Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 4, 601–611

Persistent URL: http://dml.cz/dmlcz/102414

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

ON THE ALGEBRAIC STRUCTURE ON THE JET PROLONGATIONS OF FIBRED MANIFOLDS

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(Received January 27, 1989)

We describe some algebraic properties of the jet prolongations of fibred manifolds. In the first order it is well-known that the first jet prolongation J^1Y of an arbitrary fibred manifold $Y \to X$ is an affine bundle over Y, [3]. We develop two kinds of algebraic models for the higher order case. The first concept of a graded affine map reflects directly some algebraic aspects of the coordinate expressions of the jet prolongations of certain morphisms of fibred manifolds. The second concept of an affine tower is somewhat more sophisticated, but it characterizes properly the geometric aspects of our problem.

In the last two sections we present some applications of our algebraic models. In the higher order variational calculus our theory enables us to distinguish some special classes of Lagrangians. Furthermore, we deduce a geometrical characterization of an interesting graded affine structure of the Euler morphism of an arbitrary r-th order Lagrangian on any fibred manifold. In the theory of higher order connections, [6], [8], we also determine some special classes of r-connections, which are said to be of tower type. We give a direct description of such connections and for $r = 2$ we discuss even their curvatures from our point of view.

We remark that similar algebraic models for the spaces of higher order velocities are developed in [5]. Moreover, we remark that some algebraic models for the second order jet spaces were constructed by J. Pradines, [9]. The main difference between his approach and ours is that we intend to describe directly the algebraic properties of the classical (i.e. holonomic) jet prolongations of fibred manifolds.

1. Jet prolongations of fibred manifolds. By a fibred manifold we mean a surjective submersion $p: Y \to X$. The fibre over $x \in X$ is denoted by $E_x = p^{-1}(x)$. Given another fibred manifold $q: W \to Z$ and a fibred manifold morphism $f: Y \to W$ over a base map $f_0: X \to Z$, then $f_x: Y_x \to W_{f_0(x)}$ means the restriction of f to the fibres over x and $f_0(x)$. We denote by $\mathscr{FM}$ the category of fibred manifolds and all their

*) The second author has performed this work in the framework of the project "Geometria e fisica" partially supported by Italian Ministero della Publica Istruzione.

morphisms and by $\mathscr{FM}_{\rm I}$ the subcategory in which morphisms are restricted by the property that the underlying base map is a local diffeomorphism.

For any fibred manifold Y, J^rY denotes the space of r-jets of local sections of Y. We shall consider J^rY as a fibre bundle $J^rY \to Y$. For every morphism $f \in \mathscr{FM}_{\rm I}(Y, W)$ with an underlying base map f_0 we define $J^rf\colon JY \to J^rW$ by $J^rf(j^r_xs) = j^r_{f_0}(x)(f \circ s \circ f_0^{-1})$, where f_0^{-1} is taken locally. Hence J^r is a functor from $\mathscr{FM}_{\rm I}$ into $\mathscr{FM}$.

For $r = 1$, every $J^1Y \to Y$ is an affine bundle and every $J^1f\colon J^1Y \to J^1W$, $f \in \mathscr{FM}_{\rm I}(Y, W)$ is an affine bundle morphism, [3]. To discuss the algebraic structure of the functor J^r, $r \geqq 2$, we start from the coordinate expression of J^rf. Let x^α, y^i be some local fibre coordinates on $Y \to X$. The induced coordinates on J^rY are

$$y^i_\alpha = \frac{\partial s^i(x)}{\partial x^\alpha}, \ldots, y^i_{\alpha_1\ldots\alpha_r} = \frac{\partial^r s^i(x)}{\partial x^{\alpha_1}\ldots\partial x^{\alpha_r}},$$

which are symmetric in all subscripts. Let z^α, w^p be some local fibre coordinates on $W \to Z$, $\dim Z = \dim X$, and let $w^p_\alpha, \ldots, w^p_{\alpha_1\ldots\alpha_r}$ be the induced coordinates on J^rW Let $z^\alpha = f^\alpha(x^\beta)$, $w^p = f^p(x^\alpha, y^i)$ be the coordinate expression of a map $f \in \mathscr{FM}_I(Y, W)$ and let $x^\alpha = \tilde{f}^\alpha(z^\beta)$ be the locally inverse diffeomorphism of its base map f_0. Write $\bar{f}^p(z^\alpha, y^i) = f^p(\tilde{f}^\beta(z^\alpha), y^i)$. For $r = 2$ we find the map $J^2_yf\colon J^2_yY \to J^2_f(x)\,W$ by a direct evaluation in the form

$$w^p_\alpha = a^p_i y^i_\beta \tilde{a}^\beta_\alpha + b^p_\alpha \tag{1}$$

where $a^p_i = \partial f^p/\partial y^i$, $\tilde{a}^\beta_\alpha = \partial \tilde{f}^\beta/\partial z^\alpha$, $b^p_\alpha = \partial \bar{f}^p/\partial z^\alpha$, and

