda_n - \lambda_{n+1}} (x_{n+1}(t_{i+1}) - x_{n+1}(t_i)) + \\
& + \frac{\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)}{\lambda_n - \lambda_{n+1}} (x_n(t_i) - y_n - x_{n+1}(t_i) + y_{n+1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \sum_{i=1}^M \|\hat{x}(t_{i+1}) - \hat{x}(t_i)\| \leq \sum_{i=1}^M |\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)| \|x_0 - y\| + \\
& + \sum_{n \geq k} \sum_{\{i \in \{1, \dots, M\} | \lambda_{n+1} \leq \hat{\lambda}(t_i) \leq \hat{\lambda}(t_{i+1}) \leq \lambda_n\}} \|x_n(t_{i+1}) - x_n(t_i)\| + \\
& + \sum_{n \geq k} \sum_{\{i \in \{1, \dots, M\} | \lambda_{n+1} \leq \hat{\lambda}(t_i) \leq \hat{\lambda}(t_{i+1}) \leq \lambda_n\}} \|x_{n+1}(t_{i+1}) - x_{n+1}(t_i)\| + \\
& + \sum_{n \geq k} \sum_{\{i \in \{1, \dots, M\} | \lambda_{n+1} \leq \hat{\lambda}(t_i) \leq \hat{\lambda}(t_{i+1}) \leq \lambda_n\}} \frac{\hat{\lambda}(t_{i+1}) - \hat{\lambda}(t_i)}{\lambda_n - \lambda_{n+1}} (\text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n + \\
& + \text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1}) \leq \lambda_k \|x_0 - y\| + \sum_{n \geq k} (\text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n + \\
& + \text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1} + \text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_n + \text{var}_{[t_0, t_0 + \delta(\lambda_n)]} x_{n+1}) \leq \\
& \leq \lambda_k \|x_0 - y\| + 4 \sum_{n \geq k} 1/n^2 < \varepsilon/2.
\end{aligned}$$

We proved that

$$\text{var}_{[t_0, t_0 + \delta(\lambda_k)]} \hat{x} < \varepsilon.$$

We prove that $\hat{x}(\cdot)$ is an absolutely continuous function on the interval $[t_0 + \delta(\lambda_k), t_0 + \delta(1)]$. Since $x_n(\cdot)$ is an absolutely continuous function on the interval $[0, \delta_n]$ then for $\forall \varepsilon > 0, \exists \eta > 0$ such that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (t_0 + \delta(\lambda_k) \leq t_1 \leq \tau_1 \leq \dots \leq t_m \leq \tau_m \leq \delta_n)$$

the following holds

$$\sum_{j=1}^m (\tau_j - t_j) < \eta \Rightarrow \sum_{j=1}^m \|x_n(t_j) - x_n(\tau_j)\| < \varepsilon/(4k), \quad n \leq k.$$

It follows that for every system of intervals

$$[t_1, \tau_1], \dots, [t_m, \tau_m], \quad (t_0 + \delta(\lambda_k) \leq t_1 \leq \tau_1 \leq \dots \leq t_m \leq \tau_m \leq \delta(1))$$

holds

$$\sum_{j=1}^m (\tau_j - t_j) < \eta \Rightarrow \sum_{n \leq k} \sum_{\{i \in \{1, \dots, m\} | \lambda_{n+1} \leq \hat{\lambda}(t_i) \leq \hat{\lambda}(\tau_i) \leq \lambda_n\}} \|x_n(t_i) - x_n(\tau_i)\| < \varepsilon/4.$$

We proved that $\hat{x}(\cdot)$ is an absolutely continuous on the interval $[t_0 + \delta(\lambda_k), t_0 + \delta(1)]$ and since $\text{var}_{[t_0, t_0 + \delta(\lambda_k)]} \rightarrow 0$ for $k \rightarrow \infty$, then it is absolutely continuous on the interval $[t_0, t_0 + \delta(1)]$. \square

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