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HAKE'S PROPERTY OF A MULTIDIMENSIONAL GENERALIZED RIEMANN INTEGRAL

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Throughout, $m \geq 1$ is a fixed integer. The set of all real numbers is denoted by \mathbf{R} , and the m -fold cartesian product of \mathbf{R} is denoted by \mathbf{R}^m . For $x = (\xi_1, \dots, \xi_m)$ in \mathbf{R}^m we let $|x| = \max \{|\xi_1|, \dots, |\xi_m|\}$, and in \mathbf{R}^m we use exclusively the metric induced by the norm $|x|$. The distance from a point $x \in \mathbf{R}^m$ to a set $E \subset \mathbf{R}^m$ is denoted by $\text{dist}(x, E)$. If $E \subset \mathbf{R}^m$, then E^0 , ∂E , $d(E)$, and $|E|_k$, $k = 0, \dots, m$, denote, respectively, the interior, boundary, diameter, and the k -dimensional Hausdorff measure of E (as is customary, a "measure" means an "outer measure"). We define the Hausdorff measure as in [1; Section 2.10.2, p. 171], so that $|E|_k$ is the counting measure of E if $k = 0$, and the k -dimensional Lebesgue measure of E if $k \geq 1$ and $E \subset \mathbf{R}^k$. Instead of $|E|_m$ we write $|E|$.

A k -plane, $k = 0, \dots, m-1$, is a k -dimensional linear submanifold H of \mathbf{R}^m which is parallel to k distinct coordinate axes. When the dimension k of H is not specified, we talk about a *plane* H , whose dimension is denoted by $\dim H$. A *regulator* is a pair $(\varepsilon, \mathcal{H})$ where $0 < \varepsilon < 1/2$ and \mathcal{H} is a finite family of planes.

An *interval* is a set $A = \prod_{i=1}^m [a_i, b_i]$ where $a_i, b_i \in \mathbf{R}$ and $a_i < b_i$, $i = 1, \dots, m$. Intervals are *nonoverlapping* if their interiors are disjoint. Given a regulator $(\varepsilon, \mathcal{H})$, we call an interval A an $(\varepsilon, \mathcal{H})$ -interval whenever

$$\varepsilon < \sup \left\{ \frac{|A \cap H|_{\dim H}}{[d(A)]_{\dim H}} : H \in \mathcal{H} \cup \{\mathbf{R}^m\}, A \cap H \neq \emptyset \right\}.$$

A *figure* is a finite union of intervals. A *division* of a figure F is a finite collection $\{A_1, \dots, A_n\}$ of nonoverlapping intervals whose union is F . A *partition* of an interval A is a set $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ such that $\{A_1, \dots, A_p\}$ is a division of A and $x_i \in A_i$, $i = 1, \dots, p$. When all A_1, \dots, A_p are $(\varepsilon, \mathcal{H})$ -intervals for a regulator $(\varepsilon, \mathcal{H})$, then P is called an $(\varepsilon, \mathcal{H})$ -partition. If δ is a positive function on A and $d(A_i) < \delta(x_i)$, $i = 1, \dots, p$, we say that P is δ -fine.

Let f be a function on an interval A and let $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ be a partition of A . We set

$$\sigma(f, P) = \sum_{i=1}^p f(x_i) |A_i|$$

and say that f is *integrable* on A if there is a real number I satisfying the following condition: given a regulator $(\varepsilon, \mathcal{H})$, we can find a positive function δ on A so that $|\sigma(f, P) - I| < \varepsilon$ for each δ -fine $(\varepsilon, \mathcal{H})$ -partition P of A . The number I , which is uniquely determined by f (cf. [6; Corollary 2.5]), is called the *integral* of f over A , denoted by $\int_A f$.

It was established in [6] that over intervals, the integral is a well behaved extension of the Lebesgue integral which yields a very general divergence theorem (see [6; Theorem 5.4]). Our goal is to show that under suitable conditions, the integrability on an interval A follows from the integrability on intervals contained in the interior of A . In dimension one, such a result was established by H. Hake for the Perron integral (cf. [2] and also [8; Chapter VIII, Lemma (3.1), p. 247]), whose Riemann-like definition was given by Kurzweil (see [6]) and independently by Henstock (see [3]). A direct proof for the one-dimensional generalized Riemann integral can be found, e.g., in [7; Lemma 7, (v)].

If C and D are nonempty compact subsets of \mathbf{R}^m , the positive real number

$$\varrho(C, D) = \max \left\{ \sup_{x \in C} \text{dist}(x, D), \sup_{y \in D} \text{dist}(y, C) \right\}$$

is called the *Hausdorff distance* between C and D . We say that a sequence $\{F_n\}$ of figures *converges* to an interval A if $F_n \subset A^0$, $n = 1, 2, \dots$, $\lim \varrho(F_n, A) = 0$, and $\sup |\partial F_n|_{m-1} < +\infty$.