$$\begin{aligned} w^p_{\alpha_1\alpha_2} = {} & a^p_i y^i_{\beta_1\beta_2}\tilde{a}^{\beta_1}_{\alpha_1}\tilde{a}^{\beta_2}_{\alpha_2} + a^p_{i_1i_2}y^{i_1}_{\beta_1}y^{i_2}_{\beta_2}\tilde{a}^{\beta_1}_{\alpha_1}\tilde{a}^{\beta_2}_{\alpha_2} + \\ & b^p_{i\alpha_2}y^i_\beta\tilde{a}^\beta_{\alpha_1} + b^p_{i\alpha_1}y^i_\beta\tilde{a}^\beta_{\alpha_2} + a^p_i y^i_\beta \tilde{a}^\beta_{\alpha_1\alpha_2} + b^p_{\alpha_1\alpha_2} \end{aligned} \tag{2}$$

where $a^p_{i_1i_2} = \partial^2 f^p/\partial y^{i_1}\,\partial y^{i_2}$, $\tilde{a}^\beta_{\alpha_1\alpha_2} = \partial^2\tilde{f}^\beta/\partial z^{\alpha_1}\,\partial z^{\alpha_2}$, $b^p_{i\alpha} = \partial^2\bar{f}^p/\partial y^i\,\partial z^\alpha$, $b^p_{\alpha_1\alpha_2} = \partial^2\bar{f}^p/\partial z^{\alpha_1}\,\partial z^{\alpha_2}$.

The fact that (1) is an affine map corresponds to the well-known situation in the first order case. The typical feature of (2) is that it is a polynomial map of weighted degree 2, provided y^i_α are considered with weight 1 and $y^i_{\alpha_1\alpha_2}$ with weight 2. We deduce that an analogous property holds in arbitrary order r, where we shall not need the explicit formula for $w^p_{\alpha_1\ldots\alpha_r}$.

Proposition 1. *If we consider $y^i_{\alpha_1\ldots\alpha_k}$ with weight k, $k = 1, \ldots, r$, then $w^p_{\alpha_1\ldots\alpha_r}$ is a polynomial of weighted degree r.*

Proof. Assume by induction that $w^p_{\alpha_1\ldots\alpha_{r-1}}$ is a polynomial of weighted degree $r - 1$. By definition of J^rf, we obtain $w^p_{\alpha_1\ldots\alpha_r}$ from $w^p_{\alpha_1\ldots\alpha_{r-1}}$ in such a way that in every monomial we apply consecutively to each of its terms one of the following elementary changes, and leave the remaining terms unchanged. The elementary changes mean that we replace $y^i_{\alpha_{j_1}\ldots\alpha_{j_k}}$ by $y^i_{\alpha_{j_1}\ldots\alpha_{j_k}\beta}\tilde{a}^\beta_{\alpha_r}$, $a^p_{i_{j_1}\ldots i_{j_k}}$ by $a^p_{i_{j_1}\ldots i_{j_k}i}y^i_\beta\tilde{a}^\beta_{\alpha_r} + b^p_{i_{j_1}\ldots i_{j_k}\alpha_r}$, $\tilde{a}^\beta_{\alpha_{j_1}\ldots\alpha_{j_k}}$ by $\tilde{a}^\beta_{\alpha_{j_1}\ldots\alpha_{j_k}\alpha_r}$ and $b^p_{i_{j_1}\ldots i_{j_k}\alpha_{l_1}\ldots\alpha_{l_m}}$ by $b^p_{i_{j_1}\ldots i_{j_k}i\alpha_{l_1}\ldots\alpha_{l_m}}y^i_\beta\tilde{a}^\beta_{\alpha_r} + b^p_{i_{j_1}\ldots i_{j_k}\alpha_{l_1}\ldots\alpha_{l_m}\alpha_r}$. This yields our assertion, QED.

2. Graded affine maps. We consider real vector and affine spaces of finite dimension only. We denote by DA the derived vector space of an affine space A and by $Df: DA \to DB$ the derived linear map of an affine map $f: A \to B$. Every vector space V has a canonical structure of an affine space with $DV = V$.

Taking into account Proposition 1 we introduce the following general concepts. Given a multiindex $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$ of range r, we define its weighted length by

$$\|\Lambda\| = 1\Lambda_1 + 2\Lambda_2 + \ldots + r\Lambda_r .$$

Let $A_1, \ldots, A_r, B$ be affine spaces and let $f: A_1 \times \ldots \times A_r \to B$ be a smooth map. Denote by $D_i f(a): DA_i \to DB$ its partial differential with respect to D_i at $a \in A_1 \times \ldots \times A_r$ and by $D_\Lambda f(a): DA_1 \times \ldots \times DA_r \to DB$ its iterated partial differential with respect to the multiindex Λ.

