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TRANSITIVITY OF PRINCIPAL TOLERANCES IS NOT A MAL'CEV PROPERTY

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Polynomial conditions for a variety of algebras to have transitive principal tolerances (alias to be principal tolerance trivial) were given in several papers, cf. [1], [3] and [4]. However, none of them are Mal'cev ones.

Theorem. *Transitivity of principal tolerances is not a Mal'cev property.*

Proof. Let $\mathscr{V}$ be the variety of all algebras $\langle A, \wedge, \vee, u \rangle$ of the type $(2, 2, 1)$ that satisfy the distributive lattice identities. Put $A = \{0, a, 1\}$, $0 \neq a \neq 1 \neq 0$, and define the operations $\wedge$ and $\vee$ as in the three-element distributive lattice with the least element 0 and the greatest element 1. Further, let $u = (0 \to 1, a \to a, 1 \to 0)$. In this way, we have obtained an algebra in $\mathscr{V}$. It is obvious that the principal tolerance $T(0, a) = \{0, a\}^2 \cup \{a, 1\}^2$ is not transitive. Hence $\mathscr{V}$ has not transitive principal tolerances even though it satisfies all the identities holding in the variety of all distributive lattices, which has transitive principal tolerances (see [2]). Q.E.D.

Example 1. The variety of all distributive lattices has transitive principal tolerances (cf. [2]).

Example 2. The variety of all monounary algebras $\langle A, f \rangle$ that satisfy $f(f(x)) = x$ has not transitive principal tolerances even though all its free algebras have (cf. [3]).

For the comparison's sake, we include a list of polynomial conditions for the transitivity of principal tolerances that are based on the author's result [3], Thm. 1.

Proposition. *Let $\mathscr{V}$ be a variety of algebras. The following conditions are equivalent:*

(E) *for any $n \in \mathbf{N}$, any $(n + 2)$-ary polynomials f_1, g, f_2 and any n-ary polynomials s, t, u, v such that*

$$f_1(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) = g(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x})$$
$$f_2(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) = g(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x})$$

are $\mathscr{V}$-identities there exist $(n + 2)$-ary polynomials g_1, f, g_2 such that

$$f_1(t(\boldsymbol{x}), s(\boldsymbol{x}), \boldsymbol{x}) = g_1(u(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x})$$
$$f(s(\boldsymbol{x}), t(\boldsymbol{x}), \boldsymbol{x}) = g_1(v(\boldsymbol{x}), u(\boldsymbol{x}), \boldsymbol{x})$$

$$f(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}) = g_2(u(\mathbf{x}), v(\mathbf{x}), \mathbf{x})$$
$$f_2(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}) = g_2(v(\mathbf{x}), u(\mathbf{x}), \mathbf{x})$$
are $\mathscr{V}$-identities;

(F) *for any $n \in \mathbf{N}$, any $(n+2)$-ary polynomials f_1, f_2 and any n-ary polynomials s, t there exist $(n+2)$-ary polynomials g_1, f, g_2 such that*

$$f_1(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}) = g_1(f_1(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}), f_2(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}), \mathbf{x})$$
$$f(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}) = g_1(f_2(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}), f_1(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}), \mathbf{x})$$
$$f(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}) = g_2(f_1(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}), f_2(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}), \mathbf{x})$$
$$f_2(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}) = g_2(f_2(t(\mathbf{x}), s(\mathbf{x}), \mathbf{x}), f_1(s(\mathbf{x}), t(\mathbf{x}), \mathbf{x}), \mathbf{x})$$
are $\mathscr{V}$-identities;

(G_4) *for any $n \in \mathbf{N}$ and any $(n+4)$-ary polynomials f_1, f_2 there exist $(n+4)$-ary polynomials g_1, f, g_2 such that*

$$f_1(z, y, \mathbf{w}, y, z) = g_1(f_1(y, z, \mathbf{w}, y, z), f_2(z, y, \mathbf{w}, y, z), \mathbf{w}, y, z)$$
$$f(y, z, \mathbf{w}, y, z) = g_1(f_2(z, y, \mathbf{w}, y, z), f_1(y, z, \mathbf{w}, y, z), \mathbf{w}, y, z)$$
$$f(z, y, \mathbf{w}, y, z) = g_2(f_1(y, z, \mathbf{w}, y, z), f_2(z, y, \mathbf{w}, y, z), \mathbf{w}, y, z)$$
$$f_2(y, z, \mathbf{w}, y, z) = g_2(f_2(z, y, \mathbf{w}, y, z), f_1(y, z, \mathbf{w}, y, z), \mathbf{w}, y, z)$$
are $\mathscr{V}$-identities;

(G_2) *for any $n \in \mathbf{N}$ and any $(n+2)$-ary polynomials f_1, f_2 there exist $(n+4)$-ary polynomials g_1, f, g_2 such that*

$$f_1(z, y, \mathbf{w}) = g_1(f_1(y, z, \mathbf{w}), f_2(z, y, \mathbf{w}), \mathbf{w}, y, z)$$
$$f(y, z, \mathbf{w}, y, z) = g_1(f_2(z, y, \mathbf{w}), f_1(y, z, \mathbf{w}), \mathbf{w}, y, z)$$
$$f(z, y, \mathbf{w}, y, z) = g_2(f_1(y, z, \mathbf{w}), f_2(z, y, \mathbf{w}), \mathbf{w}, y, z)$$
$$f_2(y, z, \mathbf{w}) = g_2(f_2(z, y, \mathbf{w}), f_1(y, z, \mathbf{w}), \mathbf{w}, y, z)$$
are $\mathscr{V}$-identities.

Sketch of proof. (E) $\Rightarrow$ (F): Set the first projection for g.

(F) $\Rightarrow$ (G_4): Set the sequence $\mathbf{w}, y, z$ for $\mathbf{x}$, the $(n+1)$-st projection for s and the $(n+2)$-nd projection for t.

(G_4) $\Rightarrow$ (G_2): The $(n+2)$-ary polynomials f_1, f_2 may be assumed to be $(n+4)$-ary.

(G_2) $\Rightarrow$ (E): Put $\mathbf{w} \equiv \mathbf{x}$, assume ($G_2$) yields g_1', f', g_2'. Set $s(\mathbf{x})$ for y and $t(\mathbf{x})$ for z. Take

$$g_1(p, q, \mathbf{x}) \equiv g_1'(g(p, q, \mathbf{x}), g(q, p, \mathbf{x}), \mathbf{x}, s(\mathbf{x}), t(\mathbf{x}))$$
$$f(p, q, \mathbf{x}) \equiv f'(p, q, \mathbf{x}, s(\mathbf{x}), t(\mathbf{x}))$$
$$g_2(p, q, \mathbf{x}) \equiv g_2'(g(q, p, \mathbf{x}), g(p, q, \mathbf{x}), \mathbf{x}, s(\mathbf{x}), t(\mathbf{x}))$$

and we are done.

Remark. Conditions (E), (F), and (G_2) were formulated in [3], [4], and [1] respectively, and proved to be equivalent to the transitivity of principal tolerances, condition (G_4) is new.

Boldface $\boldsymbol{x}$ stands for $x_1, \ldots, x_n$, boldface $\boldsymbol{w}$ for $w_1, \ldots, w_n$.

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