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A FAMILY OF NONREGULAR DISTANCE MONOTONE GRAPHS

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0. INTRODUCTION AND BASIC NOTIONS

The  $(O, 2)$ -graphs and the distance monotone graphs (DM-graphs) have been introduced in [1] and [2], [4] and [5], respectively, in two characterizations of hypercubes. H. M. Mulder [2] and independently J. M. Laborde-Rao Hebbare [3] proved that  $(O, 2)$ -graphs are regular. In [5] and [6] the authors introduced the following conjecture.

**Conjecture.** Every DM-graph  $G$  with minimal degree  $d(G) \geq 3$  is regular.

We exhibit a family of counterexamples to this conjecture.

We first recall some definitions introduced in [2] and [5].

For any two vertices  $u, v$  in a simple graph  $G$  the interval  $I(u, v)$  is the set of vertices lying on a shortest  $u, v$ -path. Let  $d(u, v)$  be the distance between  $u$  and  $v$ .

A graph  $G$  is distance-monotone (DM-graph for short) if each interval  $I(u, v)$  verifies  $w \in V(G) - I(u, v) \Rightarrow \exists w' \in I(u, v)$  such that  $d(w, w') > d(u, v)$ .

DM-graphs of diameter 3 except  $P_4$  can be obtained from  $K_{n,n}$  for  $n \geq 3$  by deletion of a perfect matching [5]. In [6] a matrix representation of DM-graphs of diameter 4 with  $d(G) \geq 3$  is introduced.

Consider a  $(0, 1)$ -matrix  $M = (m_{i,j})$  fulfilling the following conditions

(1)  $M$  has at least 4 rows and 4 columns.

(2) For any 3 different row indices  $i, j, k$  there are 4 column indices  $a, b, c, d$  such that

$$m_{ia} = m_{ja} \neq m_{ka},$$

$$m_{ib} \neq m_{jb} = m_{kb},$$

$$m_{ic} = m_{kc} \neq m_{jc},$$

$$m_{id} = m_{jd} = m_{kd}.$$

(2\*) For any 3 different column indices  $a, b, c$  there are 4 row indices  $i, j, k, l$  such that

$$m_{ia} = m_{ib} \neq m_{ic},$$

$$m_{ja} \neq m_{jb} = m_{jc},$$

$$m_{ka} = m_{kc} \neq m_{kb},$$

$$m_{la} = m_{lb} = m_{lc}.$$

To every such a  $m$  by  $n$  matrix  $M$  we associate a graph  $G$  of order  $2(m + n)$  in the following way:

$$V(G) = \{u_1, u_2, \dots, u_m\} \cup \{u'_1, u'_2, \dots, u'_m\} \cup \{v_1, v_2, \dots, v_n\} \cup \{v'_1, v'_2, \dots, v'_n\}$$

and all the edges of  $G$  are obtained by:

$$m_{ij} = 1 \Rightarrow \{u_i, v'_j\} \in E(G) \quad \text{and} \quad \{u'_i, v_j\} \in E(G),$$

$$m_{ij} = 0 \Rightarrow \{u_i, v_j\} \in E(G) \quad \text{and} \quad \{u'_i, v'_j\} \in E(G).$$

It is easy to verify that  $G$  is a DM-graph of diameter 4, and furthermore all DM-graphs of diameter 4 with  $d(G) \geq 3$  can be obtained in this way [6].

We are going to construct an  $m \times n$  matrix ( $m \neq n$ ) fulfilling the conditions (1), (2) and (2\*). In the associated DM-graph vertices  $u_i, u'_i$  corresponding to row indices are of degree  $n$  and vertices  $v_i, v'_i$  corresponding to column indices are of degree  $m$ , therefore, the associated DM-graph is not regular.

Let  $e_1, e_2, \dots, e_p$  be the canonic basis of  $V(p, 2)$  the vector space of dimension  $p$  over  $GF(2)$ .

Let  $u_1, u_2, \dots, u_m$  and  $v_1, v_2, \dots, v_n$  be some mutually different vectors of  $V(p, 2)$ . We can associate to these vectors a  $n \times m$  matrix  $M = (m_{ij})$  in the following way:  $m_{ij} = v_i u_j = v_{i1} u_{j1} + v_{i2} u_{j2} + \dots + v_{ip} u_{jp}$  where  $v_i = (v_{i1}, v_{i2}, \dots, v_{ip})$  and  $u_j = (u_{j1}, u_{j2}, \dots, u_{jp})$  in the basis  $e_1, e_2, \dots, e_p$ .