Definition 1. A smooth map $f: A_1 \times \ldots \times A_r \to B$ is said to be of *weighted degree k*, if $D_\Lambda f(a) = 0$ for all $\|\Lambda\| > k$ and all $a \in A$.

One sees easily that this definition is equivalent to the following coordinate characterization of f.

Proposition 2. *A map $f: A_1 \times \ldots \times A_r \to B$ is of weighted degree k, if and only if in any affine coordinates on $A_1, \ldots, A_r, B$ it is represented by polynomials of weighted degree k, provided the variables from A_i are considered with weight i, $i = 1, \ldots, r$.*

Indeed, if $D_\Lambda f = 0$ identically for all $\|\Lambda\| > k$, then all partial derivatives of f of non-weighted degree greater than k vanish identically. Hence f is a polynomial map by the Taylor theorem and one deduces Proposition 2 by discussing each monomial separately.

Let $B_1, \ldots, B_s$ be other affine spaces.

Definition 2. A map $f: A_1 \times \ldots \times A_r \to B_1 \times \ldots \times B_s$, $f = (f_1, \ldots, f_s)$, $f_i: A_1 \times \ldots \times A_r \to B_i$ is called *graded affine*, if f_i is of weighted degree i for all $i = 1, \ldots, s$.

Proposition 3. *The i-th component of a graded affine map $f_i: A_1 \times \ldots \times A_r \to B_i$ factorizes through $A_1 \times \ldots \times A_i \to B_i$ for all $i \leqq r$.*

Proof. The variables of weights higher than i cannot appear in a polynomial of weighted degree i.

Let $g: B_1 \times \ldots \times B_s \to C_1 \times \ldots \times C_t$ be another graded affine map.

Proposition 4. *The composition $g \circ f: A_1 \times \ldots \times A_r \to C_1 \times \ldots \times C_t$ is also graded affine.*

Proof. This follows directly from Proposition 2.

Proposition 4 implies that all graded affine maps form a category $\mathscr{GA}$. An object $A_1 \times \ldots \times A_r$ of this category will be called an r-graded affine space. We underline that this category cannot be characterized by its objects and that the product structure $A_1 \times \ldots \times A_r$ on an r-graded affine space is not preserved under the isomorphisms in $\mathcal{GA}$. In other words, for a $\mathcal{GA}$-isomorphism $A_1 \times \ldots \times A_r \to \to A_1 \times \ldots \times A_r$ there is generally no underlying map $A_{i_1} \times \ldots \times A_{i_k} \to \to A_{i_1} \times \ldots \times A_{i_k}$, $(i_1, \ldots, i_k) \neq (1, \ldots, k)$. (We shall overcome this disadvantage by introducing the concept of an affine tower in the next section.) Nevertheless, the r-graded affine spaces determine a full subcategory $r\mathcal{GA}$ in $\mathcal{GA}$, whose morphisms are said to be *r-graded affine*.

3. Affine towers. We first introduce two auxiliary concepts.

Definition 3. An *admissible affine bundle* is an affine bundle $E \to X$ such that the derived vector spaces DE_x of all fibres E_x, $x \in X$, are equal.

In other words, the derived vector bundle DE of E is a product bundle $X \times |E|$, where $|E|$ denotes the common value of the derived vector spaces DE_x. Every smooth section $s: X \to E$ of an admissible affine bundle $p: E \to X$ determines a trivialization $E \approx X \times |E|$, $y \mapsto (p(y), \ y - s(p(y)))$. Such a trivialization of E will be called admissible. Given an admissible trivialization, we can reconstruct the defining section as the section corresponding to the zero element of $|E|$. In the sequel we shall consider the admissible trivializations of E only.

Let $F \to Z$ be another admissible affine bundle.

Definition 4. An affine bundle morphism $f: E \to F$ over a smooth map $f_0: X \to Z$ is said to be *admissible*, if the derived linear maps $Df_x: |E| \to |F|$ coincide for all $x \in X$.

In other words, the derived vector bundle morphism $Df: DE \to DF$ is a product $f_0 \times |f|$, where $|f|: |E| \to |F|$ denotes the common value of Df_x. As a direct consequence of Definition 4, we deduce that in any admissible trivializations of both bundles f has the form

(3) $\qquad (f_0, |f| + \varphi): X \times |E| \to Z \times |F|$

where φ is an arbitrary smooth map $X \to |F|$.

We shall define the category $r\mathcal{AT}$ of affine r-towers by induction. For $r = 1$ we identify $1\mathcal{AT}$ with the category of affine spaces and affine maps. Assume by induction we have defined the category $\mathcal{AT}(r-1)$ of affine at most $(r-1)$-towers with the following properties I–III.

I. Every affine k-tower is an admissible affine bundle over an affine $(k-1)$-tower and its projection is an $\mathcal{AT}(r-1)$-morphism, $2 \leqq k \leqq r-1$.