Let f be a function on a set $E \subset \mathbf{R}^m$ such that f is integrable on each interval $A \subset E$. If $F \subset E$ is a figure, we set

$$\int_F f = \sum_{D \in \mathcal{D}} \int_D f$$

where \mathcal{D} is a division of F . By the additivity of the integral (see [6; Proposition 3.6]), the number $\int_F f$ does not depend on the choice of \mathcal{D} . Using this observation, we can conveniently formulate our result.

Proposition. *Let f be a function on an interval A which is integrable on each interval $B \subset A^0$. If a finite $\lim \int_{F_n} f$ exists for every sequence $\{F_n\}$ of figures converging to A , then all these limits have the same value I , f is integrable on A , and $\int_A f = I$.*

Proof. If $\{B_n\}$ and $\{C_n\}$ are sequences of figures converging to A , then so is the sequence $\{B_1, C_1, B_2, C_2, \dots\}$. By our assumption

$$\lim \int_{B_n} f = \lim \int_{C_n} f = I,$$

and it remains to show that f is integrable on A and $\int_A f = I$. According to [6; Corollary 4.2], we may assume that $f(x) = 0$ for each $x \in \partial A$.

Choose a regulator $(\varepsilon, \mathcal{H})$ with $\partial A \subset \bigcup \mathcal{H}$, and construct a locally finite countable family $\{K_1, K_2, \dots\}$ of nonoverlapping intervals with $\bigcup_{n=1}^{\infty} K_n = A^0$. If $x \in A^0$ then the collection $\mathcal{K}_x = \{K_n : x \in K_n\}$ is finite and $x \in (\bigcup \mathcal{K}_x)^0$. For $n = 1, 2, \dots$, let \mathcal{H}_n be a finite collection of planes which is closed with respect to nonempty intersections and assume that $\mathcal{H} \subset \mathcal{H}_n$ and $\partial K_n \subset \bigcup \mathcal{H}_n$. By Henstock's lemma

(see [6; Lemma 4.3]), there is a $\delta_n: K_n \rightarrow (0, 1)$ such that

$$\sum_{i=1}^p |f(x_i) |A_i| - \int_{A_i} f| < \frac{\varepsilon}{2^{n+1}}$$

for each δ_n -fine $(\varepsilon, \mathcal{H}_n)$ -partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ of K_n , $n = 1, 2, \dots$. Making the δ_n 's smaller, we may assume that the following conditions are satisfied:

- (1) $\delta_n(x) < \text{dist}(x, H)$ for each $H \in \mathcal{H}_n$ and each $x \in K_n - H$;
- (2) $\delta_n(x) < \text{dist}(x, A - \bigcup \mathcal{H}_x)$ for each $x \in A^0$;
- (3) $\delta_n(x) = \delta_{n'}(x)$ whenever $x \in K_n \cap K_{n'}$.

Let $h = \sum_{H \in \mathcal{H}} |A \cap H|_{\dim H}$, and find an $\eta \in (0, 1)$ so that $|\int_F f - I| < \varepsilon/2$ for each figure $F \subset A^0$ with $\varrho(A, F) < \eta$ and $|\partial F|_{m-1} < 2mh/\varepsilon$. In view of condition (3), we can define a $\delta: A \rightarrow (0, 1)$ by setting

$$\delta(x) = \begin{cases} \delta_n(x) & \text{if } x \in K_n, \quad n = 1, 2, \dots, \\ \eta & \text{if } x \in \partial A. \end{cases}$$

Let P be a δ -fine $(\varepsilon, \mathcal{H})$ -partition of A , and let

$$Q = \{(B, x) \in P : x \in \partial A\} \cup \bigcup_{x \in A^0} \{(B \cap K_n, x) : K_n \in \mathcal{H}_x, (B \cap K_n)^0 \neq \emptyset\}.$$

Then Q is a δ -fine partition of A and $\sigma(f, P) = \sigma(f, Q)$. Using [6; Lemma 3.5 and Corollary 2.5], it is easy to verify that the collection $Q_n = \{(C, x) \in Q : C \subset K_n\}$ is a subset of a δ_n -fine $(\varepsilon, \mathcal{H}_n)$ -partition of K_n , $n = 1, 2, \dots$; in particular,

$$\sum_{(C,x) \in Q_n} |f(x) |C| - \int_C f| < \frac{\varepsilon}{2^{n+1}}$$

by the choice of δ_n . The interval A is the union of nonoverlapping figures

$$F = \bigcup_{n=1}^{\infty} (\bigcup \{C : (C, x) \in Q_n\}) \quad \text{and} \quad F' = \bigcup \{B : (B, x) \in P, x \in \partial A\}.$$