Example. Let  $p = 3, m = 8$  (then the  $u_i$  are all the vectors of  $V(3, 2)$ ) and  $n = 7$  with  $v_1 = e_1, v_2 = e_2, v_3 = e_3, v_4 = e_1 + e_2, v_5 = e_1 + e_3, v_6 = e_2 + e_3, v_7 = e_1 + e_2 + e_3$ . We obtain the following  $7 \times 8$  matrix:

$$(I) \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

**Proposition 1.** *The matrix (I) verifies the properties (2) and (2\*). The associated DM-graph is therefore of order 30 with 14 vertices of degree 8 and 16 of degree 7.*

**Proposition 2.** *For  $p \geq 4$  let  $u_1, u_2, \dots, u_m$  and  $v_1, v_2, \dots, v_n$  be some mutually different vectors of  $V(p, 2)$  such that the  $p$  basis vectors  $e_1, e_2, \dots, e_p$  and the  $p(p - 1)/2$  sums of two basis vectors belong to the sets  $\{u_1, u_2, \dots, u_m\}$  and  $\{v_1, v_2, \dots, v_n\}$ .*

*Then the matrix associated to these vectors verifies the properties (2) and (2\*).*

The proof is common to the propositions:

Without loss of generality by permutations of rows and by permutation of columns we can assume that the first  $p$  rows and the first  $p$  columns are associated to the basis vectors.

We first prove the property (2\*). Choose 3 columns  $i, j, k$  and consider the values of the first  $p$  rows corresponding to the basis vectors. These are nothing else than that the vectors  $u_i, u_j, u_k$  expressed in the basis  $e_1, e_2, \dots, e_p$ .

*First case:* we have a row index  $\alpha, \alpha \leq p$ , with  $m_{\alpha i} = m_{\alpha j} = m_{\alpha k}$ . Then there is an other row  $\beta$  (still in the first  $p$  rows) such that for example  $m_{\beta i} = m_{\beta j} \neq m_{\beta k}$  (because  $u_i, u_j, u_k$  are distinct).

Therefore there is a third one  $\gamma$  with  $m_{\gamma i} \neq m_{\gamma j} = m_{\gamma k}$  or  $m_{\gamma i} \neq m_{\gamma j} \neq m_{\gamma k}$ ; in both cases the row associated to  $e_\beta + e_\gamma$  is the fourth required.

*Second case:* there is no row index  $\alpha, \alpha \leq p$  with  $m_{\alpha i} = m_{\alpha j} = m_{\alpha k}$ .

If  $p = 3$  (therefore we study the matrix (I)) we may have:

$$\begin{aligned} m_{1i} &= m_{1j} \neq m_{1k}, \\ m_{2i} &\neq m_{2j} = m_{2k}, \\ m_{3i} &= m_{3k} \neq m_{3j}. \end{aligned}$$

In this case the 7<sup>th</sup> row is associated to  $e_1 + e_2 + e_3$  and we have  $m_{7i} = m_{7j} = m_{7k}$ .

If we are not in this case, or if  $p > 3$ , we have two indices (in the first  $p$  ones), say  $\alpha$  and  $\beta$ , with

$$\begin{aligned} m_{\alpha i} &= m_{\alpha j} \neq m_{\alpha k}, \\ m_{\beta i} &= m_{\beta j} \neq m_{\beta k} \\ &(\text{possibly permuting } i, j, k). \end{aligned}$$

The vectors  $u_i$  and  $u_j$  are different, therefore there exists an index  $\gamma \leq p$  with

$$\begin{aligned} m_{\gamma i} &\neq m_{\gamma j} \neq m_{\gamma k} \\ \text{or} \\ m_{\gamma i} &\neq m_{\gamma j} = m_{\gamma k} \end{aligned}$$

and we have the property (2\*) with the row indices  $\alpha, \gamma, \delta$  and  $\varepsilon$ , with  $e_\delta = e_\alpha + e_\beta$  and  $e_\varepsilon = e_\beta + e_\gamma$ .

Thus the matrix verifies (2\*), and by transposition verifies (2).

Therefore we have the following property:

**Proposition 3.** For  $p \geq 4$  and every pair of integers  $n, m$  such that

$$\begin{aligned} p(p+1)/2 \leq n \leq 2^p, \\ p(p+1)/2 \leq m \leq 2^p \end{aligned}$$

there exists a DM-graph with some vertices of degree  $n$  and others of degree  $m$ .

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