Hence every affine k-tower $E^k \to E^{k-1}$ yields a sequence of the underlying affine i-towers $E^i \to E^{i-1}$, $i = 2, \ldots, k$.

II. For every affine k-tower $E^k \to E^{k-1}$ there exists a section $\sigma: E^{k-1} \to E^k$ which is an $\mathcal{AT}(r-1)$-morphism, $2 \leqq k \leqq r-1$.

Such sections will be called the distinguished section of the affine k-tower $E^k \to$

$\to E^{k-1}$. Every distinguished section $\sigma_k: E^{k-1} \to E^k$ determines a trivialization $E^k = E^{k-1} \times |E^k|$, which will also be said to be distinguished. Hence every sequence $\sigma_2: E^1 \to E^2, \dots, \sigma_k: E^{k-1} \to E^k$ of distinguished sections of the underlying affine towers determines a decomposition

$$E^k = |E_1| \times \dots \times |E_k| \tag{4}$$

called a *distinguished total trivialization of* E^k, provided we write $E_i = |E^i|$. Let F^l be another affine l-tower, $l \leqq r - 1$.

III. A map $f: E^k \to F^l$ is an $\mathscr{AT}(r-1)$-morphism if and only if $f: E_1 \times \dots \times E_k \to F_1 \times \dots \times F_l$ is a $\mathscr{GA}$-morphism in every distinguished total trivialization.

By III, the change of two distinguished total trivializations of the same affine k-tower, $k \leqq r - 1$, is represented by a $\mathscr{GA}$-isomorphism.

Definition 5. Let E^k be an affine k-tower, $k \leqq r - 1$, and let A be an affine space. A map $f: E^k \to A$ is said to be of *weighted degree* s, if $f: E_1 \times \dots \times E_k \to A$ is a map of weighted degree s in a distinguished total trivialization of E^k.

Using III and Proposition 2 we deduce that this concept does not depend on the choice of a distinguished total trivialization. We denote by $W^s(E^k, A)$ the set of all maps $E^k \to A$ of weighted degree s. In particular, if we have a vector space V, the $W^s(E^k, V)$ is a vector space as well.

Definition 6. An *affine r-tower* is a pair (E^r, S), where $E^r \to E^{r-1}$ is an admissible affine bundle over an affine $(r-1)$-tower and S is a set of sections $\sigma: E^{r-1} \to E^r$ such that $\sigma - \bar{\sigma} \in W^r(E^{r-1}, |E^r|)$ for every $\sigma, \bar{\sigma} \in S$ and $\sigma + \delta \in S$ for every $\sigma \in S$ and every $\delta \in W^r(E^{r-1}, |E^r|)$.

The sections of S will be called the *distinguished sections* and the corresponding trivializations $E^r \approx E^{r-1} \times |E^r|$ will also be said to be *distinguished*. Clearly, the set S is determined by a single distinguished section $\sigma: E^{r-1} \to E^r$ and the distinguished sections constitute an affine space. In every distinguished trivialization, the distinguished sections coincide with the maps of $W^r(E^{r-1}, |E^r|)$.

Let (F^r, R) be another affine r-tower over an affine $(r-1)$-tower F^{r-1}.

Definition 7. A *morphism of affine r-towers* $(E^r, S) \to (F^r, R)$ is an admissible affine bundle morphism $f: E^r \to F^r$ over a morphism of affine $(r-1)$-towers $f_0: E^{r-1} \to F^{r-1}$ such that there are distinguished trivializations of E^r and F^r with the property that $f: E^{r-1} \times |E^r| \to F^{r-1} \times |F^r|$ is of the form

$$f = (f_0, |f| + \varphi) \quad \text{with} \quad \varphi \in W^r(E^{r-1}, |F^r|). \tag{5}$$

Let $G^r \to G^{r-1}$ be a third affine r-tower with a distinguished trivialization $G^{r-1} \times |G^r|$. In general, if we have two maps of the form (5) $(f_0, |f| + \varphi): E^{r-1} \times |E^r| \to F^{r-1} \times |F^r|$ and $(g_0, |g| + \psi): F^{r-1} \times |F^r| \to G^{r-1} \times |G^r|$ with $\varphi \in W^r(E^{r-1}, |F^r|)$ and $W^r(F^{r-1}, |G^r|)$, then we evaluate directly that their composition has the form

$$(6) \qquad (g_0 \circ f_0, |g| \circ |f| + |g| \circ \varphi + \psi \circ f_0).$$

One verifies easily that $|g| \circ \varphi + \psi \circ f_0 \in W^r(E^{r-1}, |G^r|)$.