Now $\varrho(A, F) < \eta$ and

$$|\partial F|_{m-1} \leq |\partial F'|_{m-1} \leq \sum \{|\partial B|_{m-1} : (B, x) \in P, x \in \partial A\} \leq \sum_{B \in \mathcal{B}} |\partial B|_{m-1},$$

where $\mathcal{B} = \{B : (B, x) \in P, B \cap (\bigcup \mathcal{H}) \neq \emptyset\}$. Since each $B \in \mathcal{B}$ is an $(\varepsilon, \mathcal{H})$ -interval, there is a k_B -plane $H_B \in \mathcal{H}$ such that $B \cap H \neq \emptyset$ and $|B \cap H_B|_{k_B} / [d(B)]^{k_B} > \varepsilon$ (cf. [6; Section 2.1]). As $d(B) < 1$ and $k_B \leq m - 1$, we see that

$$|\partial B|_{m-1} \leq 2m[d(B)]^{m-1} \leq 2m[d(B)]^{k_B} < \frac{2m}{\varepsilon} |B \cap H_B|_{k_B}$$

and consequently,

$$\begin{aligned} \sum_{B \in \mathcal{B}} |\partial B|_{m-1} &< \frac{2m}{\varepsilon} \sum_{B \in \mathcal{B}} |B \cap H_B|_{k_B} \leq \frac{2m}{\varepsilon} \sum_{B \in \mathcal{B}} \sum_{H \in \mathcal{H}} |B \cap H|_{\dim H} = \\ &= \frac{2m}{\varepsilon} \sum_{H \in \mathcal{H}} \sum_{B \in \mathcal{B}} |B \cap H|_{\dim H} \leq \frac{2m}{\varepsilon} h. \end{aligned}$$

It follows that

$$\begin{aligned} |\sigma(f, P) - I| &\leq \left| \sum_{n=1}^{\infty} \sum_{(C,x) \in Q_n} f(x) |C| - \int_F f \right| + \left| \int_F f - I \right| < \\ &< \varepsilon/2 + \sum_{n=1}^{\infty} \sum_{(C,x) \in Q_n} |f(x) |C| - \int_C f| < \varepsilon, \end{aligned}$$

and the proof is completed.

Remark. The reader may compare the Proposition with [4; Theorem (6.3)].

If f is integrable on an interval A and $\{A_n\}$ is a sequence of intervals converging to A , it follows from [6; Proposition 4.10] that $\lim \int_{A_n} f = \int_A f$. However, in the Proposition a sequence of figures cannot be replaced by a sequence of intervals. Indeed, for $(x, y) \in \mathbf{R}^2$ set

$$f(x, y) = \begin{cases} |y|/xy & \text{if } x \neq 0 \text{ and } 0 < |y| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let $A = [0, 1] \times [-2, 2]$. Then f is integrable on each interval $B \subset A^0$ and $\lim \int_{A_n} f = 0$ for every sequence $\{A_n\}$ of intervals converging to A . Yet, f is not integrable on A because it is not integrable on $[0, 1]^2$.

On the other hand, the next example shows that if f is integrable on an interval A and $\{F_n\}$ is a sequence of figures converging to A , a finite $\lim \int_{F_n} f$ may not exist.

Example. Let $m = 2$, $A = [0, 1]^2$, and for $k = 1, 2, \dots$, let

$$\begin{aligned} A_{k+} &= [3 \cdot 2^{-k-1}, 2^{-k+1}] \times [2^{-k-1}, 2^{-k}], \\ A_{k-} &= [2^{-k-1}, 2^{-k}] \times [3 \cdot 2^{-k-1}, 2^{-k+1}]. \end{aligned}$$

By [6; Example 6.2], the function f defined by

$$f(x) = \begin{cases} \pm 2^{2(k+1)}/k & \text{if } x \in A_{k\pm}, \\ 0 & \text{if } x \in A - \bigcup_{k=1}^{\infty} A_{k\pm}, \end{cases}$$

is integrable on A . If $F_n = [2^{-2n-1}, 1]^2 - \bigcup_{k=n}^{\infty} (A_{k-})^0$, then $q(A, F_n) \leq 2^{-n}$ and $|\partial F_n|_1 \leq 6$, $n = 1, 2, \dots$. Thus $\{F_n\}$ is a sequence of figures converging to A , and $\lim \int_{F_n} = +\infty$ since

$$\int_{F_n} f = \sum_{k=n}^{2n} \int_{A_{k+}} f = \sum_{k=n}^{2n} (1/n) \geq \log n, \quad n = 1, 2, \dots$$

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