Proposition 5. *Definition 7 does not depend on the choice of the distinguished sections of S and R.*

Proof. Assume we have used $\sigma \in S$ and $\varrho \in R$ in (5) and let $\bar{\sigma} \in S$ and $\bar{\varrho} \in R$. Then the maps $(\mathrm{id}_{E^{r-1}}, \mathrm{id}_{|E^r|} + \bar{\sigma} - \sigma)$ and $(\mathrm{id}_{F^{r-1}}, \mathrm{id}_{|F^r|} + \varrho - \bar{\varrho})$ represent the changes of trivializations. By (6), in the new trivializations f is of the form

$$(7) \qquad (f_0, |f| + |f| \circ (\bar{\sigma} - \sigma) + \varphi + (\varrho - \bar{\varrho}) \circ f_0).$$

This is an expression of the same type, QED.

Formula (6) now implies that affine r-towers and their morphisms form a category $r\mathscr{A}\mathscr{T}$.

Write $|E^r| = E_r$. If we combine a distinguished trivialization $E^r = E^{r-1} \times E_r$ of an affine r-tower with a total distinguished trivialization $E^{r-1} = E_1 \times \ldots \times E_{r-1}$ of the underlying affine $(r-1)$-tower, we obtain a decomposition

$$E^r = E_1 \times \ldots \times E_r$$

called a distinguished total trivialization of E^r. By the induction hypothesis III and by the definition of $W^r(E^{r-1}, F_r)$, we obtain easily

Proposition 6. *An admissible affine bundle morphism $f: E^r \to F^r$ over an $\mathscr{A}\mathscr{T}(r-1)$-morphism is a morphism of affine r-towers if and only if f is represented by a $\mathscr{G}\mathscr{A}$-morphism $E_1 \times \ldots \times E_r \to F_1 \times \ldots \times F_r$ in every distinguished total trivialization of E^r and F^r.*

Now we can define, in a unified way, the concept of morphism between an affine k-tower E^k and an affine l-tower F^l, $k, l \leqq r$.

Definition 8. An admissible affine bundle morphism $f: E^k \to F^l$ is said to be a *morphism of affine towers*, if f is represented by a $\mathscr{G}\mathscr{A}$-morphism $E_1 \times \ldots \times E_1 \to \to F_1 \times \ldots \times F_l$ in some distinguished total trivializations of E^k and F^l, $k, l \leqq r$.

Proposition 6 and the fact that $\mathscr{G}\mathscr{A}$ is a category imply that Definition 8 does not depend on the choice of the distinguished total trivializations. Hence we obtain a category $\mathscr{A}\mathscr{T}r$ of affine at most r-towers.

Obviously, the bundle projection of an affine at most r-tower is an $\mathscr{A}\mathscr{T}r$-morphism. Further, in the category $\mathscr{A}\mathscr{T}r$ we can characterize the distinguished sections used in the definition of an affine r-tower.

Proposition 7. *A section $\sigma: E^{r-1} \to E^r$ of an affine r-tower $E^r \to E^{r-1}$ is distinguished if and only if it is an $\mathscr{A}\mathscr{T}r$-morphism.*

Proof. In a distinguished trivialization $E^r = E^{r-1} \times E_r$, σ has the form $(\mathrm{id}_{E^{r-1}}, s)$, $s: E^{r-1} \to E_r$. This is a $\mathscr{G}\mathscr{A}$-morphism in any distinguished total trivialization of E^{r-1} if an only if $s \in W^r(E^{r-1}, E_r)$, QED.

Thus, we have justified that the induction properties I–III are fulfilled in the category $\mathscr{A}\mathscr{T}r$ and our induction procedure is correct.

By construction, the full subcategory of affine at most $(r-1)$-towers in $\mathscr{A}\mathscr{T}r$ coincides with the original category $\mathscr{A}\mathscr{T}(r-1)$. Hence we can take the union of all categories $\mathscr{A}\mathscr{T}r$, which defines the category $\mathscr{A}\mathscr{T}$ of all affine towers.

Having finished with the analytic aspects, we state three simple geometric properties of affine towers.

Proposition 8. *A morphism of affine r-towers $f\colon E^r \to F^r$ over $f_0\colon E^{r-1} \to F^{r-1}$ is an isomorphism if and only if both f_0 and $|f|\colon E_r \to F_r$ are isomorphisms.*

Proof. This follows easily from (6).

Proposition 9. *For every two distinguished sections σ and $\bar{\sigma}$ of an affine r-tower $E^r \to E^{r-1}$ there exists exactly one isomorphism $f\colon E^r \to E^r$ over* $\mathrm{id}_{E^{r-1}}$ and id_{E_r} *satisfying $f_0 \circ \sigma = \bar{\sigma}$.*

Proof. This follows easily from Proposition 6.

Proposition 10. *If $f\colon E^r \to F^r$ is an admissible affine bundle morphism over a morphism of affine $(r-1)$-towers $f_0\colon E^{r-1} \to F^{r-1}$ and there exist $\sigma \in S$ and $\varrho \in R$ such that*

$$f \circ \sigma = \varrho \circ f_0 \tag{8}$$

then f is a morphism of affine towers $(E^r, S) \to (F^r, R)$.

Proof. By (3), the trivialized form of an admissible affine bundle morphism is $(f_0, |f| + \varphi)$ with a smooth map φ. In the distinguished trivializations, (8) means

$$|f| \circ \sigma + \varphi = \varrho \circ f_0 \tag{9}$$

with $\sigma \in W^r(E^{r-1}, E_r)$ and $\varrho \in W^r(F^{r-1}, F_r)$. This implies $\varphi \in W^r(E^{r-1}, F_r)$, QED.

Remark 1. Consider the inverse problem: given $\varphi \in W^r(E^{r-1}, F_r)$, a morphism $f_0\colon E^{r-1} \to F^{r-1}$ of affine $(r-1)$-towers, and a linear map $|f|\colon E_r \to F_r$, do there exist $\sigma \in W^r(E^{r-1}, E_r)$ and $\varrho \in W^r(F^{r-1}, F_r)$ such that (9) holds? This is true if both f_0 and $|f|$ are isomorphisms (and in some other interesting cases), but a direct discussion of the case $r = 2$ shows that this is not true in general. Hence we cannot use condition (8) for an equivalent definition of the morphisms of affine r-towers.

4. Affine tower bundles. We first recall the basic facts from the theory of structured bundles by A. Cabras, D. Canarutto and the second author, [1]. Given two categories $\mathscr{C}$ and $\mathscr{D}$, a functor $\Phi\colon \mathscr{C} \to \mathscr{D}$ is said to be faithful, if for every $A, B \in \mathrm{Ob}\,\mathscr{C}$ the restricted map $\Phi\colon \mathscr{C}(A, B) \to \mathscr{D}(\Phi A, \Phi B)$ is injective. Let $\mathscr{M}$ denote the category of all smooth manifolds and all smooth maps.

Definition 9. A category $\mathscr{S}$ over manifolds is a *faithful functor* $\mu\colon \mathscr{S} \to \mathscr{M}$.

An $\mathscr{S}$-structure on a manifold $\mathscr{M}$ means an element $A \in \mathrm{Ob}\,\mathscr{S}$ such that $\mu A = M$. By the injective property of μ, for every smooth map $f\colon \mu A \to \mu B$ there exists either one $\mathscr{S}$-morphism $g \in \mathscr{S}(A, B)$ satisfying $\mu g = f$ or none. In the first case it is usual to say that f is an $\mathscr{S}$-morphism of A into B.

Let $p: E \to X$ be a fibred manifold.

Definition 10. An *$\mathscr{S}$-bundle* is a pair (E, σ), where $\sigma: X \to \mathrm{Ob}\, \mathscr{S}$ is a map satisfying $\mu\, \sigma(x) = E_x$ and the following condition of local triviality: for every $x \in X$ there exists a neighbourhood U, a trivialization $\varphi: p^{-1}(U) \approx U \times F_\varphi$ and an $\mathscr{S}$-structure on F_φ such that every map $\varphi_x: E_x \to F_\varphi$ is an $\mathscr{S}$-isomorphism.

Let $q: D \to Z$ be another fibred manifold and (D, ϱ) another $\mathscr{S}$-bundle.

Definition 11. A fibred manifold morphism $f: E \to D$ over $f_0: X \to Z$ is said to be a *morphism of $\mathscr{S}$-bundles*, if every $f_x: E_x \to D_{f_0(x)}$ is an $\mathscr{S}$-morphism from $\sigma(x)$ into $\varrho(f_0(x))$.

We denote by $\mathscr{S}\mathscr{B}$ the category of all $\mathscr{S}$-bundles and their morphisms.

In particular, since $\mathscr{A}\mathscr{T}$ is a category over manifolds in a canonical way, we obtain in this way the category $\mathscr{A}\mathscr{T}\mathscr{B}$ of affine tower bundles and its full subcategory $r\mathscr{A}\mathscr{T}\mathscr{B}$ for every r.

Consider the functor J^r from Section 1. We shall show that there is a canonical structure of $r\mathscr{A}\mathscr{T}$-bundle on every J^rY. By Definition 10 we have to determine a structure of an affine r-tower on every fibre J^r_yY, $y \in Y$. Take a local fibre coordinates x^α, y^i on $Y \to X$. The induced coordinates $y^i_\alpha, \ldots, y^i_{\alpha_1 \ldots \alpha_r}$ define a decomposition

$$(10) \qquad J^r_yY = L(\mathbf{R}^m, \mathbf{R}^n) \times L^2(\mathbf{R}^m, \mathbf{R}^n) \times \ldots \times L^r(\mathbf{R}^m, \mathbf{R}^n)$$

where $m = \dim X$, $m + n = \dim Y$ and $L^i(\mathbf{R}^m, \mathbf{R}^n)$ means the set of all polynomial homogeneous maps of degree i from $\mathbf{R}^m$ into $\mathbf{R}^n$. This trivialization defines a structure of an affine r-tower on J^r_yY. Obviously, the condition of local triviality from Definition 10 is fulfilled. Of course, we have to prove that the structure of an affine r-tower on J^r_yY does not depend on the choice of the coordinate system. But this is a special consequence of a more general result we deduce below.

Consider another fibred manifold $W \to Z$, $\dim Z = m$, $\dim W = m + k$, and $\mathscr{F}\mathscr{M}_1$-morphism $f: Y \to W$ and some local fibre coordinates z^α, w^p on a neighbourhood of $f(x)$. This defines a decomposition

$$(11) \qquad J^r_{f(x)}W = L(\mathbf{R}^m, \mathbf{R}^k) \times L^2(\mathbf{R}^m, \mathbf{R}^k) \times \ldots \times L^r(\mathbf{R}^m, \mathbf{R}^k)\,.$$

By Proposition 1 $J^r_yf: J^r_yY \to J^r_{f(x)}W$ is a graded affine map. In particular, if f is the change of a coordinate system on Y, then both decompositions (10) and (11) are related by an $r\mathscr{G}\mathscr{A}$-morphism, so that they determine the same structure of an affine r-tower. Hence we have proved

Proposition 11. *J^r is a functor $\mathscr{F}\mathscr{M}_1 \to r\mathscr{A}\mathscr{T}\mathscr{B}$.*

Since every J^rY is an affine r-tower bundle, we have specified some geometrically interesting morphisms between the jet prologations of various orders of fibred manifolds. In the next two sections we present some examples.

5. Applications to the higher order variational calculus. An r-th order Lagrangian on a fibred manifold $Y \to X$ is a base-preserving morphism of fibred manifolds

$$\lambda: J^r Y \to \Lambda^m T^* X\,, \quad m = \dim X\,,$$

see e.g. [4], [10]. Since $\Lambda^m T^* X$ is a vector bundle, it has a canonical structure of an affine bundle, i.e. of an affine 1-tower. Hence the affine tower morphisms $\lambda: J^r Y \to \Lambda^m T^* X$ determine a special geometric class of r-th order Lagrangians on Y. Thus, our theory presents a contribution to the problem of the classification of all Lagrangians up to the isomorphisms of fibred manifolds, see R. B. Gardner and W. F Shadwick, [2]. For $r = 1$ we obtain the affine Lagrangians of the form

$$(12) \qquad (a_i^\alpha(x^\beta, y^j)\, y_\alpha^i + b(x^\beta, y^i))\, \omega\,, \quad \omega = dx^1 \wedge \ldots \wedge dx^m\,.$$

In the first order, one can further characterize e.g. the quadratic affine Lagrangians of the form

$$(13) \qquad (a_{ij}^{\alpha\beta} y_\alpha^i y_\beta^j + b_i^\alpha y_\alpha^i + c)\, \omega$$

where the coefficients are some smooth functions on Y. Especially these Lagrangians seem to be interesting for various applications. In the second order, the affine tower morphisms specify the following class of Lagrangians:

$$(14) \qquad (a_i^{\alpha\beta} y_{\alpha\beta}^i + b_{ij}^{\alpha\beta} y_\alpha^i y_\beta^j + c_i^\alpha y_\alpha^i + d)\, \omega$$

where the coefficients are smooth functions on Y. But even here one can define the bimorphisms or multimorphisms of the 2-graded affine type.

Further, it is very interesting that our ideas can characterize an important property of the Euler morphism of an arbitrary Lagrangian. The Euler morphism of a Lagrangian $\lambda: J^r Y \to \Lambda^m T^* X$ is a base-preserving morphism

$$E(\lambda): J^{2r} Y \to V^* Y \otimes \Lambda^m T^* X$$

where VY denotes the vertical tangent bundle of Y, see e.g. [4]. The equation $E(\lambda) = 0$ characterizes the critical sections of λ. The coordinate expression $E_i\, dy^i \otimes \omega$ of $E(\lambda)$ has the form

$$(15) \qquad E_i = \sum_{|A| \leqq r} (-1)^{|A|}\, \mathbf{D}_A(\partial_i^A L)$$

where $L\omega$ is the coordinate expression of λ, $|A|$ is the usual length of a multiindex A of range m, ∂_i^A means the partial derivative with respect to y_A^i and $\mathbf{D}_A$ denotes the iterated formal (or total) derivative. For $|A| = 1$ we have

$$(16) \qquad \mathbf{D}_\alpha L = \partial_\alpha L + \sum_{|A| \leqq r} (\partial_i^A L)\, y_{A+\alpha}^i \quad \alpha = 1, \ldots, m\,.$$

The canonical inclusion $J^{2r} Y \subset J^r(J^r Y)$ defines a structure of an affine r-tower bundle on $J^{2r} Y \to J^r Y$. Applying our ideas to (15) and (16), we deduce

Proposition 12. *For every r-th order Lagrangian $\lambda: J^r Y \to \Lambda^m T^* X$, its Euler morphism $E(\lambda): J^{2r} Y \to V^* Y \otimes \Lambda^m T^* X$ is a morphism of affine towers over $J^r Y$.*

In other words, the coordinate expression of $E(\lambda)$ is of weighted degree r in the derivatives of order $r + 1, \ldots, 2r$, provided the derivatives of order $r + k$ are considered with weight k.

There is a more geometrical approach to the proof of Proposition 12. In general, for every base-preserving morphism

$$\psi: J^l Y \to \bigwedge^k T^* X$$

one defines its formal exterior differential $\mathbf{D}\psi: J^{l+1} Y \to \bigwedge^{k+1} T^* X$ by setting, for every local section s of Y,

$$\mathbf{D}\psi(j^{l+1}s) := \mathrm{d}(\psi(j^l s)) \tag{17}$$

where $\psi(j^l s)$ is an exterior k-form on X and d means the exterior differential, [4], [10].

Proposition 13. *If $\psi: J^l Y \to \bigwedge^k T^* X$ is a morphism of affine towers over Y, then $\mathbf{D}\psi: J^{l+1} Y \to \bigwedge^{k+1} T^* X$ is also a morphism of affine towers over Y.*

Proof consists in a straightforward analysis of the defining formula (17).

If we now use a geometrical approach to the definition of the Euler morphism explained in [4] and apply Proposition 13, we obtain a more conceptual proof of Proposition 12.

6. Applications to higher order connections. Generalizing some investigations by P. Libermann related with the vector bundle case, [6], the second author with L. Mangiarotti defined an r-th order connection (in short: r-connection) on an arbitrary fibred manifold $p: Y \to X$ as a smooth section

$$\Gamma: J^{r-1} Y \to J^r Y, \tag{18}$$

[8]. For $r = 1$ one obtains the general first order connections studied by several authors. Our theory characterizes directly a special class of r-connections.

Definition 12. If (18) is a morphism of affine tower bundles over Y, then Γ is called an *r-connection of tower type*, $r \geqq 2$.

Since the bundles in question are over Y, we obtain no special type in the first order case $Y \to J^1 Y$. On the other hand, the coordinate expression of a second order connection of tower type is

$$y^i_{\alpha\beta} = A^i_{jk} y^j_\alpha y^k_\beta + B^i_{j\alpha} y^j_\beta + B^i_{j\beta} y^j_\alpha + C^i_{\alpha\beta} \tag{19}$$

where the coefficients are some smooth functions on Y.

The r-connections of tower type are closely related with our theory of affine r-towers. The restriction of such an r-connection on Y to $J^{r-1}_y Y$ is just a distinguished section of the affine r-tower $J^r_y Y$. Let us denote by $CT^r_y Y$ the set of all such sections. Definition 6 implies that $CT^r_y Y$ is an affine space, whose derived vector space is the space of all affine tower morphisms of $J^{r-1}_y Y$ into $V_y Y \otimes S^r T^*_x X$, $x = p(y)$, where S^r denotes the r-th symmetric tensor power. Consider $CT^r Y := \bigcup_{y \in Y} CT^r_y Y$ with the canonical structure of an affine bundle over Y. Then we can summarize by

Proposition 14. *The r-connections of tower type on Y coincide with the sections of the affine bundle* $CT^rY \to Y$.

We conclude the paper with a concrete example illustrating that r-connections of tower type have several specific properties. Consider first an arbitrary 2-connection $\Gamma: J^1Y \to J^2Y$ on Y. The coordinate expression of Γ is $y^i_{\alpha\beta} = \Gamma^i_{\alpha\beta}(x^\gamma, y^j, y^k_\delta)$ with $\Gamma^i_{\alpha\beta}$ symmetric in both subscripts. The corresponding horizontal distribution on J^1Y is

$$\text{(20)} \qquad \mathrm{d}y^i = y^i_\alpha \, \mathrm{d}x^\alpha \,, \quad \mathrm{d}y^i_\alpha = \Gamma^i_{\alpha\beta} \, \mathrm{d}x^\beta \,.$$

Taking into account the well-known exact sequence

$$0 \to VY \otimes_{J^1Y} T^*X \to VJ^1Y \to VY \to 0$$

we find that the values of the curvature of Γ lie in the tensor product of the pullback of $VY \otimes T^*X$ over J^1Y with $\bigwedge^2 T^*X$. However, if Γ is a 2-connection of tower type, its curvature factorizes through a map

$$\text{(21)} \qquad J^1Y \to VY \otimes T^*X \otimes \textstyle\bigwedge^2 T^*X \,.$$

Since we have affine bundles over Y on both sides of the arrow, Definition 1 gives a well-defined concept of a morphism of degree k.

Proposition 15. *The curvature* (21) *of a 2-connection of tower type is a bundle morphism of the third degree.*

Proof. This can be derived from (19) and (20) by a standard evaluation, QED.

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